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Theory of Liquid He^3

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Following London (1939) in the case of liquid He^4, a theory of liquid He^3 has been developed. We describe liquid He^3 by picturing it as a metal in which the ions and the electrons are replaced by particles of the same kind viz. He^3 atoms. A fraction of the total number of atoms corresponds to the free electrons in the metal and behaves as an ideal F. D. gas in a potential well, while the rest of the atoms constitute a quasi-crystalline lattice. Agreement between theory and experiment concerning the specific heat, entropy, vapour pressure and nuclear spin magnetic susceptibility (and its variation with pressure) is found to be satisfactory.

§ 1. Introduction

It is natural to attempt an explanation of the properties of liquid He^3 on the basis of models similar to those which have been tried in the case of liquid He^4, for both are essentially quantum liquids. The success of the London's[1)] interpretation of the behaviour of liquid He^4 in terms of the condensation of an ideal B. E. gas clearly suggested a similar approach to the problem of liquid He^3. Possibly because there was no evidence to the contrary, this idea was generally accepted and in the beginning some authors[2),3),4)] applied the ideal F. D. gas model to discuss the properties of liquid He^3. However, there appears to be no valid basis for treating liquid He^3 as an ideal F. D. gas, especially because the recent measurements of the specific heat and entropy,[5),6)] vapour pressure,[7),8)] nuclear spin magnetic susceptibility[9),10),11)] and the surface tension[12),13)] of liquid He^3 are found to be in complete disagreement with the predictions of this simple limiting model (See in this connection references 14), 15)). This led Price,[16)] Rice[17)] and Temperley[18)] to give up the above approach altogether and use the "cell model" of the liquid to explain the observed properties of liquid He^3.

In the present paper we have attempted to show that the possibilities of a F. D. model for liquid He^3 have not been altogether exhausted. In fact, a F. D. gas model, modified on the lines of a corresponding model for liquid He^4, leads to a fairly satisfactory explanation of the properties of liquid He^3. An inevitable consequence of a model of the above type is the existence of a partial nuclear spin allignment in the liquid at all temperatures.

Following London[19)] in the case of liquid He^4, we describe[20)] liquid He^3 by picturing it as a metal in which the ions and the electrons are replaced by particles of the same kind viz. He^3 atoms. A fraction $f(=1/3.8)$ of the total number of atoms corresponds to the free electrons in the metal and behaves as an ideal F. D. gas in a potential well of depth ψ, while the rest of the atoms constitute a quasi-crystalline lattice.

We have taken the F. D. degeneracy temperature to be 0.54°K and the Debye temperature as 8.5°K. The agreement between theoretically calculated and the experimentally observed values of specific heat, entropy and vapour pressure of liquid He^3 is found to be fairly satisfactory. According to the present theory, the measured nuclear spin magnetic susceptibility of liquid He^3 is due only to the fraction which behaves as an ideal F. D. gas with a degeneracy temperature of 0.54°K. This is very close to the degeneracy temperature of 0.45°K required to fit the experimental results of Fairbank et al.[9),10)] with the theoretical F. D. curve. The recent measurements of Walters and Fairbank[11)] concerning the variation of the nuclear spin magnetic susceptibility of liquid He^3 as a function of temperature and pressure are also found to be in fairly good agreement with the present theory.

§ 2. Specific heat and entropy of liquid He^3

We shall now (for the sake of completeness) deduce the expressions for the specific heat and entropy for an assembly of particles obeying F. D. statistics as well as for a quasicrystalline lattice and present them in the form best suited for our purposes, althoughmore general and elegant formulations are well known.

2 A. For an assembly of particles obeying F. D. statistics, the number of particles possessing energy in the range ϵ and $\epsilon+d\epsilon$ is given by

$$n(\epsilon)\cdot d\epsilon = D\cdot\frac{\epsilon^{1/2}\cdot d\epsilon}{1/A\cdot e^{\epsilon/kT}+1}, \tag{2·1}$$

where $D=(2\pi gV/h^3)\cdot(2m)^{3/2}$, g being the weight factor ($g=2$ for He^3 atoms). A is the degeneracy parameter, the other symbols have their usual meanings.

The total number of the particles in the assembly is

$$N=\int_0^\infty n(\epsilon)\cdot d\epsilon = D(kT)^{3/2}\cdot F_{1/2}(\eta), \tag{2·2}$$

where $\eta=\ln A$ and $F_\sigma(\eta)$ is the well-known F. D. integral given by

$$F_\sigma(\eta)=\int_0^\infty\frac{x^\sigma\cdot dx}{e^{x-\eta}+1}. \tag{2·3}$$

The total energy of the assembly is thus given by

$$E=\int_0^\infty \epsilon\cdot n(\epsilon)\cdot d\epsilon = D(kT)^{5/2}\cdot F_{3/2}(\eta) = NkT\cdot F_{3/2}(\eta)/F_{1/2}(\eta). \tag{2·4}$$

The Gibbs free energy of the assembly is related to the degeneracy parameter A by the relation

$$G=NkT\ln A = NkT\cdot\eta, \tag{2·5}$$

and since

$$E=\tfrac{3}{2}pV, \tag{2·6}$$

where p is the vapour pressure, the Helmholtz free energy is expressed as

$$F=G-pV=NkT[\eta-(\tfrac{2}{3})\cdot F_{3/2}(\eta)/F_{1/2}(\eta)]. \tag{2·7}$$

The expression for entropy obtained by using equations (2·4) and (2·7) is

$$S=(E-F)/T=Nk\left[(5/2)\cdot\frac{(2/3)F_{3/2}(\eta)}{F_{1/2}(\eta)}-\eta\right], \tag{2·8}$$

while the specific heat is given by $C=T(\partial S/\partial T)$, or

$$C=Nk\left[(15/4)\cdot\frac{(2/3)F_{3/2}(\eta)}{F_{1/2}(\eta)}-(9/4)\cdot\frac{F_{1/2}(\eta)}{d/d\eta\cdot F_{1/2}(\eta)}\right]. \tag{2·9}$$

Let us now consider the two extreme cases in which the evaluation of the F. D. integral is not very difficult.

(i) *For the case of non-degeneracy*, $A\ll 1$, we have

$$F_\sigma(\eta)=\Gamma_{(\sigma+1)}\cdot\left\{A-\frac{A^2}{2^{\sigma+1}}+\frac{A^3}{3^{\sigma+1}}-\frac{A^4}{4^{\sigma+1}}+\cdots\right\}, \tag{2·10}$$

wheae $\Gamma_{(\sigma)}$ is the well-known 'Gamma-Function'. The total number of particles in the assembly is thus given by

$$N=D(kT)^{3/2}\cdot\Gamma_{(3/2)}\cdot\left\{A-\frac{A^2}{2^{3/2}}+\frac{A^3}{3^{3/2}}-\frac{A^4}{4^{3/2}}+\cdots\right\}. \tag{2·11}$$

And, therefore,

$$A_1\equiv\frac{N}{D(kT)^{3/2}\cdot\Gamma_{(3/2)}}=\sum_{n=1}^{\infty}\frac{A^n}{n^{3/2}}\cdot(-1)^{n-1}. \tag{2·12}$$

The total energy of the assembly is thus given by

$$E=\tfrac{3}{2}NkT[1+\alpha A_1-bA_1^2+cA_1^3+\cdots], \tag{2·13}$$

where

$$\alpha=\frac{1}{2^{5/2}},\ b=\left(\frac{2}{3^{5/2}}-\frac{1}{4^{3/2}}\right),\ c=\left(\frac{3}{4^{5/2}}+\frac{5}{2\times 8^{3/2}}-\frac{3}{6^{3/2}}\right).$$

The Gibbs free energy for the non-degenerate case is thus given by

$$G=NkT\cdot\ln A=NkT[\ln A_1+2\alpha A_1-\tfrac{3}{2}bA_1^2+\tfrac{4}{3}cA_1^3-\cdots]. \tag{2·14}$$

Using equations (2·13) and (2·14) and the relation

$$E=\tfrac{3}{2}pV, \tag{2·6}$$

the expression for the Helmholtz efre energy $F(=G-pV)$ is given by

$$F=NkT[\ln A_1-1+\alpha A_1-\tfrac{1}{2}bA_1^2+\tfrac{1}{3}cA_1^3-\cdots]. \tag{2·15}$$

The entropy is thus given by

$$S=(E-F)/T=Nk[\tfrac{5}{2}-\ln A_1+\tfrac{1}{2}\alpha A_1-bA_1^2+\tfrac{7}{6}cA_1^3-\cdots], \tag{2·16}$$

while the specific heat is given by

$$C=T(\partial S/\partial T)=\tfrac{3}{2}Nk[1-\tfrac{1}{2}\alpha A_1+2bA_1^2-\tfrac{7}{2}cA_1^3+\cdots]. \tag{2·17}$$

(ii) *For the completely degenerate case,* $A\gg 1$.

The integrations in this case are carried out by using Sommerfeld's formula according to which, for large A

$$F_\sigma(\eta)=\frac{\eta^{\sigma+1}}{(\sigma+1)}\cdot\Big\{1+(\pi^2/6)\cdot\frac{\sigma(\sigma+1)}{\eta^2}+(7\pi^4/360)\cdot\frac{(\sigma+1)\sigma(\sigma-1)(\sigma-2)}{\eta^4}+\cdots\Big\}. \tag{2·18}$$

Equation (2·2) thus yields

$$N=D(kT)^{3/2}\cdot(2/3)\eta^{3/2}\Big\{1+(\pi^2/8)\cdot\frac{1}{\eta^2}+(7\pi^4/640)\cdot\frac{1}{\eta^4}+\cdots\Big\}. \tag{2·19}$$

And, defining A_1, by relation (2·12), we have

$$\eta=\ln A=\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\cdot\Big\{1-(\pi^2/12)\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{4/3}}-(\pi^4/80)\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{8/3}}-\cdots\Big\}. \tag{2·20}$$

Similarly, equation (2·4) gives us by using equations (2·18), (2·19) and (2·20)

$$E=(3/5)NkT\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\cdot\Big\{1+(5\pi^2/12)\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{4/3}}-\cdots\Big\}. \tag{2·21}$$

The expression for the Gibbs free energy is thus given by

$$G=NkT\cdot\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\cdot\Big\{1-(\pi^2/12)\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{4/3}}-\cdots\Big\}, \tag{2·22}$$

from which we can easily obtain as before

$$F=(3/5)NkT\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\cdot\Big\{1-(5\pi^2/12)\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{4/3}}-\cdots\Big\}, \tag{2·23}$$

$$S=(\pi^2/2)Nk\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{2/3}}, \tag{2·24}$$

and

$$C=(\pi^2/2)Nk\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{2/3}}. \tag{2·25}$$

In these equations A_1 is given by the expression

$$A_1=\frac{N}{D(kT)^{3/2}\cdot\Gamma_{(5/2)}}=\frac{Nh^3}{gV(2\pi mkT)^{3/2}}. \tag{2·26}$$

The parameter A_1 can also be expressed in terms of the F. D. degeneracy temperature

$$T^*=\epsilon_0/k=(h^2/2mk)\,(3N/4\pi gV)^{2/3}, \tag{2·27}$$

by the relation

$$A_1=1/\Gamma_{(5/2)}\cdot(T^*/T)^{3/2}. \tag{2·28}$$

Thus for large values of A (i.e. for the completely degenerate case), equations (2·24) and (2·25) for entropy and specific heat may be written in the usual form as

$$S=C=(\pi^2/2)\cdot(NkT/T^*)=(\pi^2/2)\cdot(Nk^2T/\epsilon_0)=4.93\,R(kT/\epsilon_0), \tag{2·29}$$

where $R(=Nk)$ is the gas constant.

2B. Specific heat and entropy of a quasi-crystalline lattice

We assume that liquid He^3 forms a quasi-crystalline lattice in which only longitudinal elastic vibrations can be excited. This phonon excitation contributes to the specific heat as well as to the entropy, the expressions for which can be derived on the basis of Debye's[21)] theory for specific heat of solids.

We assume that the vibrations of the liquid (which is treated as a solid) form a continuous spectrum and are identical with the elastic vibrations of a continuous solid. Further, we assume that in the liquid only the longitudinal elastic waves can be propagated and these waves travel with the velocity $C_l=\{(K'+\frac{4}{3}n')/\rho\}^{1/2}$, where n' and K' are the coefficient of rigidity and the bulk modulus respectively while ρ is the density of the liquid.

The number of modes of longitudinal vibrations contained between the frequency limits ν and $\nu+d\nu$ and enclosed in volume V is

$$4\pi\nu^2\cdot d\nu/c_l^3\cdot V. \tag{2·30}$$

The total number of independent modes of vibration is thus given by

$$\int_0^{\nu_m}4\pi V\cdot 1/c_l^3\cdot\nu^2\cdot d\nu. \tag{2·31}$$

The upper limit of integration cannot be infinite, as it would make the number of degrees of freedom infinite. Actually in the present case this number is N, where N is the total number of particles. This enables us to obtain ν_m by putting the integral (2·31) equal to N. This gives us

$$\nu_m^3=\frac{3N}{4\pi V\cdot(1/c_l^3)}. \tag{2·32}$$

Thus following Debye, we cut off the elastic spectrum at an upper limit ν_m which has a definite value for liquid He^3 depending upon the velocity of the longitudinal waves. Since the mean energy of a wave of frequency ν is, by Planck's theorem, $h\nu/(e^{h\nu/kT}-1)$, the energy of the liquid on account of longitudinal excitations is

$$E=4\pi V\cdot\frac{1}{c_l^3}\int_0^{\nu_m}\frac{h\nu^3\cdot d\nu}{e^{h\nu/kT}-1}. \tag{2·33}$$

Combining it with relation (2·32), we have

$$E=\frac{3N}{\nu_m^3}\int_0^{\nu_m}\frac{h\nu^3\cdot d\nu}{e^{h\nu/kT}-1}. \tag{2·34}$$

To evaluate this expression, let us put

$$h\nu/kT=\xi,\ h\nu_m/k=\theta,\ \text{and}\ h\nu_m/kT=\theta/T=\xi_m=x. \tag{2·35}$$

The expression for energy of the liquid on account of longitudinal excitations can thus be written as

$$E=3NkT(T/\theta)^3\int_0^x\frac{\xi^3\cdot d\xi}{e^\xi-1}. \tag{2·36}$$

This expression cannot be integrated in finite terms. Again, differentiating (2·34) with respect to T and making substitutions as above, we have

$$C_v=Nk\left[(12/x^3)\int_0^x\frac{\xi^3\cdot d\xi}{e^\xi-1}-\frac{3x}{e^x-1}\right]\equiv R\cdot D(x). \tag{2·37}$$

This is the expression for the specific heat of liquid He^3 by treating it as a quasi-crystalline solid. It cannot be evaluated in finite terms. The expression within square brackets is a function of $x(=\theta/T)$ only and is usually denoted by $D(x)$. It is known as the Debye function[21)] and has been calculated for values of x ranging from 0 to ∞. In the two extreme cases when $T\gg\theta$, $C_v\approx R$ while for $T\ll\theta$, it simplifies to $C_v=(4\pi^4/5)R(T/\theta)^3$, which is known as "Debye's T^3-Law".

The corresponding expression for the entropy of liquid He^3 treated as a quasi-crystalline solid may be obtained as follows.

Since by definition, the entropy

$$S=\int_0^T(C_v/T)\cdot dT=\int_0^T 1/T\cdot(\partial E/\partial T)\cdot dT, \tag{2·38}$$

the expression for entropy in terms of the energy becomes

$$S=E/T+\int_0^T(E/T^2)\cdot dT. \tag{2·39}$$

This result follows by integrating the integral in (2·38) by parts, and from the fact that (E/T) vanishes as the absolute zero is approached.

Further we try to express E in terms of the Debye function, $D(x)$, the values of which have been calculated for various values of x ranging from 0 to ∞. Equation (2·37) for specific heat may be re-written as

$$\left(\frac{\partial E}{\partial T}\right)_V + \frac{3Nkx}{e^x-1} = \frac{12}{x^3}\cdot Nk\int_0^x \frac{\xi^3\cdot d\xi}{e^\xi-1}. \qquad (2\cdot40)$$

Substituting equation (2·40) in equation (2·36) for E, we have

$$E=\frac{1}{4}\left[T\left(\frac{\partial E}{\partial T}\right)_V + 3NkT\cdot\frac{x}{e^x-1}\right]. \qquad (2\cdot41)$$

Thus we have

$$\int_0^T\left(\frac{E}{T^2}\right)\cdot dT = \frac{1}{4}\left[\frac{E}{T}+\int_0^T\left(\frac{E}{T^2}\right)\cdot dT - 3Nk\int_{x=\infty}^{x=\theta/T}\frac{dx}{e^x-1}\right]. \qquad (2\cdot42)$$

Equation (2·42) can be re-written as

$$\int_0^T\left(\frac{E}{T^2}\right)\cdot dT = \frac{1}{3}\cdot\frac{E}{T} - Nk\ln\left(1-e^{-x}\right). \qquad (2\cdot43)$$

Using equations (2·41) and (2·43) in (2·39), the expression for entropy is given by

$$S=\frac{1}{3}\left(\frac{\partial E}{\partial T}\right)_V + Nk\cdot\frac{x}{e^x-1} - Nk\ln\left(1-e^{-x}\right).$$

Or

$$S=R\left[\frac{1}{3}D(x)+\frac{x}{e^x-1}-\ln\left(1-e^{-x}\right)\right], \qquad (2\cdot44)$$

where $x=\theta/T$, while θ is the Debye temperature. The Helmboltz free energy of liquid He^3 treated as a quasi-crystalline solid is thus given by (using equations (2·41) and (2·44))

$$F=E-TS=-RT\left[\frac{1}{12}D(x)+\frac{1}{4}\cdot\frac{x}{e^x-1}-\ln\left(1-e^{-x}\right)\right]. \qquad (2\cdot45)$$

2C. Comparison with experimental results of specific heat and entropy of liquid He^3

According to the present theory, we describe liquid He^3 by picturing it as a metal in which the ions and the electrons are replaced by particles of the same kind viz. He^3 atoms. Since a fraction f of the total number of atoms corresponds to the free electrons in a metal and behaves as an ideal F. D. gas in a potential well, while the rest of the atoms constitute a quasi-crystalline lattice, we have according to the present model, the following expressions for specific heat and entropy of liquid He^3.

(i) *For the case of non-degeneracy* $A\ll 1$ *or* $T\gg T^*$

Equations (2·17) and (2·37) give us

$$C=\left(\frac{3}{2}\right)Rf\left[1-\frac{1}{2}\alpha A_1+2bA_1^2-\frac{7}{2}cA_1^3+\cdots\right]+R\cdot D(x), \qquad (2\cdot 46)$$

while equations (2·16) and (2·44) give

$$S=Rf\left[\frac{5}{2}-\ln A_1+\frac{1}{2}\alpha A_1-bA_1^2+\frac{7}{6}cA_1^3-\cdots\right]$$
$$+R\left[\frac{1}{3}D(x)+\frac{x}{e^x-1}-\ln(1-e^{-x})\right], \qquad (2\cdot 47)$$

where

$$\alpha=\frac{1}{2^{5/2}},\quad b=\left(\frac{2}{3^{5/2}}-\frac{1}{4^{5/2}}\right),\quad c=\left(\frac{3}{4^{5/2}}+\frac{5}{2\times 8^{5/2}}-\frac{3}{6^{5/2}}\right),$$

while $x=\theta/T$, θ being the Debye temperature. R is the gas constant and has a value 1.986 cal mole^{-1} deg^{-1}. $D(x)$ is the Debye function, the values of which are given in reference (21) for various values of x ranging from 0 to ∞. The parameter A_1 is expressed in terms of the F. D. degeneracy temperature T^* by the relation (2·28).

(ii) *For the case of partial degeneracy,* $A\sim 1$ *or* $T\sim T^*$

In this temperature range He^3 liquid is partially degenerate and, therefore, we have to use the exact formulas. Equations (2·9) and (2·37) give us

$$C=Rf\left[\left(\frac{15}{4}\right)\cdot\frac{(2/3)F_{3/2}(\eta)}{F_{1/2}(\eta)}-\left(\frac{9}{4}\right)\cdot\frac{F_{1/2}(\eta)}{d/d\eta\cdot F_{1/2}(\eta)}\right]+R\cdot D(x), \qquad (2\cdot 48)$$

while the equations (2·8) and (2·44) give us the relation

$$S=Rf\left[\left(\frac{5}{2}\right)\cdot\frac{(2/3)F_{3/2}(\eta)}{F_{1/2}(\eta)}-\eta\right]+R\left[\frac{1}{3}D(x)+\frac{x}{e^x-1}-\ln(1-e^{-x})\right], \qquad (2\cdot 49)$$

where $F_\sigma(\eta)$ is the well-known F. D. integral (defined by equation (2·3)) which has been computed by McDougall and Stoner[22] for various values of η ranging from 20 to -4.0. For a given value of η, the temperature can be calculated from the corresponding values of $kT/\epsilon_0(=T/T^*)$ given in Table 8·1 of reference (22).*

(iii) *For the completely degenerate case,* $A\gg 1$ *or* $T\ll T^*$

In this case the equations (2·29) and (2·37) give us

$$C=4.93\,Rf(T/T^*)+R\cdot D(x), \qquad (2\cdot 50)$$

while the equations (2·29) and (2·44) give

$$S=4.93\,Rf(T/T^*)+R\left[\frac{1}{3}D(x)+\frac{x}{e^x-1}-\ln(1-e^{-x})\right]. \qquad (2\cdot 51)$$

We now try to fix the various arbitrary parameters θ, f and T^* which occur in the theory. The choice of θ is governed by the shape of the specific heat curve above 1°K. According to Debye's theory of specific heat of solids, it has a value given by

---

* Singwi[9] used the relation $F_{1/2}(\eta)=(2/3)(\epsilon_0/kT)^{3/2}=(2/3)(T^*/T)^{3/2}$ to calculate the temperature corresponding to a given value of η [See equation (1·11), reference (22)].

$$\theta = h\nu_m/k = (hc_l/k)\,(3N/4\pi V)^{1/3}, \tag{2·52}$$

in terms of the velocity c_l of the longitudinal waves. Mikura[23)] has calculated the velocity of sound (longitudinal waves) in liquid He^3 by making use of the empirical relation discovered by Rao,[24)]

$$Mc_l^{1/3}\cdot 1/\rho = \gamma\cdot V_c(T_c/M)^{1/6}, \tag{2·53}$$

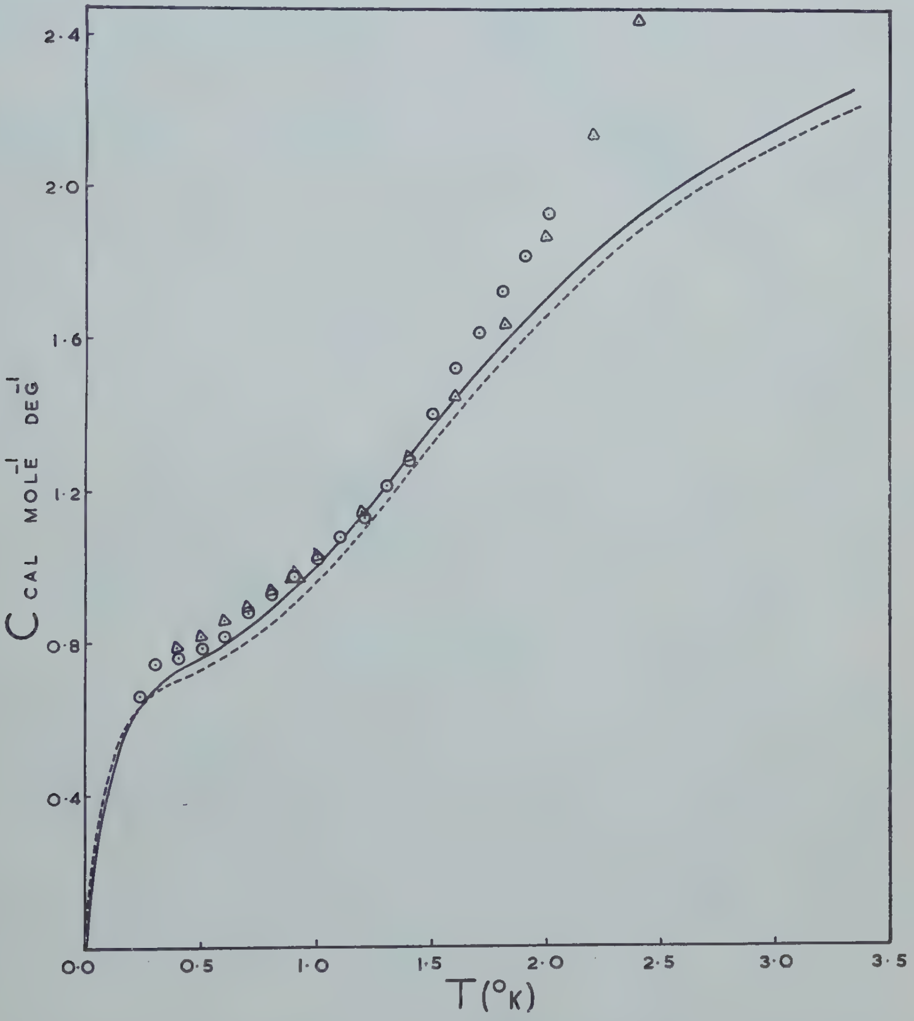

Fig. 1. The specific heat, C cal mole^{-1} deg^{-1}, of liquid He^3 as a function of absolute temperature, T(°K).
——: Present Theory of liquid He^3 with $T^*=0.54$°K, $f=1/3.8$.
- - -: Present Theory of liquid He^3 with $T^*=0.45$°K, $f=1/4.0$.
⊙: Smoothed experimental results of Abraham et al.[5)]
△: Experimental results of Roberts and Sydoriak.[6)]

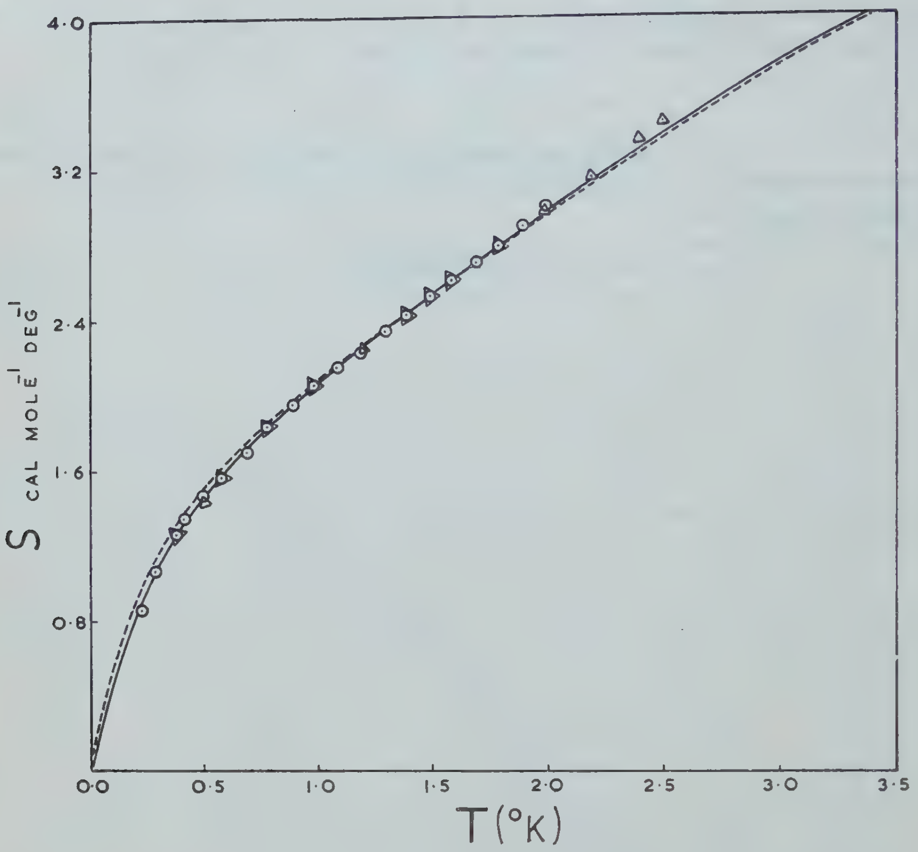

Fig. 2. The entropy, S cal mole^{-1} deg^{-1}, of liquid He^3 as a function of absolute temperature, T(°K).
——: Present Theory of liquid He^3 with $T^*=0.54$°K, $f=1/3.8$.
- - -: Present Theory of liquid He^3 with $T^*=0.45$°K, $f=1/4.0$.
⊙: Smoothed experimental results of Abraham et al.[5)]
△: Experimental results of Roberts and Sydoriak.[6)]

where M and ρ are the molecular weight and the density of the liquid respectively while c_l is the velocity of sound. T_c and V_c are the critical temperature and volume (c.c./mole) respectively. γ is a universal constant, for which, when c_l is measured in m/sec, Mikura[23)] used a value 2.936, which Nomoto and Kishimoto[25)] found to be the most appropriate for liquid He^4. Mikura[23)] found that θ decreases from 9.25 to 7.52°K corresponding to a decrease of c_l from 136 to 113 m/sec as the temperature increases from $T\leq 1.5$ to 2.5°K. We have chosen the Debye temperature of liquid He^3 to be $\theta=8.5$°K.** The parameter $f(=1/3.8)$ is fixed from the shape of the specific heat

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** It has also been pointed out by Temperley[26)] that a Debye temperature of the order of 9°K gives a satisfactory representation of the high temperature part of the specific heat curve for liquid He^3. However, Walters and Fairbank[11)] have estimated the velocity of sound at 1.2°K in liquid He^3 to be 195 meters per second. It implies a Debye temperature ~ 15°K. More experimental data for the coefficient of isothermal compressibility and the velocity of sound at different temperatures is highly desirable so as to give us a better value of θ.

curve below 1°K while $T^*(=0.54°\text{K})$ is chosen so as to get the observed value of entropy5),6) of liquid He^3 at 1.5°K. The calculated values of the specific heat and entropy of liquid He^3, as a function of temperature, by using equations (2·46) to (2·51) have been compared in Fig. 1 and Fig. 2 with the corresponding experimental results.5),6) It is evident from Figures 1 and 2 that the agreement is fairly satisfactory. The discrepancy above 2°K in the case of the specific heat can be attributed to the lowering of the Debye temperature, as has been pointed out by Mikura.23)*

Fairbank et al.,10) however, found that their nuclear spin magnetic susceptibility data of liquid He^3 could be fitted in with the susceptibility of an ideal F. D. gas with $T^*=0.45°\text{K}$. Retaining the original value of $\theta(=8.5°\text{K})$, the observed value of entropy5),6) at 1.5°K implies in this case a value of f equals to 1/4.0. We have, therefore, shown in Figures 1 and 2 the specific heat and entropy computations with these values of T^* and f (dotted curves). The agreement in this case, between theoretically calculated and experimentally observed5),6) values, is not so good as compared to what we have with our first choice of T^* and f. However, the calculated values in this case are still in reasonable accord with the experimental results.5),6)

§ 3. Vapour pressure of liquid He^3

For an independent non-localized ideal gas, the Helmholtz free energy27) is given by

$$(F_v)_{\text{ideal}}=-kT\ln[(p\cdot f\cdot)^N/N!], \qquad (3\cdot1)$$

where

$$(p\cdot f\cdot)=gV_v(2\pi mkT/h^2)^{3/2}. \qquad (3\cdot2)$$

Here g is the weight factor, V_v is the molar volume of He^3 in the vapour phase while the other symbols have their usual meanings.

Using Stirling's formula

$$\ln(N!)=N(\ln N-1),$$

which is valid for large values of N, and $g=2$ for He^3 atom, the expression (3·1) for Helmholtz's free energy reduces to

$$(F_v)_{\text{ideal}}=-NkT[\ln V_v-\ln N+\tfrac{3}{2}\ln T+\ln(2\pi mk/h^2)^{3/2}+\ln 2+1]. \qquad (3\cdot3)$$

Thus the Gibbs free energy for an independent non-localized ideal gas is given by the expression

$$(G_v)_{\text{ideal}}=(F_v)_{\text{ideal}}+PV_v, \qquad (3\cdot4)$$

which by using the relation for an ideal gas

$$PV_v=RT, \qquad (3\cdot5)$$

reduces to

$$(G_v)_{\text{ideal}}=RT[\ln P-\tfrac{5}{2}\ln T-\ln\{(2\pi m/h^2)^{3/2}\cdot k^{5/2}\}-\ln 2]. \qquad (3\cdot6)$$

---

* The results of computations, with variation of θ and T^* with respect to T being taken into account, will be published subsequently.

Assuming vapours of liquid He^3 to be slightly imperfect, the equation of state for the vapours of liquid He^3 can be expressed in the form (ignoring third and higher virial coefficients)

$$V_v = RT/P \cdot [1 + B(T)/V_v], \tag{3·7}$$

where $B(T)$, the second virial coefficient of He^3, is a function of T only. Equation (3·7) gives

$$V_v = RT/P + B - B^2P/RT, \tag{3·8}$$

where the terms involving higher powers of the second virial coefficient have been dropped.

It is difficult to compute directly the partition function ($p.f.$) and thus the Gibbs function of a non-ideal gas.* However, the $p.f.$, F_v and G_v for a slightly imperfect gas must be such as to give us the expression (3·8). For any assembly we have the following thermodynamical relation for volume

$$V = (\partial G/\partial P)_{T,N}. \tag{3·9}$$

Using this relation in equation (3·8) and integrating we get

$$G_v = RT \ln P + BP - \tfrac{1}{2} \cdot (BP)^2/RT + f(T), \tag{3·10}$$

where $f(T)$ is a function that depends on T but not on P, and its form can be determined because G_v must approach $(G_v)_{\text{ideal}}$ when P is very small. Hence the Gibbs free energy for a slightly imperfect gas, up to the second power of the second virial coefficient and by neglecting the higher virial coefficients, can be written as

$$G_v = RT[\ln P - \tfrac{5}{2} \ln T - \ln\{(2\pi m/h^2)^{3/2} \cdot k^{5/2}\} - \ln 2] + BP - \tfrac{1}{2} \cdot (BP)^2/RT. \tag{3·11}$$

With the atomic weight 3.0162 of He^3, the vapour pressure constant in (3·11), including the spin, amounts to

$$\ln[2(2\pi m/h^2)^{3/2} \cdot k^{5/2}] = 5.325,$$

if the pressure is expressed in millimeters of mercury. Accordingly equation (3·11) reads

$$G_v = RT[\ln(P_{mm}/T^{5/2}) - 5.325] + BP - \tfrac{1}{2} \cdot (BP)^2/RT, \tag{3·12}$$

which is precisely the same as that deduced by Chen and London.[28)] This expression is valid until the third and higher virial coefficients of the vapour become significant (probably at about 2.5°K).

Along the vapour pressure line the Gibbs free energy of the vapour is equal to that of the liquid. Thus the vapour pressure equation for He^3 is given by

$$G_v = G_{\text{liq}}. \tag{3·13}$$

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* We are thankful to Professor H. N. V. Temperley for suggesting an indirect way out of this difficulty (Private communication).

The total Gibbs free energy of liquid He^3 is given by

$$G_{\text{liq}}=F_{\text{liq}}+PV_l-\psi, \tag{3·14}$$

where $F_{\text{liq}}[=f(F_{\text{F.D.}})+(F_{\text{phonon}})]$ is the total Helmholtz free energy of liquid He^3 and V_l is the molar volume of He^3 in the liquid phase. Here ψ is a constant which is added to fix up the lowest energy level. (In fact ψ is the " latent heat at absolute zero ").

Using equation (3·12) for G_v and the relations for $F_{\text{F.D.}}$ and F_{phonon} derived in Sec. 2, the vapour pressure of liquid He^3 is thus given by the following expressions :

(i) *For $A\ll 1$ or $T\gg T^*$*

$$\begin{aligned}\log_{10}P_{mm}=&\,2.3126-\frac{\psi}{2.3026\,RT}+2.5\log_{10}T+\log_{10}(1-e^{-x})+f\log_{10}A_1\\&+\frac{f}{2.3026}\left[-1+\alpha A_1-\frac{1}{2}bA_1^2+\frac{1}{3}cA_1^3-\cdots\right]\\&-\frac{1}{2.3026}\left[\frac{1}{12}D(x)+\frac{1}{4}\cdot\frac{x}{e^x-1}\right]\\&+\frac{1}{2.3026\,RT}\left[P(V_l-B)+\frac{(BP)^2}{2RT}\right].\end{aligned}\tag{3·15}$$

(ii) *For $A\sim 1$ or $T\sim T^*$*

$$\begin{aligned}\log_{10}P_{mm}=&\,2.3126-\frac{\psi}{2.3026\,RT}+2.5\log_{10}T+\log_{10}(1-e^{-x})\\&+\frac{f}{2.3026}\left[\eta-\left(\frac{2}{3}\right)\cdot\frac{F_{3/2}(\eta)}{F_{1/2}(\eta)}\right]-\frac{1}{2.3026}\left[\frac{1}{12}D(x)+\frac{1}{4}\cdot\frac{x}{e^x-1}\right]\\&+\frac{1}{2.3026\,RT}\left[P(V_l-B)+\frac{(BP)^2}{2RT}\right].\end{aligned}\tag{3·16}$$

(iii) *For $A\gg 1$ or $T\ll T^*$*

$$\begin{aligned}\log_{10}P_{mm}=&\,2\cdot 3126-\frac{\psi}{2.3026\,RT}+2.5\log_{10}T+\log_{10}(1-e^{-x})\\&+\frac{f}{2.3026}\left(\frac{3}{5}\right)\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\cdot\left\{1-\left(\frac{5\pi^2}{12}\right)\cdot\frac{1}{[\Gamma_{(5/2)}\cdot A_1]^{4/3}}-\cdots\right\}\\&-\frac{1}{2.3026}\left[\frac{1}{12}D(x)+\frac{1}{4}\cdot\frac{x}{e^x-1}\right]\\&+\frac{1}{2.3026\,RT}\left[P(V_l-B)+\frac{(BP)^2}{2RT}\right].\end{aligned}\tag{3·17}$$

We have calculated the vapour pressure of liquid He^3, as a function of temperature, by using equations (3·15), (3·16) and (3·17) and both the values of f. We choose ψ arbitrarily in order to get agreement with the observed vapour pressure[7),8)] at 1.5°K. The values of ψ are found to be 5.21 and 5.12 cal mole^{-1} respectively for $f=1/3.8$

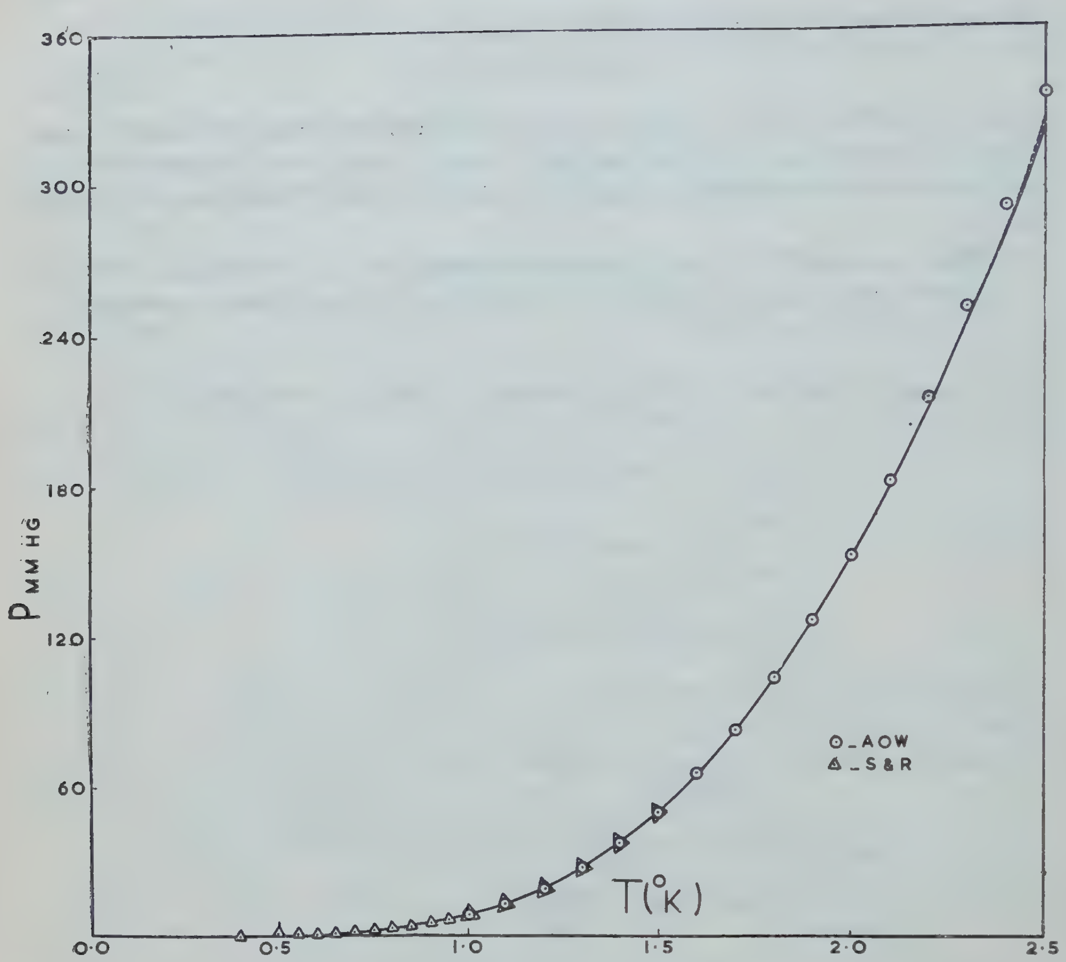

Fig. 3. The vapour pressure, P_{mmHg}, of liquid He^3 as a function of absolute temperature, T(°K).
——: Present Theory of Liquid He^3 with $T^*=0.54$°K, $f=1/3.8$.
– – –: Present Theory of Liquid He^3 with $T^*=0.45$°K, $f=1/4.0$.
⊙: Smoothed experimental results of Abraham et al.[7]
△: Smoothed experimental results of Sydoriak and Roberts.[6]

and $f=1/4.0$. These values of ψ are in fairly good agreement with the value 5.25 cal mole^{-1} for the "latent heat of liquid He^3 at absolute zero", computed by Trikha and Nanda[29] from the Chen and London[28] vapour pressure equation for liquid He^3. The vapour imperfection corrections have been applied using the second virial coefficients computed by Kilpatrick et al.[10] For the molar volume, V_l, of He^3 in the liquid phase, we have used the recently determined data of densities of liquid He^3 by Kerr.[11]

In Fig. 3 we have compared the vapour pressure of liquid He^3 on the basis of the present theory (by using both the values of f) with the corresponding experimental results.[7],[8] No calculations have been made above 2.5°K because the equation of state (3·7), is inadequate for the saturated vapour above this temperature. The higher virial coefficients of liquid He^3, about which we have no knowledge so far, should be included for higher temperatures. It is evident from Fig. 3 that the agreement between theory and experimental data is fairly satisfactory.

§ 4. Nuclear spin magnetic susceptibility of liquid He^3

4 A. Spin magnetic moment of an assembly of particles obeying F. D. statistics.

When an external magnetic field H acts on an assembly of atoms of an ideal F. D. gas, the magnetic energy of an atom will have only two values—either $+H\mu$ if the spin is parallel to H, or $-H\mu$ if the spin is anti-parallel to H, where μ denotes the nuclear magnetic moment of the atom. The magnetic moment M produced by the magnetic field H can be determined as usual from the F, the Helmholtz free energy, by the relation

$$M = -(\partial F/\partial H)_{T,V}, \tag{4·1}$$

and we obtain, proceeding in the usual way,[2] the result

$$M = \frac{N\mu^2 H}{kT}\left[\frac{F'_{1/2}(\eta) + (\beta_1^2/6)F'''_{1/2}(\eta) + \cdots}{F_{1/2}(\eta) + (\beta_1^2/2)F''_{1/2}(\eta) + \cdots}\right], \tag{4·2}$$

where

$$\beta_1 = \mu H/kT,\ F'_{1/2}(\eta) = dF_{1/2}(\eta)/d\eta,\ F''_{1/2}(\eta) = d^2F_{1/2}(\eta)/d\eta^2$$

and

$$F'''_{1/2}(\eta) = d^3F_{1/2}(\eta)/d\eta^3.$$

However, the expressions for magnetic moment can also be determined in a more straightforward way by using the Gibbs free energy[3] instead of the Helmholtz free energy, and this is equivalent to using the generalized form of Boltzmann's law. Again, for the sake of completeness, we will give a brief account of the relations we require for our purposes.

In the presence of the magnetic field, let (N_+) denote the number of atoms in the assembly possessing spins parallel to H and (N_-) possessing spins antiparallel to H, and let the difference between them be denoted by x_1, i. e. $(N_+) = (N_-) - x_1$. Then from a well-known theorem in thermodynamics $[G = NkT \ln A = NkT \cdot \eta]$ we have

$$(\eta_+) - (\eta_-) = -2\mu H/kT,$$

or

$$(A_+)/(A_-) = e^{-2\mu H/kT}, \tag{4·3}$$

where (A_+) and (A_-) are the degeneracy parameters (absolute activities) for the (N_+) atoms and (N_-) atoms respectively.

(*a*) *For the case of non-degeneracy*, $A \ll 1$ or $T \gg T^*$,

$2\mu H/kT \ll 1$, and the terms in equation (4·3) can be expanded in a series. Now inverting the series in equation (2·12) we have

$$(A_+) = (A_{1+})[1+\alpha_1(A_{1+})+\alpha_2(A_{1+})^2+\cdots], \tag{4·4}$$

and

$$(A_-) = (A_{1-})[1+\alpha_1(A_{1-})+\alpha_2(A_{1-})^2+\cdots], \tag{4·5}$$

where

$$\alpha_1=\frac{1}{2^{3/2}}, \quad \alpha_2=\left(\frac{2}{4^{3/2}}-\frac{1}{3^{3/2}}\right),$$

and

$$(A_{1+}) = \frac{2(N_+)}{D(kT)^{3/2}\cdot\Gamma_{(3/2)}} = \frac{(N_+)}{N}\cdot 2A_1. \tag{4·6}$$

Similarly

$$(A_{1-}) = \frac{(N_-)}{N}\cdot 2A_1. \tag{4·7}$$

Therefore,

$$\left(\frac{A_+}{A_-}\right)=\frac{(N_+)}{(N_-)}\left[1+\left\{\frac{(N_+)-(N_-)}{N}\right\}2\alpha_1 A_1+\left\{\frac{(N_-)^2-(N_-)^2}{N^2}\right\}4\alpha_2 A_1^2\right.$$
$$\left.+\left\{\frac{(N_-)^2-(N_+)(N_-)}{N^2}\right\}4\alpha_1^2 A_1^2+\cdots\right]. \tag{4·8}$$

But,

$$(N_+) = \frac{N}{2}\left[1-\frac{x_1}{N}\right] \text{ and } (N_-) = \frac{N}{2}\left[1+\frac{x_1}{N}\right]. \tag{4·9}$$

Hence

$$\frac{(A_+)}{(A_-)}=1-\frac{2x_1}{N}[1+\alpha_1 A_1+(2\alpha_2-\alpha_1^2)A_1^2+\cdots]+\frac{2x_1^2}{N^2}-\frac{2x_1^3}{N^3}+\cdots, \tag{4·10}$$

$$=e^{-2\mu H/kT}=1-\frac{2\mu H}{kT}+\frac{2\mu^2H^2}{k^2T^2}-\frac{4}{3}\cdot\frac{\mu^3H^3}{k^3T^3}+\cdots, \tag{4·11}$$

which gives, after a little algebraic simplification

$$M=\mu x_1=\frac{\mu^2 HN}{kT}\left[1-\alpha_1 A_1-2(\alpha_2-\alpha_1^2)A_1^2-\frac{1}{3}\left(\frac{\mu H}{kT}\right)^2-\cdots\right], \tag{4·12}$$

because x_1 is the number of atoms (each of nuclear magnetic moment μ) which contribute to the magnetism of the assembly.

(*b*) *For the completely degenerate case*, $A \gg 1$ *or* $T \ll T^*$, we have from equation (4·3)

$$\ln(A_+)-\ln(A_-)=-2\mu H/kT. \tag{4·13}$$

Now according to equation (2·20),

$$\ln A=\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\cdot\left[1-\left(\frac{\pi^2}{12}\right)\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{4/3}}-\left(\frac{\pi^4}{80}\right)\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{8/3}}-\cdots\right]$$
$$=f(N), \text{ say.} \tag{4·14}$$

Then

$$\ln(A_+)=f\left\{\frac{N}{2}\left(1-\frac{x_1}{N}\right)\right\} \text{ and } \ln(A_-)=f\left\{\frac{N}{2}\left(1+\frac{x_1}{N}\right)\right\},$$

and

$$\ln(A_+)-\ln(A_-)=-x_1\frac{\partial f(N/2)}{\partial(N/2)}-\frac{x_1^3}{24}\cdot\frac{\partial^3 f(N/2)}{\partial(N/2)^3}-\cdots. \qquad (4\cdot15)$$

Noting that

$$\frac{\partial A_1^t}{\partial N}=\left(\frac{t}{N}\right)A_1^t,$$

we have from equations (4·13), (4·14) and (4·15)

$$\frac{2\mu H}{kT}=\left(\frac{4}{3}\right)\cdot\left(\frac{x_1}{N}\right)\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}\left[1+\left(\frac{\pi^2}{12}\right)\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{4/3}}\right.$$
$$\left.+\left(\frac{3\pi^4}{80}\right)\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{8/3}}+\cdots\right]+\frac{8}{81}\left(\frac{x_1}{N}\right)^3\cdot\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}+\cdots, \qquad (4\cdot16)$$

so that

$$M=\mu x_1=\left(\frac{3}{2}\right)\frac{\mu^2 HN}{kT}\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{2/3}}\left[1-\left(\frac{\pi^2}{12}\right)\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{4/3}}\right.$$
$$\left.-\left(\frac{11\pi^4}{360}\right)\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^{8/3}}-\cdots\right]-\frac{1}{4}\mu N\left(\frac{\mu H}{kT}\right)^3\cdot\frac{1}{\{\Gamma_{(5/2)}\cdot A_1\}^2}-\cdots, \qquad (4\cdot17)$$

or

$$M=\left(\frac{3}{2}\right)\frac{\mu^2 HN}{kT^*}\left[1-\left(\frac{\pi^2}{12}\right)\left(\frac{T}{T^*}\right)^2-\left(\frac{11\pi^4}{360}\right)\cdot\left(\frac{T}{T^*}\right)^4\right.$$
$$\left.-\frac{1}{6}\left(\frac{\mu H}{kT}\right)^2\cdot\left(\frac{T}{T^*}\right)^2-\cdots\right], \qquad (4\cdot18)$$

where use has been made of equation (2·28) in writing this equation in terms of T^*.

The expressions (4·12) and (4·18) obtained by Singh[33] for the paramagnetic moments for the non-degenerate and degenerate cases respectively, using the Gibbs free energy are the same as obtained by Stoner[32] using the Helmholtz free energy. In fact, the presently quoted expressions are given to a higher order term for the variation of M with temperature.

4 B. Comparison with experimental results of nuclear spin magnetic susceptibility of liquid He³.

It is significant to note that in our computations of the thermodynamic quantities of liquid He³ we have not taken into account the contribution to the total energy of the liquid arising out of the possible random alignment of the nuclear spin of atoms which constitute the quasi-crystalline lattice. Thus, according to the present theory the nuclear spins of these atoms are aligned for all temperatures and therefore make no contribution to the nuclear spin magnetic (paramagnetic) susceptibility. The measured

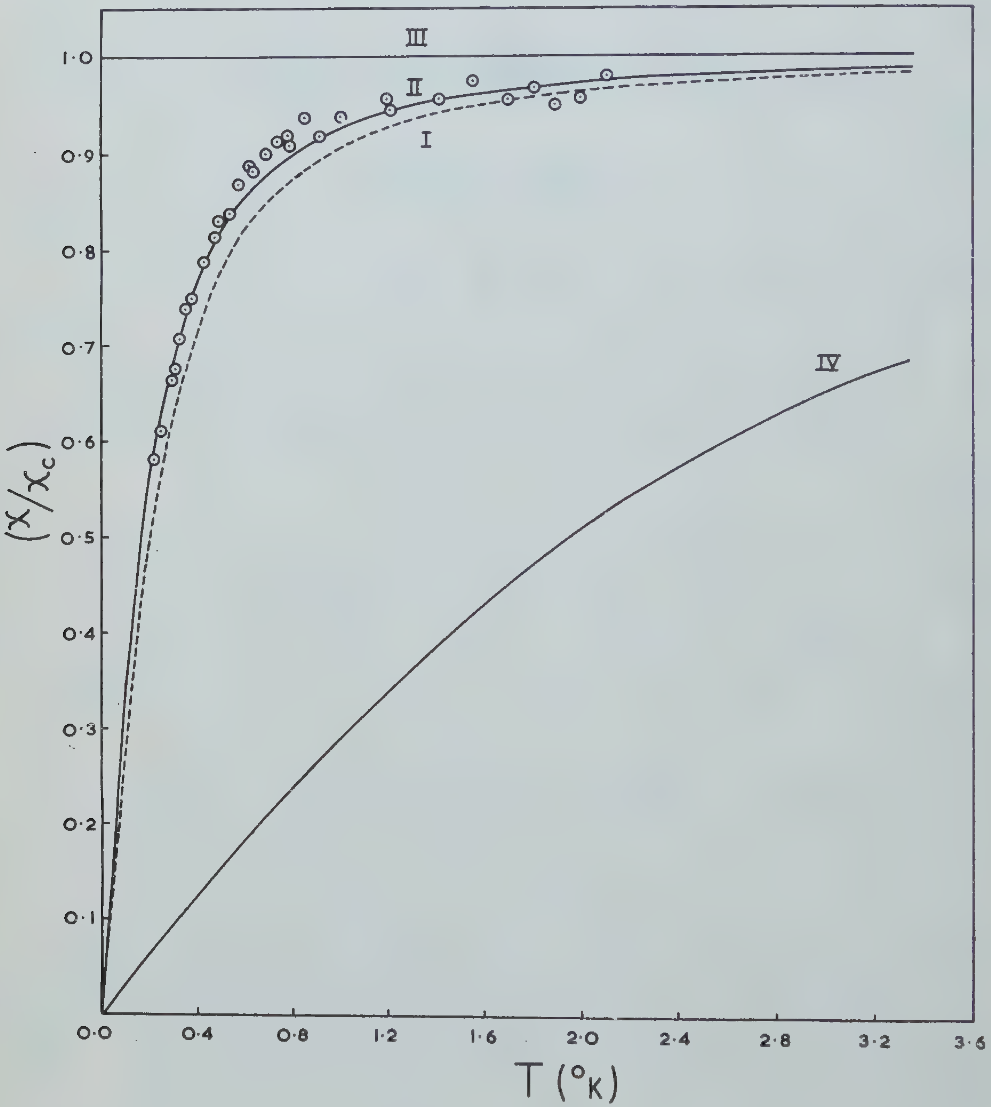

Fig. 4. The ratio of the nuclear spin magnetic susceptibility, χ, of liquid He^3, to the classical (i.e. for very large T) nuclear spin magnetic susceptibility, χ_c, as a function of absolute temperature, T(°K).

Curve I: Present Theory of Liquid He^3 with $T^*=0.54$°K, $f=1/3.8$.
Curve II: Present Theory of Liquid He^3 with $T^*=0.45$°K, $f=1/4.0$.
Curve III: Boltzmann's Statistics (Curie's paramagnetic susceptibility).
Curve IV: Ideal F. D. gas model with $T^*=5$°K.
⊙: Experimental results of Fairbank et al.[10]

paramagnetic susceptibility is, therefore, due only to the fraction f of He^3 atoms which behaves as an ideal F. D. gas. Thus according to the present theory, the paramagnetic (nuclear spin magnetic) volume susceptibility $\chi(=M/HV)$ is given by (ignoring terms of higher power in H) the following expressions :

(i) *For* $A \ll 1$ *or* $T \gg T^*$

$$\chi = f(n\mu^2/kT)[1-\alpha_1 A_1 - 2(\alpha_2-\alpha_1^2)A_1^2 - \cdots], \tag{4·19}$$

where n is the number of He^3 atoms per unit volume.

(ii) *For* $A \sim 1$ *or* $T \sim T^*$

$$\chi = f(n\mu^2/kT)\cdot F'_{1/2}(\eta)/F_{1/2}(\eta). \tag{4·20}$$

In this case of partial degeneracy exact values of the F. D. functions, computed by McDougall and Stoner[22] have to be used.

(iii) *For* $A \gg 1$ *or* $T \ll T^*$

$$\chi = (3/2)f(n\mu^2/kT^*)[1-(\pi^2/12)\cdot(T/T^*)^2 - (11\pi^4/360)\cdot(T/T^*)^4 - \cdots]. \tag{4·21}$$

We have calculated the nuclear spin magnetic susceptibility of liquid He^3, as a function of temperature, by using equations (4·19), (4·20) and (4·21) and F. D. degeneracy temperature of 0.54°K. The values so obtained for $(\chi/\chi_c) = [\chi(kT/fn\mu^2)]$ are compared in Fig. 4 with the experimental results.[10] This degeneracy temperature of 0.54°K is very close to the degeneracy temperature of 0.45°K required by Fairbank et al.[10] to fit their experimental results with the theoretical F. D. curve. In Fig. 4 we have also included the theoretically computed values of (χ/χ_c) for $T^*=0.45$°K. For the sake of comparison, we have also plotted in Fig. 4 the theoretical results of liquid He^3 treated as an ideal F. D. gas with the same density and atomic mass as liquid He^3 ($T^*=5$°K).

It is evident from Fig. 4 that the nuclear spin magnetic susceptibility data of liquid He^3, as reported by Fairbank et al.[10] is in fairly good agreement with the present theory rather than with the ideal F. D. gas model.

§ 5. Variation with pressure of nuclear spin magnetic susceptibility of liquid He^3

Recently, Walters and Fairbank[11] have reported the measurements of the nuclear spin magnetic susceptibility of liquid He^3 as a function of pressure. In Fig. 5 we have plotted the nuclear spin magnetic susceptibility data of liquid He^3 at 1.2°K as a function of density of liquid He^3. It is evident from Fig. 5 that, at constant temperature, the nuclear spin magnetic susceptibility varies linearly with the density of liquid He.^{3} This is in agreement with the predictions of the present theory according to which we have, at constant temperature,

$$\chi = n \times \text{constant} = (\rho/m) \times \text{constant}, \tag{5·1}$$

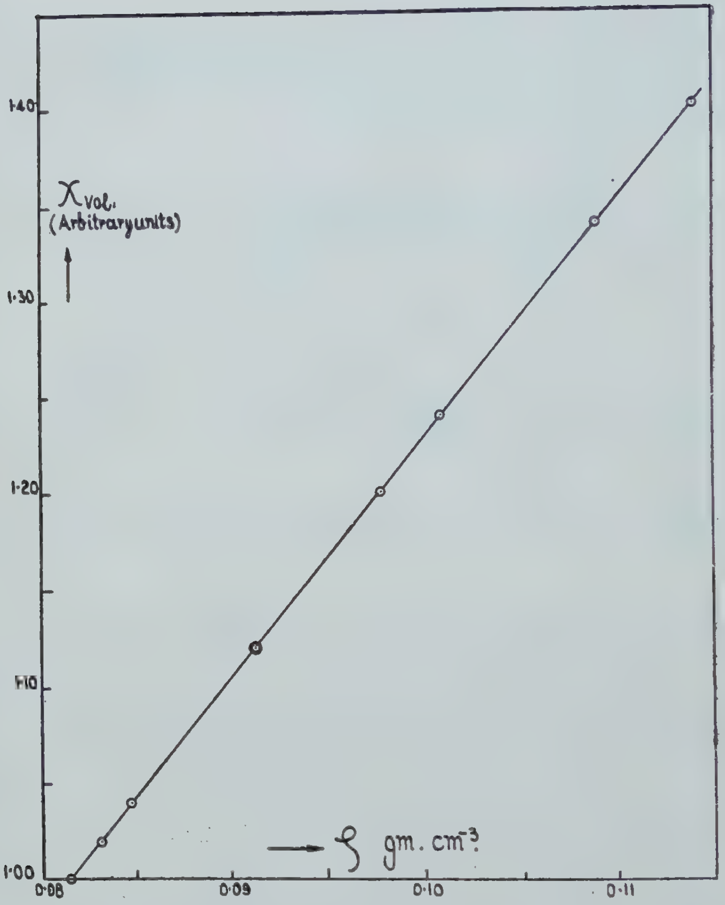

Fig. 5. The nuclear spin magnetic susceptibility, χ_{vol} (arbitrary units), of liquid He^3 at 1.2°K as a function of density, ρ gm·cm^{-3}, of liquid He^3.

——: Present Theory.

⊙: Experimental results of Walters and Fairbank.[11)]

where ρ is the density of liquid He^3 while m is the mass of a He^3 atom.

The observed fact (as pointed out by Walters and Fairbank[11)]) that the deviations of the nuclear spin magnetic susceptibility of liquid He^3 from the Curie susceptibility law become less pronounced as the pressure over the liquid is increased, seems to be in agreement with the predictions of the present model. In Table 1 we have compared the experimental data, of the variation of χ with pressure, with the calculated values on the basis of the present theory as well as on the basis of an ideal F. D. gas model (with $T^*=5$°K). It is evident from Table 1 that the experimental data, of the variation with pressure of nuclear spin magnetic susceptibility of liquid He^3 as reported by Walters and Fairbank,[11)] is in fairly good agreement with the present theory rather than with the ideal F. D. gas model.

Table 1. The nuclear spin magnetic susceptibility, χ, of liquid He^3 as a function of pressure and temperature.

T(°K)	Pressure	χ			
		Ideal F. D. gas model with $T^*=5$°K	Present Theory with		Exptal.
			$T^*=0.54$°K, $f=1/3.8$	$T^*=0.45$°K, $f=1/4.0$	
0.45	Sat. V. P.	0.14	0.75	0.79	0.80
0.33	21.7 atmos.	0.14	0.86	0.93	0.80

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ADDENDA

On Solutions of He^3 and He^4
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Prog. Theor. Phys. **17** (1957), 679.

Concerning equation (2. 1),* p.681:

The assumption (i) that "He^3 mixes with the whole of He^4" implies that the normal part of He^4 cannot be considered as separated in space from the superfluid part. In fact, we take for granted that the normal as well as the superfluid parts of He^4 are superposed in space. Following Landau (L. Landau, J. Phys. U.S.S.R. **5** (1953), 1941), 71.), we imagine that the normal particles move about, collide with each other and the walls, and imitate the appearance that in a perfect background superfluid there is another normal fluid. Then following Feynman (Phys. Rev. **91** (1953), 1291; **91** (1953), 1301; **94** (1954), 262), who has extended these ideas for the case of He^3 and He^4 solutions, we assume that in the perfect background superfluid there is another fluid consisting of normal part particles of He^4 and He^3 atoms. Thus the superfluid part of He^4 contributes to the density, because it occupies the background and fills the space of the container.

The assumption (ii) means that as far as the entropy of mixing is concerned, the contribution of the superfluid part is zero. As the superfluid part of He^4 forms the background in which the normal part particles of He^4 and He^3 atoms move about and mix together, we expect no contribution to the entropy of mixing from the background fluid which has zero entropy of its own. Thus the third term (mixing term) in eq. (2·1) follows immediately. The details will be published subsequently.

* This note was originally sent as an added footnote, but unfortunately it came in too late for the proof reading.

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On the Electron-Lattice Interaction in Non-Polar Semiconductors

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The two-phonon process in non-polar semiconductors is discussed and the acoustical mode-scattering mobility is derived from this type of electron-lattice interaction. The obtained mobility depends on T^{-2}. This result is combined with the usual one which depends on $T^{-1.5}$, and it is concluded that the one-phonon process treated in the usual theory is predominant at lower temperature while the two-phonon process is predominant at higher temperature. Then it is shown that the transition of mobility from the one-phonon region to the two-phonon region well accounts for the transition of temperature dependence observed on the lattice scattering mobility in each case of electrons and holes in germanium and silicon. The small deviations of temperature dependence of the observed mobilities from T^{-2} or $T^{-1.5}$ seem to be explained with the optical mode-scattering.

§ 1. Introduction

The temperature dependence of lattice scattering mobility in non-polar semiconductors has been reported[1)-4)] as being different from the theoretical prediction obtained by assuming that a) the lattice vibration is acoustical, b) energy surfaces near the band edge which locates at $\boldsymbol{k}=0$ have spherical symmetry and c) the electron-lattice interaction Hamiltonian is linearly dependent on the lattice displacement, in other words, the one-phonon process alone takes place. Because of this disagreement, it is necessary to reconsider the three conditions listed above. The optical mode scattering and the interband transition have been discussed by several authors,[5)-8)] and their results are still far from the perfect solution of the problem. The purpose of the present paper is to discuss the two-phonon process and explain the temperature dependence of lattice scattering mobility by the consideration of this process.

The two-phonon process has been considered to be insignificant in the usual theory, which is successful in metals. In non-degenerate semiconductors, however, it can not always be considered to be so in view of the following fact. Since the conduction electrons are thermal in these materials, their wave numbers are very small compared with the maximum wave number of phonons at usual temperature, and the number of phonons to scatter conduction electrons is a very small fraction of the total number of phonons in the case of one-phonon process because of the conservation of energy and momentum. On the other hand, all phonons can interact with conduction electrons in the case of two-phonon process.

In § 2, the transition probability is derived from the second order term of electron-lattice interaction Hamiltonian, and the mean lifetime is calculated in § 3. In § 4 the mobility is obtained and compared with the observed one.

§ 2. Transition probability

According to the deformable atom hypothesis, the perturbing potential for electrons is given, including the second order term by

$$-\boldsymbol{R}\cdot\mathrm{grad}\, V(\boldsymbol{r})+\tfrac{1}{2}(\boldsymbol{R}\cdot\mathrm{grad})^2 V(\boldsymbol{r}), \tag{1}$$

where $V(\boldsymbol{r})$ is the potential in the normal lattice, $\boldsymbol{r}$ is the electronic coordinate and

$$\boldsymbol{R}(\boldsymbol{r})=\frac{1}{\sqrt{2\rho}}\sum_{q}\sum_{j}\frac{1}{\sqrt{\omega(\boldsymbol{q},j)}}\boldsymbol{e}(\boldsymbol{q},j)\,\{a(\boldsymbol{q},j)e^{i\boldsymbol{q}\cdot\boldsymbol{r}}+a^*(\boldsymbol{q},j)e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\}, \tag{2}$$

in which ρ is the density of the crystal, $\omega(\boldsymbol{q},j)$ is the frequency of vibration having wave number $\boldsymbol{q}$ and polarization index j, $\boldsymbol{e}(\boldsymbol{q},j)$ is the polarization vector of this mode of vibration, and $a(\boldsymbol{q},j)$ and $a^*(\boldsymbol{q},j)$ are the amplitude and its conjugate of this mode, respectively. In this paper we exclude the optical mode of vibration and consider (2) to represent the acoustical mode. Since the first term of (1) has been discussed in the usual theory,[9] only the second term is discussed here, which is

$$H=\tfrac{1}{2}(\boldsymbol{R}\cdot\mathrm{grad})^2 V(\boldsymbol{r}). \tag{3}$$

Although the magnitude of this term is always smaller than that of the first term of (1), the effect of this term on the scattering of electrons will be possible to be dominant according to the discussion in § 1.

The electronic wave function is expressed as

$$\psi_k(\boldsymbol{r})=(1/\sqrt{N})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\chi_k(\boldsymbol{r}), \tag{4}$$

where N is the number of atoms in the specimen of crystal and $\chi_k(\boldsymbol{r})$ is the function having the periodicity of the lattice.

Eq. (3) includes two groups of terms, one of which depends on $(\boldsymbol{e}\cdot\boldsymbol{q}')(\boldsymbol{e}'\cdot\mathrm{grad}\, V)$ and the other $\boldsymbol{e}\cdot\mathrm{grad}(\boldsymbol{e}'\cdot\mathrm{grad}\, V)$, where $\boldsymbol{e}$ and $\boldsymbol{e}'$ are the abbreviations of $\boldsymbol{e}(\boldsymbol{q},j)$ and $\boldsymbol{e}(\boldsymbol{q}',j')$, respectively. The former group results from the fact that the lattice displacement is not uniform and the latter group is the usual term from the second derivative of potential. Since the spatial change of lattice waves is usually much smaller than that of electronic potential, the former group of terms mentioned above is much smaller than the latter and may be neglected as shown in Appendix.

Using (4) and neglecting terms depending on $(\boldsymbol{e}\cdot\boldsymbol{q}')(\boldsymbol{e}'\cdot\mathrm{grad}\, V)$, we obtain the matrix element of (3), without time factor,

$$\begin{aligned}H(\boldsymbol{k},\boldsymbol{k}')=\frac{\hbar}{4N\rho}\sum_{q}\sum_{q'}\sum_{j}\sum_{j'}\left(\frac{nn'}{\omega\omega'}\right)^{1/2}\int d\tau\, e^{i(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{r}}\cdot\chi_{k'}^*\chi_k\boldsymbol{e}\cdot\mathrm{grad}(\boldsymbol{e}'\cdot\mathrm{grad}\, V)\\ +\text{three terms depending respectively on } n(n'+1),\\ (n+1)n' \text{ and } (n+1)(n'+1),\end{aligned} \tag{5}$$

where $\hbar$ is Planck's constant devided by 2π, n is the abbreviation of $n(\boldsymbol{q},j)$ which is the number of phonons specified with $(\boldsymbol{q},j)$, and n' is the abbreviation of $n(\boldsymbol{q}',j')$.

The integral in the first term of (5) vanishes, unless $\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}+\boldsymbol{q}'=\boldsymbol{K}$, where $\boldsymbol{K}$ is a basic vector of reciprocal lattice in which the zero-vector is included. Here we neglect the "Umklapp"-process ($\boldsymbol{K}\neq 0$) as usual, hence we obtain the law of momentum conservation

$$\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}+\boldsymbol{q}'=0. \tag{6}$$

When (6) is satisfied, the integral in the first term of (5) is reduced to

$$N\int d\tau_0 \chi_{k'}^{*}\chi_k \boldsymbol{e}\cdot\mathrm{grad}\,(\boldsymbol{e}'\cdot\mathrm{grad}\,V), \tag{7}$$

where the integral is to be performed over full space within a Wingner-Seitz sphere. Assuming that χ_k has spherical symmetry and eliminating V by the use of the Schrödinger equation, we can reduce (7) to approximately

$$\tfrac{8}{3}ND(\boldsymbol{e}\cdot\boldsymbol{e}'), \tag{8}$$

where

$$D=\frac{1}{4}\int d\tau_0\left\{\frac{\hbar^2}{2m}\Delta\chi_k\left(\Delta\chi_k^{*}+\frac{1}{\chi_k}|\mathrm{grad}\,\chi_k|^2\right)+2E_0\chi_k^{*}\Delta\chi_k\right\}, \tag{9}$$

in which m is the electronic true mass, Δ is Laplacian operator and E_0 is the total energy of the electron with $\boldsymbol{k}=0$. Since we do not have reliable values of χ_k, we shall treat D as an unknown parameter in the following discussion.

From (5) and (8), we obtain the matrix element

$$H(\boldsymbol{k},\boldsymbol{k}')=\frac{2\hbar D}{3\rho}\sum_{q}\sum_{j}\sum_{j'}\left(\frac{nn'}{\omega\omega'}\right)^{1/2}(\boldsymbol{e}\cdot\boldsymbol{e}') \tag{10}$$

for the transition with simultaneous absorption of two phonons, where

$$\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}+\boldsymbol{q}'=0. \tag{11}$$

The probability of this transition may be calculated with the use of the conventional time-dependent perturbation theory which was proved to be valid by Seitz,9) and it becomes

$$P(\boldsymbol{k},\boldsymbol{k}')=\frac{8\pi\hbar D^2}{9\rho^2}\sum\sum\sum\frac{nn'}{\omega\omega'}(\boldsymbol{e}\cdot\boldsymbol{e}')^2\delta(\mathcal{E}_1), \tag{12}$$

where

$$\mathcal{E}_1=E-E'+\hbar\omega+\hbar\omega', \tag{13}$$

in which

$$E=(\hbar^2/2m^*)k^2,\quad E'=(\hbar^2/2m^*)k'^2 \tag{14}$$

and m^* is the effective mass of the electron. Provided that the frequency of lattice vibration is independent of the direction of polarization, the summation over all values of j and j' can be performed easily, and

$$\sum_{j}\sum_{j'}(\boldsymbol{e}\cdot\boldsymbol{e}')^2=3. \tag{15}$$

Then Eq. (12) is reduced to

$$P(\boldsymbol{k}, \boldsymbol{k}') = \frac{8\pi\hbar D^2}{3\rho^2}\sum_q \frac{nn'}{\omega\omega'}\delta(\mathcal{E}_1). \tag{16}$$

Three other terms of Eq. (5) may be calculated in the same way. The second term and the third term of (5) represent the same process in which one phonon is absorbed and the other is emitted simultaneously. The last term represents the process in which two phonons are emitted simultaneously. The probability of the former process is

$$P(\boldsymbol{k}, \boldsymbol{k}') = \frac{16\pi\hbar D^2}{3\rho^2}\sum_q \frac{n(n'+1)}{\omega\omega'}\delta(\mathcal{E}_2), \tag{17}$$

where

$$\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}-\boldsymbol{q}'=0 \tag{18}$$

and

$$\mathcal{E}_2 = E-E'+\hbar\omega-\hbar\omega'. \tag{19}$$

The probability of the latter process is

$$P(\boldsymbol{k}, \boldsymbol{k}') = \frac{8\pi\hbar D^2}{3\rho^2}\sum_q \frac{(n+1)\,(n'+1)}{\omega\omega'}\delta(\mathcal{E}_3), \tag{20}$$

where

$$\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{q}-\boldsymbol{q}'=0 \tag{21}$$

and

$$\mathcal{E}_3 = E-E'-\hbar\omega-\hbar\omega'. \tag{22}$$

§ 3. Mean lifetime

The mean lifetime τ is defined by

$$(f-f_0)/\tau = \sum_{k'}\{W(\boldsymbol{k}, \boldsymbol{k}')f[1-f'] - W(\boldsymbol{k}', \boldsymbol{k})[1-f]f'\}, \tag{23}$$

where f and f' are the distribution functions of electrons in $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively, f_0 is the equilibrium distribution function and $W(\boldsymbol{k}, \boldsymbol{k}')$ is the total transition probability of the electron from $\boldsymbol{k}$ to $\boldsymbol{k}'$.

When the temperature is below the Debye temperature as in the following discussion, the magnitude of $(\hbar\omega+\hbar\omega')$ included in (13) and (22) is usually comparable to the magnitude of E or E', and on the other hand, the magnitude of $(\hbar\omega-\hbar\omega')$ included in (19) is negligible compared with that of E or E' because of the laws of momentum and energy conservation. Then the difference between E and E' satisfying the law of energy conservation is usually much larger in the process in which two phonons are simultaneously absorbed or emitted than in the process in which one phonon is absorbed and the other phonon is simultaneously emitted, and hence the difference of the value of f_0 from that of f_0' is much greater in the former process than in the latter process. Therefore the latter process may be considered to take place much more frequently than the former process. In the following discussion, we shall neglect the process in which two phonons are simulianeously absorbed or emitted.

Now we can use the following approximation,

$$\mathcal{E}_2 = E - E' \, (\equiv \mathcal{E}) . \tag{24}$$

In this case, according to Fröhlich,[10] the mean lifetime is given by

$$\frac{1}{\tau} = \sum_{k'} \frac{|\boldsymbol{k} - \boldsymbol{k}'|^2}{2k^2} P(\boldsymbol{k}, \boldsymbol{k}') .$$

Introducing (17) into this equation, we obtain

$$\frac{1}{\tau} = \frac{8\pi\hbar D^2}{3\rho^2 k^2} \sum_{k'} |\boldsymbol{k} - \boldsymbol{k}'|^2 \delta(\mathcal{E}) \sum_q \frac{1}{\omega\omega'} n(n'+1) . \tag{25}$$

For convenience, we replace the summation over $\boldsymbol{k}'$ by over $\boldsymbol{Q} \equiv \boldsymbol{k} - \boldsymbol{k}'$, and we replace all summations by integrations. Then Eq. (25) is reduced to

$$\frac{1}{\tau} = \frac{\hbar D^2}{24\pi^5 \rho^2 k^2} \int d\boldsymbol{Q} Q^2 \delta(\mathcal{E}) \int \frac{d\boldsymbol{q}}{\omega\omega'} (nn' + n) . \tag{26}$$

The frequency of lattice vibration may be considered to satisfy approximaiely

$$\omega = vq , \tag{27}$$

where v is the mean sound velocity averaged over three directions of polarization. Now we assume that, when $T < \theta$,

$$n = \begin{cases} k_0 T / \hbar v q & (\hbar v q < k_0 T) , \\ e^{-\hbar v q / k_0 T} & (k_0 T < \hbar v q < k_0 \theta) , \end{cases} \tag{28}$$

where k_0 is Boltzmann's constant and θ is the Debye temperature. By the use of (27) and (28) the first term of (26) is reduced to

$$\frac{\hbar D^2}{24\pi^5 \rho^2 v^2 k^2} \int d\boldsymbol{Q} Q^2 \delta(\mathcal{E}) \left\{ \frac{(k_0 T)^2}{\hbar^2 v^2} \int \frac{d\boldsymbol{q}}{q^2 q'^2} + \int \frac{d\boldsymbol{q}}{qq'} e^{-\hbar v (q+q') / k_0 T} \right\} , \tag{29}$$

where the integral over q in the first term of (29) is to be performed from 0 to q_T and in the second term from q_T to q_m, in which $q_T = k_0 T / \hbar v$, $q_m = k_0 \theta / \hbar v$. Using the relation $q'^2 = Q^2 + q^2 - 2Qq \cos \alpha$ (α is the angle between $\boldsymbol{Q}$ and $\boldsymbol{q}$), we integrate the first term in the bracket of (29) over all allowed values of $\boldsymbol{q}$. Since q' is nearly equal to q in the second term, we replace q' by q there and integrate this term over all allowed values of $\boldsymbol{q}$. The bracket becomes

$$\pi \frac{k_0 T}{\hbar v} \left\{ \frac{k_0 T}{\hbar v} \left(\frac{\pi^2}{Q} + \frac{4\pi}{q_T} \right) + 2 \left(e^{-1} - e^{-\theta/T} \right) \right\} . \tag{30}$$

When $T > \theta$, the third and the fourth terms of (30) vanish and q_T in the second term is replaced by q_m.

When (30) is introduced into (29), the integration over $\boldsymbol{Q}$ in (29) is performed by Seitz's method,[9] and (29) is reduced to

$$\frac{2D^2m^*k_0T}{3\pi^3\hbar^2\rho^2v^3}\left\{\frac{\pi^2k_0T}{3\hbar v}+(2\pi+e^{-1}-e^{-\theta/T})\,k\right\}$$

$$\simeq\frac{2D^2m^*k_0T}{3\pi^2\hbar^2\rho^2v^3}\left\{\frac{\pi k_0T}{3\hbar v}+2k\right\}. \tag{31}$$

The second term of (31) is negligible compared with the first term because of the following relation,

$$\frac{6\hbar v\bar{k}}{\pi k_0T}=\frac{12\sqrt{2}\,vm^{*1/2}}{\pi(k_0T)^{1/2}}<\frac{1}{10}T^{-1/2},$$

where $\bar{k}$ is the mean value of k averaged over the Boltzmann distribution.

The next problem is to calculate the second term of (26), which can be done in the similar manner as in the derivation of (31). The result is approximately

$$\frac{2D^2m^*\log(k_0/\hbar v)\,k_0Tk}{3\pi^3\hbar^2\rho^2v^3}. \tag{32}$$

The ratio of the mean value of (32) to the value of (31) is

$$\frac{6\sqrt{2}\,m^{*1/2}v\log(k_0/\hbar v)}{\pi^{5/2}(k_0T)^{1/2}}<5T^{-1/2},$$

which shows that (32) may be neglected when $T>30$°K. When $T<30$°K, the one phonon process is always predominant as it will be shown in the next section. Finally we obtain

$$\tau=\frac{9\pi\hbar^3\rho^2v^4}{2D^2m^*k_0^2T^2}, \tag{33}$$

which is independent of energy.

§ 4. Mobility

The mobility μ is related to the mean lifetime by the equation

$$\mu=\frac{e}{m^*}\tau, \tag{34}$$

where e is the electronic charge. Using (33) we obtain the mobility

$$\mu_{II}=\frac{9\pi e\hbar^3\rho^2v^4}{2D^2m^{*2}k_0^2}T^{-2}, \tag{35}$$

where the suffix II denotes the two-phonons process. The temperature dependence of this mobility is different from the result of the usual theory, which is[9)]

$$\mu_I=\frac{3(2\pi)^{1/2}e\hbar^4\rho v_l^2}{2C^2m^{*5/2}k_0^{3/2}}T^{-3/2}, \tag{36}$$

where the suffix I denotes the one-phonon process, v_l is the velocity of the longitudinal sound wave and

$$C=\frac{\hbar^2}{2m}\int d\tau_0|\text{grad}\,\chi_k|^2. \tag{37}$$

We calculate the total acoustical mode-scattering mobility by the use of the approximate relation $1/\mu=1/\mu_I+1/\mu_{II}$. Although this relation is a rough approximation, we employ it because of the difficulty of the exact calculation. The obtained mobility is

$$\mu=\frac{9\pi e\hbar^3\rho^2v^4}{2m^{*2}k_0{}^2D^2}\cdot\frac{1}{T_c{}^{1/2}T^{3/2}+T^2}, \tag{38}$$

where

$$T_c=\frac{9\pi m^*\rho^2v^8C^4}{2\hbar^2k_0v_l{}^4D^4}. \tag{39}$$

It is concluded from (38) that the acoustical mode-scattering mobility depends on T^{-2} when $T>T_c$, and that it depends on $T^{-1.5}$ when $T<T_c$.

Before the numerical calculation of (38) and (39), it is necessary to estimate the value of D, which is from (9)

$$D=\frac{m}{2\hbar^2}\int d\tau_0\left\{\frac{\hbar^4}{4m^2}\varDelta\chi_k\left(\varDelta\chi_k^*+\frac{1}{x_k}|\text{grad}\,\chi_k|^2\right)+\frac{\hbar^2}{m}E_0\chi_k^*\varDelta\chi_k\right\}. \tag{40}$$

Here $-\frac{\hbar^2}{2m}\int d\tau_0\chi^*\varDelta\chi$ is the kinetic energy of the electron and it is nearly equal to the absolute value of E_0 when the electron is in s-state. Although the electronic state is not s but the linear conbination of s and p in the present case, we consider the following relation to be an acceptable approximation,

$$E_0=\frac{\hbar^2}{2m}\int d\tau_0\chi_k^*\varDelta\chi_k. \tag{41}$$

From (40) and (41) we obtain the relation between D and C, which is approximately

$$D=\frac{2m}{\hbar^2}C^2. \tag{42}$$

Then (35) and (39) are reduced to

$$\mu_{II}=\frac{9\pi e\hbar^7\rho^2v^4}{8(mm^*k_0)^2C^4}T^{-2} \tag{43}$$

and

$$T_c=\frac{9\pi\hbar^6m^*\rho^2v^8}{32k_0(mv_lC)^4}. \tag{44}$$

It is to be noted that the two-phonon process is expressed with the same parameter C as the one-phonon process. Since the value of C cannot be obtained theoretically, it is inevitable for us to determine it from the observed value. It may be questionable to assume that the observed mobility depending on $T^{-1.5}$ equals to μ_I, since the observed mobility will also be affected by the impurity scattering. But, in order to estimate the order of magnitude of T_c and μ_{II}, we employ values of μ_I given in Table I as observed mobilities. Table I also contains values of C determined from them for three samples. The fact that the values of C are of the order of e.v. proves that our choice of μ_I is not so wrong. Values of m^* used in this calculation are also given in Table I, and they are nearly equal to the mean values of cyclotron resonance data[11),12)] averaged arithmetically over three directions of principal axes.

By the use of values of C given in Table I we obtain values of T_c and μ_{II}, which are given in Table III. Here for electrons in Si both the calculated and observed values of T_c may be doubtful because of the large contribution of impurity scattering, and only the order of magnitude will be significant.

Table I. Values of m^*, μ_I and C.

	μ_I (cm^2/volt-sec.)	m^*/m	C (e.v.)
elec. in Ge	$2.6\times10^7\cdot T^{-1.5}$	0.6	3.6
hole in Ge	$2.6\times10^7\cdot T^{-1.5}$	0.4	5.9
elec. in Si	$1.3\times10^7\cdot T^{-1.5}$	0.5	7.2

Table II. Values of ρ, v_l and v,[13), 14)]

	ρ (g/cm^3)	v_l (cm/sec.)	v (cm/sec.)
Ge	5.32	4.94×10^5	2.40×10^5
Si	2.33	8.46×10^5	4.05×10^5

Table III. Values of T_c and μ_{II} calculated and observed.[2), 3)]

	T_c (°K)		μ_{II} (cm^2/volt-sec.)	
	calc.	obs.	calc.	obs.
elec. in Ge	1000	>400	$8.3\times10^8\cdot T^{-2}$	
hole in Ge	90	100	$2.5\times10^8\cdot T^{-2}$	$1.05\times10^9\cdot T^{-2.33}$
elec. in Si	110	<160	$1.4\times10^8\cdot T^{-2}$	$7.2\times10^8\cdot T^{-2.3}$*)

*) Morin and Maita's estimation is too extreme, because they employed the relation $\mu_T=(\sigma_T/\sigma_{300})\,\mu_{300}$. We employed here the relation $\mu_T=(\sigma_T/\sigma_{200})\,\mu_{200}$.

It may be seen that the agreement between the observed and calculated values of T_c is fairly good. It proves the validity of the main part of our theory. On the other hand, there are small deviations of the temperature dependence between the observed mobilities and our results. This is not surprising in view of the fact that the optical mode-scattering as well as the acoustical mode-scattering contributes to the mobility as shown by several authors.[5)—7)] In fact, if we adopt for optical mode-scattering mobilities the values given in Table IV which are not unreasonable values,[5)—7)] the calculated mobilities agree with the observed mobilities.

Since the mobility depending on $T^{-1.5}$ has not been observed yet for holes in silicon,[3),4)] we can not estimate, in this case, the value of C by the method used above. Instead, we estimate it and values of other quantities in a self-consistent way, in other words, we estimate the value of μ_I which gives the same value of T_c for both the observed and the calculated mobilities. The obtained values of quantities are given in Table V, and the value of optical mode-scattering mobility to account for the observed mobility is given in Table IV. The value of m^* used in these calculations is $0.5m$.

Table IV. Optical mode-scattering mobilities (in units of cm²/volt-sec.)

	electron	hole
Ge	$2.6\times10^5\cdot T^{-0.5}\cdot(e^{400/T}-1)$	$3.7\times10^4\cdot T^{-0.5}\cdot(e^{400/T}-1)$
Si	$6.0\times10^4\cdot T^{-0.5}\cdot(e^{600/T}-1)$	$3.1\times10^3\cdot T^{-0.5}\cdot(e^{600/T}-1)$

Table V. Values of quantities for holes in silicon.

μ_I (cm²/volt-sec.)	μ_{II} (cm²/volt-sec.)		T_c (°K)
	calc.	obs.	
$1.0\times10^7\cdot T^{-1.5}$	$7.4\times10^7\cdot T^{-2}$	$2.5\times10^8\cdot T^{-2.3}$	55

§ 5. Conclusion

The two-phonon process which has been neglected in the usual theory plays an important role at least in non-degenerate semiconductors. The acoustical mode-scattering mobility governed by the two-phonon process depends on T^{-2}, which is different from the usual one depending on $T^{-1.5}$. The former is predominant at higher temperature and the latter is predominant at lower temperature. The transition temperature from the latter to the former agrees fairly well with that observed on the lattice scattering mobility. The small deviations of temperature dependence of the observed mobilities from T^{-2} or $T^{-1.5}$ seems to be the effect of the optical mode scattering. Therefore, the two-phonon process accounts for the main part of deviation of the observed temperature dependence from that predicted in the usual theory. The two-phonon process must be an important mechanism for other phenomena or in semi-metals. These problems are, however, not discussed here, but will be done in future.

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Appendix

There are four terms depending on $(\boldsymbol{e}\cdot\boldsymbol{q}')(\boldsymbol{e}'\cdot\operatorname{grad}V)$ in (3), one of which, for example, is

$$\frac{\hbar}{2\rho}\sum_q\sum_{q'}\sum_j\sum_{j'}\left(\frac{nn'}{\omega\omega'}\right)^{1/2}e^{i(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{r}}(\boldsymbol{e}\cdot\boldsymbol{q}')(\boldsymbol{e}'\cdot\operatorname{grad}V). \tag{A1}$$

In the similar manner as in § 2, we obtain the transition probability,

$$P(\boldsymbol{k}, \boldsymbol{k}') = \frac{\pi\hbar C^2}{36\rho^2} \sum_q \sum_j \sum_{j'} \frac{nn'}{\omega\omega'} (\boldsymbol{e}\cdot\boldsymbol{q}')^2 (\boldsymbol{e}'\cdot\boldsymbol{Q})^2 \delta(\mathcal{E}_1), \tag{A2}$$

where

$$\boldsymbol{k} - \boldsymbol{k}' + \boldsymbol{q} + \boldsymbol{q}' = 0. \tag{A3}$$

In the similar manner as in § 3, we obtain the mean lifetime, which is approximately

$$\tau = \frac{9\pi^3\hbar^4\rho^2v^5}{2m^*C^2(k_0Tk)^3}. \tag{A4}$$

Eq. (A2) and Eq. (A4) include the coupling constant C, and it is due to the fact that (A1) includes only the first derivative of potential. It is also noted that the mean lifetime (A4) depends strongly on the electronic energy.

From (A4) and the relation

$$\mu = \frac{2}{3}\frac{e}{m^*k_0T}\overline{E\tau}$$

we obtain the mobility

$$\mu = \frac{9\pi^3e\hbar^7\rho^2v^5}{2\sqrt{2}\,C^2m^{*7/2}(k_0T)^{9/2}}. \tag{A5}$$

The ratio of (A5) to (43) is

$$\frac{2\sqrt{2}\,\pi^2m^2vC^2}{m^{*3/2}(k_0T)^{5/2}} > 2\times10^9\,T^{-5/2},$$

which shows that (A5) is negligible compared with (43) at least when $T < 5000$°K.

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Remarks on Mayer's Reduced Density Matrix Method

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Mayer has proposed a variation method of finding the two-body reduced density matrix ρ_2 for the ground state of a many fermion system. In this paper it is shown explicitly that his variation method is imperfect and sometimes leads to quite unsatisfactory results. Further, his trial form of ρ_2 for electron gas is translated into that for the π electron system of benzene molecule and it is shown that such a trial form is unsatisfactory in several respects.

§ 1. Introduction

The rigorous analysis of an assembly of mutually interacting particles by means of quantum theory constitutes a very difficult problem; therefore, we must resort to some approximate methods. One of the most useful approximate methods is the well-known variation method based on the Schrödinger variation principle. In this method an appropriate trial form is assumed for the total wave function and the best is sought of all the wave functions of this type. The configuration interaction (CI) procedure in orbital theories is a typical example of this method. It is, however, usually necessary to take into account a very large number of configurations in order to obtain satisfactory results by means of CI procedure. Accordingly, it cannot be said to be a very useful procedure. In order to avoid this difficulty, another procedure has been proposed by Mott,[1] Feynman[2] and others. They assumed a single Slater determinant multiplied by a certain trial function which represents the correlation among particles. However, such treatments are also intractable, whenever the density of particles is large, because of the considerable computational difficulty involved in deriving the functional forms of one- and two-body density matrices from the total wave functions given by such trial forms. Consequently, it is desirable to find a variation method in which a trial form is assumed directly for the two-body density matrix of assembly itself. Such an investigation was done by J. E. Mayer[3] for the ground state of electron gas. His method, however, seems to be open to several difficulties. In this paper, therefore, we want to make some remarks on its validity.

§ 2. Remarks on Mayer's general method

Let us consider an assembly of N fermions. If we write the normalized total wave function of this assembly as $\Psi(q_1, q_2, \cdots, q_N)$, the one-body and two-body reduced density matrices for the pure state described by this wave function are given by

$$\langle q_1'|\rho_1|q_1''\rangle = N\int dq_2\cdots dq_N\cdot\Psi(q_1'q_2\cdots q_N)\cdot\Psi^*(q_1''q_2\cdots q_N)$$

and

$$\langle q_1'q_2'|\rho_2|q_1''q_2''\rangle = (N(N-1)/2)\int dq_3\cdots dq_N\cdot\Psi(q_1'q_2'q_3\cdots q_N)\cdot\Psi^*(q_1''q_2''q_3\cdots q_N), \tag{2·1}$$

espectively. The density matrices for mixed states are given by replacing $\Psi\rangle\langle\Psi$ with $\sum_\mu c_\mu\Psi_\mu\rangle\langle\Psi_\mu$ in Eq. (2·1), where $c_\mu\geqq 0$ and $\sum_\mu c_\mu=1$. Then the total energy E as a functional of ρ_1 and ρ_2 can be expressed as

$$E[\rho_1,\rho_2]=\mathrm{Tr}[\rho_1\mathcal{H}_1]+\mathrm{Tr}[\rho_2\mathcal{H}_2], \tag{2·2}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ denote one-body and two-body operators, respectively. From the definitions of ρ_1 and ρ_2 one can easily derive the following five conditions

$$\begin{array}{lll} \text{Condition} & \text{I.} & \mathrm{Tr}\,\rho_1=N \\ \quad ,, & \text{II.} & \int\langle q_1'q_2|\rho_2|q_1''q_2\rangle dq_2=(N-1)/2\cdot\langle q_1'|\rho_1|q_1''\rangle \\ \quad ,, & \text{III.} & \rho_2 \text{ must be a positive operator.} \\ \quad ,, & \text{IV.} & \langle q_1'q_2'|\rho_2|q_1''q_2''\rangle=-\langle q_2'q_1'|\rho_2|q_1''q_2''\rangle \\ & & \qquad =-\langle q_1'q_2'|\rho_2|q_2''q_1''\rangle \\ \quad ,, & \text{V.} & 0\leqq \text{each eigenvalue of } \rho_1\leqq 1. \end{array} \tag{2·3}$$

Mayer has considered that the ρ_1 and ρ_2 which obey these five conditions (2·3) and give the lowest value to the expression (2·2) would be equal to the real ρ_1 and ρ_2 which are derived from the exact wave function Ψ of the ground state by Eq. (2·1).

We can, however, show several examples of the matrix ρ_2, which satisfies all these conditions (2·3) but does not correspond to any of the actual reduced density matrix, when $N>2$.

Example 1.

$$\rho_2=(1/2)\sum_{i=1}^{N(N-1)}|\varphi_{2i-1}\varphi_{2i}|\rangle\langle|\varphi_{2i-1}\varphi_{2i}|, \tag{2·4}$$

where

$$\langle q_1q_2||\varphi_i\varphi_j|\rangle=(1/\sqrt{2})\{\langle q_1|\varphi_i\rangle\langle q_2|\varphi_j\rangle-\langle q_1|\varphi_j\rangle\langle q_2|\varphi_i\rangle\}$$

and φ's constitute a set of normalized spin-orbitals which are mutually orthogonal. Conditions III and IV are obviously satisfied. Further, I and V are also satisfied, because ρ_1 is given by

$$\rho_1=(1/2(N-1))\sum_{i=1}^{2N(N-1)}\varphi_i\rangle\langle\varphi_i$$

on account of Condition II. However, such ρ_2 should be excluded by the reason mentioned below. Namely, N-body density matrix can always be expanded in terms of Slater determinants Ψ_λ^0 as follows

$$\rho_N = \sum_{\lambda,\mu} \Psi_\lambda^0\rangle c_{\lambda\mu} \langle \Psi_\mu^0.$$

Then the corresponding ρ_2 can be written as

$$\rho_2 = (1/2)\sum_\lambda c_{\lambda\lambda} \sum_{i,j=1}^{N} |\varphi_i^\lambda \varphi_j^\lambda|\rangle\langle|\varphi_i^\lambda \varphi_j^\lambda|$$
$$+ (\text{non diagonal terms})$$

where φ_i^λ means an orbital contained in Ψ_λ^0, and further $c_{\lambda\lambda} \geqq 0$. Therefore, any orbital φ_i^λ in the above expression of ρ_2 must be found in $N-1$ diagonal terms at least. In $(2\cdot4)$, however, each φ_i appears just once.

Example 2.

$$\rho_2 = \rho_2^0 + |\varphi_i \varphi_j|\rangle \lambda \langle|\varphi_k \varphi_l| + |\varphi_k \varphi_l|\rangle \lambda^* \langle|\varphi_i \varphi_j| \qquad (2\cdot5)$$
$$(i,\ j,\ k, \text{ and } l \text{ are all different})$$

where ρ_2^0 is the two-body density matrix reduced from a single Slater determinant, φ_i, φ_j, φ_k and φ_l are any of the orbitals contained in the determinant, and λ is any number whose absolute value is smaller than unity. Obviously, Condition IV is satisfied by $(2\cdot5)$. Condition III is also satisfied, as can easily be shown by diagonalizing $(2\cdot5)$. And further we see that I and V are also satisfied, because the ρ_1 reduced from $(2\cdot5)$ by means of Eq. II in $(2\cdot3)$ is equal to the ρ_1^0 reduced from ρ_2^0. On the other hand, $(2\cdot5)$ cannot be actual two-body density matrix, because the wave function which generates the above ρ_1^0 must clearly be a single determinant so that the ρ_2^0 which is obtained from this determinant must be the only actual two-body density matrix from which the present ρ_1^0 can be reduced.

The existence of such matrices means that the lowest possible value of the expression $(2\cdot2)$ found by Mayer's method will in general be lower than the exact energy eigenvalue of the ground state and the corresponding ρ_2 will be also different from the exact one. Further we can present an example which shows that this discrepancy is sometimes too large to be neglected in discussing the properties of many-fermion system, such as the correlation energy.

Example 3. For the He_2 molecule at the infinite internuclear separation, we may assume trial density matrix of the form

$$\rho_2 = \rho_2^0 + |g\alpha,\ g\beta|\rangle \lambda \langle|u\alpha,\ u\beta| + |u\alpha,\ u\beta|\rangle \lambda^* \langle|g\alpha,\ g\beta|$$
$$+ |g\alpha,\ u\beta|\rangle \mu \langle|u\alpha,\ g\beta| + |u\alpha,\ g\beta|\rangle \mu^* \langle|g\alpha,\ u\beta| \qquad (2\cdot6)$$
$$(0 \leqq |\lambda|,\ |\mu| \leqq 1)$$

where ρ_2^0 is the two-body density matrix reduced from the Slater determinant $(1/\sqrt{4!}) \times |g\alpha, g\beta, u\alpha, u\beta|$, g and u being molecular orbitals with the symmetry of *gerade* and *ungerade*, and α and β being spin functions. The trial form $(2\cdot6)$ is a special form of **Example 2** and is not acceptable unless $\lambda=\mu=0$. Now, it can easily be seen that the lowest value of energy can be obtained by putting $\lambda=\mu=-1$ and further $g=(a+b)/\sqrt{2}$ and $u=(a-b)/\sqrt{2}$, a and b being respectively the $1s$ orbitals of He^+ atoms situated at A and B. In this case we obtain the one-electron energy $\mathrm{Tr}[\rho_1\mathcal{H}_1]=4E(\mathrm{He}^+)$, the two-electron (repulsive) energy $\mathrm{Tr}[\rho_2\mathcal{H}_2]=0$ and then the total energy $E[\rho_1, \rho_2]=4E(\mathrm{He}^+)$. Needless to say, the exact value of the total energy is $2E(\mathrm{He})$.

Accordingly the validity of the general method based on the subsidiary conditions (2·3) seems to us quite questionable.

§ 3. Remarks on the special trial form adopted by Mayer

Although Mayer's general method is not satisfactory as mentioned above, the special trial form of ρ_2 adopted by him in treating the ground state of electron gas seems somewhat plausible. So it is interesting to examine the validity of this kind of trial form quantitatively. For this purpose we wish to apply it to the ground state of the six π electron system of the benzene molecule, because in this case the calculation is manageable and its results can be compared with those of the CI treatments.[4] We will here confine ourselves to the approximation in which the trial forms are constructed from the six $2p\pi$ AO's. Then the trial form which is an adaptation from Mayer's one may be written as follows,

$$\rho_2=(1/2)\begin{vmatrix}\langle q_1'|\rho_1|q_1''\rangle & \langle q_1'|\rho_1|q_2''\rangle\\ \langle q_2'|\rho_1|q_1''\rangle & \langle q_2'|\rho_1|q_2''\rangle\end{vmatrix}$$
$$-(k/2)\sum_i\begin{vmatrix}\langle q_1'|w_i\alpha\rangle & \langle q_1'|w_i\beta\rangle\\ \langle q_2'|w_i\alpha\rangle & \langle q_2'|w_i\beta\rangle\end{vmatrix}\times\begin{vmatrix}\langle w_i\alpha|q_1''\rangle & \langle w_i\alpha|q_2''\rangle\\ \langle w_i\beta|q_1''\rangle & \langle w_i\beta|q_2''\rangle\end{vmatrix}. \qquad (3\cdot1)$$

In the above, k is a c-number and w_i is the Wannier function which is constructed from the six $2p\pi$ AO's and is localized on the i-th carbon atom. If $k=0$, (3·1) gives the two-body density matrix corresponding to the Hartree-Fock approximation, as is well known. By this density matrix, however, the correlation between electrons with α spin and with β spin is not included, and then the ionic character is contained too much. The second term on the right side of (3·1) has the effect of diminishing this ionic character. For this purpose it would be better to adopt as w_i the i-th $2p\pi$ AO itself, but we here adopt the Wannier function only for mathematical convenience.

In the ground state of benzene molecule, the one-body density matrix ρ_1 is invariant under all the symmetry operations of this molecule. Therefore, it may be expressed as

$$\rho_1=\sum_l\varphi_l\alpha\rangle\lambda_l\langle\varphi_l\alpha+\sum_l\varphi_l\beta\rangle\lambda_l\langle\varphi_l\beta, \qquad (3\cdot2)$$

where φ_l is a *molecular* orbital given by

$$\varphi_l\rangle=(1/\sqrt{6})\sum_{\mu=1}^{6}\exp[2\pi il\mu/6]\cdot w_\mu\rangle,$$

and λ's, the eigenvalues of ρ_1, are variational parameters, together with k in Eq. (3·1).

Now, setting $q_2' = q_2'' = q_2$ in the first term of $(3\cdot 1)$ and integrating over dq_2, we obtain

$$(N/2)\cdot\rho_1 - (1/2)\sum_l \lambda_l^2 \{\varphi_l\alpha\rangle\langle\varphi_l\alpha+\varphi_l\beta\rangle\langle\varphi_l\beta\}.$$

Similarly, we obtain from the second term

$$\begin{aligned} &-(k/2)\sum_i \{w_i\alpha\rangle\langle w_i\alpha+w_i\beta\rangle\langle w_i\beta\} \\ &= -(k/2)\sum_l \{\varphi_l\alpha\rangle\langle\varphi_l\alpha+\varphi_l\beta\rangle\langle\varphi_l\beta\}. \end{aligned}$$

Therefore, Eq. II in $(2\cdot 3)$ gives

$$\lambda_l^2 - \lambda_l + k = 0. \tag{3$\cdot$3}$$

From this equation and Conditions I and V we obtain the following relations,

$$\begin{cases} \lambda_0 = \lambda_1 = \lambda_{-1} = 1-x, \\ \lambda_3 = \lambda_2 = \lambda_{-2} = x, \\ k = x - x^2, \ \ 0 \leqq x \leqq 1. \end{cases}$$

Hereafter we use x as the variational parameter.

In our calculation we have used the values of integrals tabulated by Parr, Craig and Ross.[4] The results are summarized in the second column of Table I, where N_α and N_β mean respectively the numbers of electrons with α and β spins around one electron with α spin, and ΔW is the depression of the energy from that given by the single determinant approximation in the LCAO theory. If we impose on ρ_2 another new condition $N_\alpha = (N/2)-1$ and $N_\beta = (N/2)$ which must always be satisfied by ρ_2 of singlet states, k should be zero and then the corresponding ρ_2 reduces to that of the single Slater determinant. Therefore, the attempt is meaningless, in usual sense, to calculate the correlation energy by using such a trial form as $(3\cdot 1)$.

In the above treatment we have considered all the conditions $(2\cdot 3)$ strictly, but Mayer replaced Condition II by the less stringent one

$$\mathrm{Tr}\,\rho_2 = (1/2)\cdot N(N-1). \tag{II$'$}$$

Substituting $(3\cdot 1)$ in Eq. (II$'$), we get

$$\sum_l \lambda_l^2 = 3(1-2k). \tag{3$\cdot$4}$$

From this equation and Conditions I and V, that is,

$$\sum_l \lambda_l = 3 \ \text{ and } \ 0 \leqq \lambda_l \leqq 1,$$

we obtain the results as shown in the third column of Table I. Here we notice that these results differ from those listed in the former column by a considerable amount.

The last column of Table I shows the results of the CI calculation performed by Parr, Craig and Ross[4] which may be regarded practically as the exact solution of our present problem. From Table I we see that the agreement of the values of λ's and ΔW

written in the second and third column with those in the last column is rather unsatisfactory, not to mention the essential disagreement in N_α's explained previously. In order to improve these results, it is necessary to introduce some new freedoms into the trial form $(3\cdot1)$. If we do so, however, we encounter a difficulty arising from the fact that some subsidiary conditions are still unknown.

Table I

λ_3	0.096	0.000	0.029
$\lambda_{\pm2}$	0.096	0.202	0.071
$\lambda_{\pm1}$	0.904	0.798	0.930
λ_0	0.904	1.000	0.970
k	0.087	0.108	
N_α	2.174	2.215	2.000
N_3	2.826	2.785	3.000
ΔW	0.528 eV	1.892 eV	2.7 eV

§ 4. Conclusion

From the considerations in § 2 and § 3, we have seen that Mayer's variation method is not perfect, that is, the subsidiary condition $(2\cdot3)$ is only a very poor approximation to the true one to be satisfied by actual one- and two-body reduced density matrices. In several respects, however, his method is certainly superior to the ordinary variation method, as stated in § 1. Accordingly, it seems desirable to improve his method by investigating the general properties of reduced density matrices in more detail.

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Note added in proof

In the course of proof, a similar study by R.H. Tredgold has appeared on Phys. Rev. **105** (1957), 1421.

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Parity Nonconservation and Decay of Hyperon

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Under the assumption that the parity conservation does not hold for the decay processes of strange particles, we examine both the angular distribution of the final decay product for the cascade decay of Ξ hyperon and the asymmetrical distribution of decay product about the production plane of hyperons which are produced by $K^- - p$ interaction.

So far as our kinematical study for these processes is concerned, it is impossible to know whether the decay interaction is invariant under the time reversal. However, there exists a possibility of solving this problem by making use of the values of phase shifts for $\pi - N$ scattering. Namely, if the two decay interactions, $\Xi \to \Lambda + \pi$ and $\Lambda \to N + \pi$, are invariant under the time reversal, we may expect that the fore-aft asymmetry (with respect to the direction of motion of Λ) of the final decay product for the cascade decay will be observed, while if they are simultaneously invariant undet the charge conjugation, we cannot probably confirm the existence of asymmetrical angular distribution.

§ 1. Introduction

It has recently been established that the parity conservation[1),2)] does not hold for the processes of the nuclear β-decay,[3)] $\pi - \mu$ decay and $\mu - e$ decay.[4)] Also, there is the possibility that the parity conservation does not hold for the decay processes of strange particles. Therefore, in this paper, we shall kinematically investigate from this viewpoint the cascade decay process of hyperons and the decay processes of hyperons which are produced by $K^- - p$ interaction.[5)] The problems with respect to heavy mesons will be treated in other place.

Let us assume that the decay interaction is invariant under the strong reflection,[6)] that is, PTA = invariant, where we denote the space reflection, the particle-antiparticle conjugation and Wigner's time reversal as P, A and T, respectively. Then, for the interaction Hamiltonian H_d responsible for the decay of strange particles, we have two possibilities, that is,

(i) H_d is invariant under PA,

or (ii) H_d is invariant under PT.

We intend to examine the angular distributions for the decay processes of hyperons when we assumed (i) or (ii) instead of the parity conservation.*

First of all, let us pay our attention to the processes of cascade decay $\Xi \to \Lambda + \pi$ and $\Lambda \to N + \pi$. By observing the angular distribution of the final decay product π, it may be possible, as in the case of $\pi-\mu-e$ decay, to know whether the decay interactions are invariant under the space reflection.

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* Although there is some possibility of a linear combination of (i) and (ii), we here restrict our examination to the case (i) or (ii).

Even in the case in which the spins of both Λ and Ξ are respectively of an arbitrary half-number, the expression for the angular distribution can easily be written down, but we now deal only with the following cases in order to avoid complication:

(1) S_Λ (spin of Λ) $=1/2$ and S_Ξ (spin of Ξ) $=l+1/2$,

(2) $S_\Lambda=l+1/2$ and $S_\Xi=1/2$, and (3) $S_\Lambda=S_\Xi=3/2$.

So far as the above kinematical study is concerned, it is impossible to know the difference between (i) $PA=$inv. and (ii) $PT=$inv. Then, we discuss in § 3 the possibility of discriminating these two cases by making use of the values of the phase shifts for $\pi-N$ scattering.

In § 4 we examine the decay process of hyperons which are produced by K^--p interaction[5] in flight. Since the hyperons are produced through the strong interaction, the method mentioned in § 2 is now useless for inspecting the parity nonconservation of decay interaction. Hence we shall examine the asymmetrical distribution of decay product π with respect to the production plane of hyperon. For simplicity we deal firstly with the case in which the spins of K^- and hyperon are respectively 0 and 1/2 and secondly with the case in which the spins of K^- and hyperon are respectively 0 and 3/2.

§ 2. Cascade dacay of hyperon

We are interested in the cascade decay of Ξ hyperon because its process passes through weak interactions two times. Paying our attention to the phenomena in which Ξ decays at rest and taking z-axis along the direction of Λ, we examine the angular distribution of the final decay product π. In this case there exists only Y_L^0 among Y_L^m which represents the relative angular momentum eigenfunction of π in the process $\Xi\rightarrow\Lambda+\pi$. In other words, if only the z component of the spin of Ξ is designated, we can predict the spin polarization of Λ without its measurement.

We denote below the R-matrices for the first and second decay processes as A_L and R_L respectively, where L shows the orbital angular momentum of decay product π.

§ 2·1. The case of $S_\Xi=l+1/2$ and $S_\Lambda=1/2$

As the parity conservation does not hold for our process, the cascade decay process can now be expressed in terms of A_l, A_{l+1} and R_0, R_1. Here it must be noted that there exist only two values of $\pm1/2$ as the z component of the spin of the Ξ-particle by the reason that z- axis is chosen along the line of Λ and that the quantization is done with respect to this direction. Since it may be expected that Ξ-particles are produced with the polarization of the spin, we write down the expressions of angular distribution in the following three cases:

[1] When z component of the spin of Ξ, $(S_\Xi)_z$, is equal to 1/2,

$$\{(l+1)/(4\pi)^2\}[A_l{}^*A_l+A_{l+1}^*A_{l+1}+(A_l{}^*A_{l+1}+A_lA_{l+1}^*)]$$
$$\times[R_0{}^*R_0+R_1{}^*R_1-(R_0{}^*R_1+R_0R_1{}^*)\cos\theta]. \qquad (1)$$

[2] When $(S_\Xi)_z=-1/2$,

$$\{(l+1)/(4\pi)^2\}[A_l^* A_l+A_{l+1}^* A_{l+1}-(A_l^* A_{l+1}+A_l A_{l+1}^*)]$$
$$\times[R_0^* R_0+R_1^* R_1+(R_0^* R_1+R_0 R_1^*)\cos\theta]. \tag{2}$$

[3] When there is no polarization of the spin of Ξ,

$$\{(l+1)/(4\pi)^2\}[(A_l^* A_l+A_{l+1}^* A_{l+1})(R_0^* R_0+R_1^* R_1)$$
$$-(A_l^* A_{l+1}+A_l A_{l+1}^*)(R_0^* R_1+R_0 R_1^*)\cos\theta]. \tag{3}$$

Thus we are able to expect the appearance of asymmetry, the odd term in $\cos\theta$ in the angular distribution. Nevertheless, we can say that the angular distribution does not depend on the spin of Ξ-particle.

§ 2·2. The case of $S_\Xi=1/2$ and $S_\Lambda=l+1/2$

In a similar manner as above, we can easily obtain the angular distribution for the final decay product π.

[1] When $(S_\Xi)_z=1/2$,

$$\{(l+1)/(4\pi)^2\}[A_l^* A_l+A_{l+1}^* A_{l+1}+(A_l^* A_{l+1}+A_l A_{l+1}^*)]$$
$$\times[(R_l^* R_l+R_{l+1}^* R_{l+1})\xi-(R_l^* R_{l+1}+R_l R_{l+1}^*)\eta]. \tag{4}$$

[2] When $(S_\Xi)_z=-1/2$,

$$\{(l+1)/(4\pi)^2\}[A_l^* A_l+A_{l+1}^* A_{l+1}-(A_l^* A_{l+1}+A_l A_{l+1}^*)]$$
$$\times[(R_l^* R_l+R_{l+1}^* R_{l+1})\xi+(R_l^* R_{l+1}+R_l R_{l+1}^*)\eta]. \tag{5}$$

[3] When there is no polarization for the spin of Ξ,

$$\{(l+1)/(4\pi)^2\}[(A_l^* A_l+A_{l+1}^* A_{l+1})(R_l^* R_l+R_{l+1}^* R_{l+1})\xi$$
$$-(A_l^* A_{l+1}+A_l A_{l+1}^*)(R_l^* R_{l+1}+R_l R_{l+1}^*)\eta], \tag{6}$$

where

$$\xi=[|P_l|^2+|P_{l+1}|^2-2P_l P_{l+1}\cos\theta]/\sin^2\theta,$$
$$\eta=[2P_l P_{l+1}-\cos\theta(|P_l|^2+|P_{l+1}|^2)]/\sin^2\theta,$$

and the argument of Legendre function P_l is naturally $\cos\theta$. η consists of the odd powers of $\cos\theta$ and gives the asymmetry of the angular distribution. To facilitate the analyses of experimental data, we here write down the explicit expressions of ξ and η in the cases of $l=0$, 1, and 2.

$$\xi=\begin{cases}1 & \text{for } l=0,\\ (1/4)(1+3\cos^2\theta) & \text{for } l=1,\\ (1/4)(1-2\cos^2\theta+5\cos^4\theta) & \text{for } l=2,\end{cases} \tag{7}$$

$$\eta=\begin{cases}\cos\theta & \text{for } l=0,\\ (1/4)\,(9\cos^3\theta-5\cos\theta) & \text{for } l=1,\\ (1/4)\,(25\cos^5\theta-26\cos^3\theta+5\cos\theta) & \text{for } l=2.\end{cases} \tag{8}$$

The highest order in odd and even powers of $\cos\theta$ are generally given by $(2l+1)$ — and $(2l)$, respectively.

§ 2·3. The case of $S_\Xi=S_\Lambda=3/2$

In this case the cascade decay process can be described in terms of both A_0, A_1, A_2, A_3 and R_1, R_2, and the angular distribution has the following form when the spin of the initial Ξ is unpolarized:

$$\begin{aligned}\{(1/2)/(4\pi)^2\}\Big(&(R_1{}^*R_1+R_2{}^*R_2)[2(A_0{}^*A_0+A_2{}^*A_2)+(2/5)A_1{}^*A_1(7-6\cos^2\theta)\\ &+(6/5)A_3{}^*A_3(1+2\cos^2\theta)+\{(A_0{}^*A_2+A_0A_2{}^*)\\ &+(3/5)(A_1{}^*A_3+A_1A_3{}^*)\}(1-3\cos^2\theta)]\\ -\sqrt{1/5}\,&(R_1{}^*R_2+R_1R_2{}^*)[2(A_0{}^*A_1+A_0A_1{}^*)\cos\theta+(A_1{}^*A_2+A_1A_2{}^*)(7\cos\theta-9\cos^3\theta)\\ &+3(A_0{}^*A_3+A_0A_3{}^*)(3\cos\theta-5\cos^3\theta)+6(A_2{}^*A_3+A_2A_3{}^*)(2\cos^3\theta-\cos\theta)]\Big).\end{aligned} \tag{9}$$

From our kinematical investigation for the cascade decay of Ξ hyperon, we may conclude as follows:

(1) If the asymmetry of angular distribution for the final decay product π is confirmed, the parity conservation does not hold for the decay process of hyperon.

(2) When the form of asymmetry for angular distribution is $\cos\theta$, there is a large possibility of $S_\Lambda=1/2$ as is seen from eqs. (1)~(3). We cannot, however, conclude that $S_\Lambda=1/2$, because the cross term between A_0 and A_1 in the case of § 2·3 has the form of $\cos\theta$. In order to remove this ambiguity, the angular distribution must be folded about 90°. If the folded angular distribution is isotropic, we may probably conclude $S_\Lambda=1/2$.

§ 3. Whether PA=inv. or PT=inv.

In this section we concentrate our attention to the following problem, that is,

(i) decay interaction is invariant under PA,

or (ii) decay interaction is invariant under PT.

In the cascade decay process of Ξ hyperon there are two favourable facts which make our analyses very easy.

For the process $\Xi\rightarrow\Lambda+\pi$ the isotopic spins of the initial and final states are respectively 1/2 and 1, and for the process $\Lambda\rightarrow N+\pi$ the isotopic spin of the initial state is 0 and that of the final state is 1/2 under the assumption of $|\Delta I|=1/2$.[7),8),9)] Thus each decay process is the reaction between the pure states with regard to isotopic spin. Therefore A_L or R_L can be decomposed into a real amplitude and a phase, namely

$$\left.\begin{aligned} A_L &= |A_L| \exp(2i\beta_L), \quad (\text{for } \Xi\to\Lambda+\pi) \\ R_L &= |R_L| \exp(2i\delta_L). \quad (\text{for } \Lambda\to N+\pi) \end{aligned}\right\} \tag{10}$$

And

$$\left.\begin{aligned} A_l^* A_{l+1} + A_l A_{l+1}^* &= 2|A_l|\,|A_{l+1}| \cos 2(\beta_l - \beta_{l+1}), \\ R_l^* R_{l+1} + R_l R_{l+1}^* &= 2|R_l|\,|R_{l+1}| \cos 2(\delta_l - \delta_{l+1}). \end{aligned}\right\} \tag{11}$$

It is one of the most favourable facts that make possible to describe each decay process by just one term. On the contrary, if the decay process is the reaction between mixed states with regard to isotopic spin, many terms appear in the description of the decay process. Then there is some possibility of mutual cancellation among these terms, and as the result it will be difficult to obtain any definite conclusions.

Since Q-values in the processes of $\Xi\to\Lambda+\pi$ and $\Lambda\to N+\pi$ are respectively 66 Mev and 37 Mev, the emitted pion in each decay process has not so high energy. This is another favourable point.

First of all, let us consider the $\Lambda\to N+\pi$ process. We now denote the phase shifts for $\pi-N$ scattering as $\alpha_L^{\pm}$, where the suffix L is used to show the state of the orbital angular momentum L, and the superscripts ± show the states of $J=L\pm 1/2$, respectively. Here we have assumed the total isotopic spin of the final $N+\pi$ system to be $I=1/2$. Then we can easily find the following relation between δ_L and $\alpha_L^{\pm}$ using the unitarity of S-matrix.[10]

(i) In the case of $PA=\text{inv.}$ (in other words, $T=\text{inv.}$)

$$\left.\begin{aligned} 2\delta_l &= \alpha_l^{+} + n_l\pi, \\ 2\delta_{l+1} &= \alpha_{l+1}^{-} + n_{l+1}\pi, \end{aligned}\right\} \tag{12)*$$

where n_l and n_{l+1} are integer.

(ii) In the case of $PT=\text{inv.}$ (in other words, $A=\text{inv.}$)

There are two possibilities, where R_l or R_{l+1} represents the R-matrix element responsible to the decay process, which violates the parity conservation.

(ii–1) When R_l is the R-matrix for the decay process of parity nonconservation,

$$\left.\begin{aligned} 2\delta_l &= \alpha_l^{+} + (n_l \pm 1/2)\pi, \\ 2\delta_{l+1} &= \alpha_{l+1}^{-} + n_{l+1}\pi. \end{aligned}\right\} \tag{13}$$

(ii–2) When R_{l+1} is the R-matrix for the decay process of parity nonconservation,

$$\left.\begin{aligned} 2\delta_l &= \alpha_l^{+} + n_l\pi, \\ 2\delta_{l+1} &= \alpha_{l+1}^{-} + (n_{l+1} \pm 1/2)\pi. \end{aligned}\right\} \tag{14}$$

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* In the expressions of $2\delta_l$ and $2\delta_{l+1}$ in eqs. (12), (13) and (14), we have omitted the additional common magnitude φ which may depend on the spin or parity of hyperon, because this common φ has no effect on eqs. (15) and (16) from which our conclusion is derived.

Therefore, (i) in the case of PA=inv.,

$$R_l^* R_{l+1} + R_l R_{l+1}^* = \pm 2 |R_l| |R_{l+1}| \cos(\alpha_l^+ - \alpha_{l+1}^-), \tag{15}$$

(ii) in the case of PT=inv.,

$$R_l^* R_{l+1} + R_l R_{l+1}^* = \pm 2 |R_l| |R_{l+1}| \sin(\alpha_l^+ - \alpha_{l+1}^-). \tag{16}$$

The phenomenon $\Lambda \to N + \pi + 37$ Mev corresponds energetically to $\pi - N$ scattering at 43 Mev (in the nucleon rest system). If we adopt the Orear's formula for s-wave phase shift of $\pi - N$ scattering α_0^+ (in Orear's notation $\alpha_1) = 0.16\ \eta$,[11] the value of α_0^+ at 43 Mev becomes 6.5°. Moreover the following facts are well known by the phase shift analyses for $\pi - N$ scattering at 40 Mev, (1) the p-wave phase shifts $|\alpha_1^-|$ (α_1^- is usually denoted as α_{11}) and $|\alpha_1^+|$ $(=\alpha_{13})$ are at most of the magnitudes of 1°, (2) we may neglect the phase shifts for the states with larger angular momenta than 1. Therefore we can obtain the following results:

a) When $S_\Lambda \geqq 3/2$,

(i) in the case of PA=inv.,

$$R_l^* R_{l+1} + R_l R_{l+1}^* \simeq \pm 2 |R_l| |R_{l+1}|, \tag{17}$$

(ii) in the case of PT=inv.,

$$R_l^* R_{l+1} + R_l R_{l+1}^* \simeq 0. \tag{18}$$

b) When $S_\Lambda = 1/2$,

(i) in the case of PA=inv.,

$$R_l^* R_{l+1} + R_l R_{l+1}^* \simeq \pm 2 \times 0.99 |R_l| |R_{l+1}|, \tag{19}$$

(ii) in the case of PT=inv.,

$$R_l^* R_{l+1} + R_l R_{l+1}^* \simeq \pm 2 \times 0.11 |R_l| |R_{l+1}|. \tag{20}$$

As the spin of Ξ-particle generally ought to be polarized when Ξ is produced, let us examine first of all the cases of [1] and [2] in § 2·1 or § 2·2. Because the term of $(A_l^* A_l + A_{l+1}^* A_{l+1}) \times (R_l^* R_{l+1} + R_l R_{l+1}^*)$ is contained in the expression for angular distribution, we can derive the following conclusions without any assumption about $\Xi \to \Lambda + \pi$ process.

Even if the decay interaction responsible for the process $\Lambda \to N + \pi$ is invariant under PT, it will be difficult or impossible to confirm the existence of asymmetrical angular distribution for the final decay product because of quite small coefficients in the odd power terms in $\cos\theta$, while if the decay interaction responsible for the process $\Lambda \to N + \pi$ is invariant under PA, it may be expected that the asymmetrical angular distribution will be observed.

In order to discuss the angular distribution in the case where the spin of Ξ is unpolarized, we are obliged to consider the process $\Xi \to \Lambda + \pi$ together with $\Lambda \to N + \pi$. A same analysis for the former as that for the latter may be performed by employing the phase shifts for $\pi - \Lambda^0$ scattering. But we now have no experimental data for $\pi - \Lambda^0$

scattering, so that we shall carry on our discussion under the assumption that the values of phase shifts for $\pi-\Lambda^0$ scattering at 76 Mev are not larger than those for $\pi-N$ scattering at the same energy. This assumption will be a rather natural one because we cannot expect that $\pi-\Lambda^0$ interaction will be stronger than $\pi-N$ interaction.* Then we can estimate $(A_l^* A_{l+1}+A_l A_{l+1}^*)$ in the similar way as $(R_l^* R_{l+1}+R_l R_{l+1}^*)$.

(i) In the case of $PA=$inv.,

$$A_l^* A_{l+1}+A_l A_{l+1}^* \cong \pm 2\,|A_l|\,|A_{l+1}|,$$

and

$$R_L^* R_{L+1}+R_L R_{L+1}^* \cong \pm 2\,|R_L|\,|R_{L+1}|. \tag{21}$$

(ii) In the case of $PT=$inv.,

$$|A_l^* A_{l+1}+A_l A_{l+1}^*| \ll 2\,|A_l|\,|A_{l+1}|,$$

and

$$|R_L^* R_{L+1}+R_L R_{L+1}^*| \cong 0 \quad \text{or} \quad \leqq 2\times 0.11\,|R_L|\,|R_{L+1}|. \tag{22}$$

Thus we can derive the same conclusion as in the case where the spin of Ξ is polarized. Namely, as was stated before if decay interaction is invariant under PT, it will be difficult or impossible to confirm the existence of asymmetrical angular distribution for the final decay product, while if the decay interaction is invariant under PA, it may be expected that the asymmetrical angular distribution will be observed.

Even if there exist some bound states in $\pi-\Lambda^0$ system, our conclusion mentioned above does not suffer any change. The reason is as follows: In the case where $\pi-\Lambda^0$ system has n bound states, the value of phase shift generally varies with the incident energy from the value of $n\pi$ at zero energy. The addition of $n\pi$ to β_l or β_{l+1}, however, does not give rise to any change to our previous discussion.

Finally let us append our discussion for the case where our assumption about $\pi-\Lambda^0$ interaction is not valid. If $\pi-\Lambda^0$ interaction at 76 Mev has a particular behavior** as (3/2, 3/2) resonance in $\pi-N$ scattering at 190 Mev, there is some possibility of giving rise to the following situation:

(i) In the case of $PA=$inv.,

$$|A_l^* A_{l+1}+A_l A_{l+1}^*| \ll 2\,|A_l|\,|A_{l+1}|.$$

(ii) In the case of $PT=$inv.,

$$|A_l^* A_{l+1}+A_l A_{l+1}^*| \cong 2\,|A_l|\,|A_{l+1}|.$$

In such a particular case, our conclusion becomes as follows: Even if both weak interactions responsible for the cascade decay process are not invariant under space reflection, it will not be easy to confirm the existence of the asymmetrical angular distribution.

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* We shall later append the discussion for the case where our assumption is not valid.

** It will probably be difficult to expect such a circumstance.

§ 4. On $K^- + p \to \Sigma$ (or Λ) $+ \pi$ and Σ (or Λ) $\to N + \pi$

In this section, the angular distribution of decay product for the direction perpendicular to the production plane is examined under the assumption that the parity conservation does not hold for the decay interaction of hyperons. For simplicity we deal only with the cases of S_K (the spin of K^-) $=0$ and $S_{\Sigma(\text{or }\Lambda)}$ (the spin of Σ or Λ) $=1/2$ or $3/2$.

§ 4·1. The case of $K^- = 0^{+(-)}$ and hyperon $= 1/2^{-(+)}$

Since hyperons are produced through the strong interaction, we may think in this case that the orbital angular momentum of K^- in the initial state is equal to that of π in the final state of the production process. We now denote the R-matrix with regard to the state of orbital angular momentum L of incident K^- and that with regard to the state of orbital angular momentum L of the final decay product π as A_L and R_L, respectively. Averaging over the initial spin states of nucleon and summing over the final spin states of nucleon, we can easily write down the general expression for the angular distribution in the hyperon rest system.

$$(1/4\pi)[(\xi_1+\xi_2)(R_0^* R_0 + R_1^* R_1) + (\eta_1+\eta_2)(R_0^* R_1 + R_0 R_1^*)\cos\theta], \tag{23}$$

$$\left.\begin{aligned}
\xi_1 &= \sum_{l,l'} (l+1)(l'+1)(A_l^* A_{l'} + A_{l+1}^* A_{l'+1})[P_l P_{l'} + P_{l+1} P_{l'+1} \\
&\qquad -\cos\theta' (P_l P_{l'+1} + P_{l'} P_{l+1})]/(4k^2 \sin^2\theta'), \\
\xi_2 &= \sum_{l,l'} (l+1)(l'+1)(A_l^* A_{l'+1} + A_{l+1}^* A_{l'})[P_l P_{l'+1} + P_{l'} P_{l+1} \\
&\qquad -\cos\theta' (P_l P_{l'} + P_{l+1} P_{l'-1})]/(4k^2 \sin^2\theta'), \\
\eta_1 &= \sum_{l,l'} i(l+1)(l'+1)(A_l^* A_{l'} - A_{l+1}^* A_{l'+1})[P_l P_{l'+1} - P_{l+1} P_{l'}]/(4k^2 \sin\theta'), \\
\eta_2 &= \sum_{l,l'} i(l+1)(l'+1)(A_l^* A_{l'+1} - A_{l+1}^* A_{l'})[P_l P_{l'} - P_{l+1} P_{l'+1}]/(4k^2 \sin\theta'),
\end{aligned}\right\} \tag{24}$$

where θ' means the angle between the direction of π produced in the first reaction and that of incident K^- in the center of mass system, and θ the angle between the direction of the decay product π and that perpendicular to the production plane. Moreover, the argument of Legendre function P_L is $\cos\theta'$.

§ 4·2. The case of $K^- = 0^{+(-)}$ and hyperon $= 1/2^{+(-)}$

When the orbital angular momenta of K^- are l and $(l+1)$, those of pion produced in the first reaction are $(l+1)$ and l respectively. We now indicate the R-matrices with regard to these states as B_l and B_{l+1} respectively, then the angular distribution is given by the following substitution in eqs. (23) and (24):

$$\left.\begin{aligned}
A_l &\to -B_{l+1}, \\
A_{l+1} &\to -B_l.
\end{aligned}\right\} \tag{25}$$

Here it must be noted that the angular distribution of the final decay product π does not depend on whether the spin of hyperon has been observed before the decay. This is due to the property of $S_{\Sigma \text{ or } \Lambda}=1, 2$ and is concerned with the result that there is no angular correlation, except the θ'-dependence of angular distribution, between pion produced in the first reaction and pion as the decay product.

§ 4·3. The case of $S_K=0$ and $S_{\Sigma \text{ or } \Lambda}=3/2$

In such an energy region where the experiments are now being performed it may be a good approximation to take into account angular momentum up to *p*-wave. We show in Table 1 the states which must be considered.

Table 1

Case	$K^- - p$ interaction	Total angular momentum	Orbital angular momentum of pion produced in the first reaction	Reaction amplitude for the production process
(I) : $K^-=0^{+(-)}$ hyperon$=3/2^{-(+)}$	*s*	1/2	2	C_{s2} (½)
	p	1/2	1	C_{p1} (½)
	p	3/2	1	C_{p1} (3/2)
			3	C_{p3} (3/2)
(II) : $K^-=0^{+(-)}$ hyperon$=3/2^{+(-)}$	*s*	1/2	1	C_{s1} (½)
	p	1/2	2	C_{p2} (½)
	p	3/2	0	C_{p0} (3/2)
			2	C_{p2} (3/2)

At first we write the representation of intrinsic spin operators M_x, M_y, M_z* of spin 3/2 in which M_z is diagonal.

$$M_x=\begin{pmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix}, \quad M_y=\begin{pmatrix} 0 & -\sqrt{3}\,i/2 & 0 & 0 \\ \sqrt{3}\,i/2 & 0 & -i & 0 \\ 0 & i & 0 & -\sqrt{3}\,i/2 \\ 0 & 0 & \sqrt{3}\,i/2 & 0 \end{pmatrix},$$

and

$$M_z=\begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}. \tag{26}$$

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* We have chosen the *z*-axis along the line of the incident K^--meson and the *y*-axis along the direction perpendicular to the production plane.

We denote the spin wave functions corresponding to the eigenvalues $M_z=3/2,\ 1/2,\ -1/2$ and $-3/2$ as α, β, γ and δ, and those corresponding to the eigenvalues $M_y=3/2,\ 1/2,\ -1/2$ and $-3/2$ as α', β', γ' and δ' respectively, then

$$\left.\begin{aligned}\alpha'&=\sqrt{1/8}\,(\alpha+\sqrt{3}\,i\beta-\sqrt{3}\,\gamma-i\delta)\,,\\ \beta'&=\sqrt{1/8}\,(\sqrt{3}\,\alpha+i\beta+\gamma+\sqrt{3}\,i\delta)\,,\\ \gamma'&=\sqrt{1/8}\,(\sqrt{3}\,\alpha-i\beta+\gamma-\sqrt{3}\,i\delta)\,,\\ \delta'&=\sqrt{1/8}\,(\alpha-\sqrt{3}\,i\beta-\sqrt{3}\,\gamma+i\delta)\,.\end{aligned}\right\}\qquad(27)$$

Taking advantage of these relations, we can obtain the angular distribution of the decay product for the y-direction. Although the angle φ in the production plane can be measured from the direction of the hyperon, we now write for simplicity only the expressions in which the differential cross sections for this process have already been integrated with respect to φ. Further, in order to avoid the complexity only the terms of odd powers for $\cos\theta$ in the expressions will be shown below and those of even powers for $\cos\theta$ in them will be shown in Appendix.

Case (I) : the case of $K=0^{+(-)}$ and hyperon$=3/2^{-(+)}$

$$\begin{aligned}&(1/32k^2)\,(R_1^*R_2+R_1R_2^*)\Big(\big[i\{C_{s2}^*(1/2)\,C_{p1}(1/2)-C_{s2}(1/2)\,C_{p1}^*(1/2)\}\times\cos\theta\,(9\cos^2\theta-7)\\ &\quad+\sqrt{2/5}\,i\{C_{s2}^*(1/2)\,C_{p1}(3/2)-C_{s2}(1/2)\,C_{p1}^*(3/2)\}\times 2\cos\theta\,(9\cos^2\theta-4)\\ &\quad+\sqrt{2/5}\,i\{C_{s2}^*(1/2)\,C_{p3}(3/2)-C_{s2}(1/2)\,C_{p3}^*(3/2)\}\times 9\sin^2\theta\cos\theta\big]\sin\theta'\\ &\quad+\big[\sqrt{2/5}\,i\{C_{p1}^*(1/2)\,C_{p1}(3/2)-C_{p1}(1/2)\,C_{p1}^*(3/2)\}\times 3\cos\theta\,(9\cos^2\theta-5)\\ &\quad+\sqrt{2/5}\,i\{C_{p1}^*(1/2)\,C_{p3}(3/2)-C_{p1}(1/2)\,C_{p3}^*(3/2)\}\times 6\cos\theta\,(5-6\cos^2\theta)\\ &\quad+(2/5)\,i\{C_{p1}^*(3/2)\,C_{p3}(3/2)-C_{p1}(3/2)\,C_{p3}^*(3/2)\}\\ &\qquad\qquad\times 15\cos\theta\,(1-3\cos^2\theta)\big]\sin\theta'\cos\theta'\Big).\end{aligned}\qquad(28)$$

Case (II) : the case of $K=0^{+(-)}$ and hyperon$=3/2^{+(-)}$

$$\begin{aligned}&(1/32k^2)\,(R_1^*R_2+R_1R_2^*)\Big(\big[i\{C_{s1}^*(1/2)\,C_{p2}(1/2)-C_{s1}(1/2)\,C_{p2}^*(1/2)\}\times\cos\theta\,(7-9\cos^2\theta)\\ &\quad+\sqrt{2}\,i\{C_{s1}^*(1/2)\,C_{p0}(3/2)-C_{s1}(1/2)\,C_{p0}^*(3/2)\}\times 2\cos\theta\\ &\quad+\sqrt{2}\,i\{C_{s1}^*(1/2)\,C_{p2}(3/2)-C_{s1}(1/2)\,C_{p2}^*(3/2)\}\times\cos\theta\,(9\cos^2\theta-5)\big]\sin\theta'\\ &\quad+\big[\sqrt{2}\,i\{C_{p2}^*(1/2)\,C_{p0}(3/2)-C_{p2}(1/2)\,C_{p0}^*(3/2)\}\times 9\sin^2\theta\cos\theta\\ &\quad+\sqrt{2}\,i\{C_{p2}^*(1/2)\,C_{p2}(3/2)-C_{p2}(1/2)\,C_{p2}^*(3/2)\}\times 6\cos\theta\,(3\cos^2\theta-2)\\ &\quad+2i\{C_{p0}^*(3/2)\,C_{p2}(3/2)-C_{p0}(3/2)\,C_{p2}^*(3/2)\}\\ &\qquad\qquad\times 3\cos\theta\,(1-3\cos^2\theta)\big]\sin\theta'\cos\theta'\Big).\end{aligned}\qquad(29)$$

From the above examination we may state the following results.

(1) If an asymmetrical distribution of decay products about the production plane is observed, we can conclude that the parity conservation does not hold for the decay process of hyperon.

(2) Let us discuss the form of asymmetry taking up the angular distributions in the cases of $\theta'=90°$. If it is $\sin^2\theta\cos\theta$, this shows that the interference of $S_{1/2}$* and $P_{3/2}$ will play an important role in $K^- - p$ interaction, and if it is $\cos\theta\ (7-9\cos^2\theta)$, this shows that the interference of $S_{1/2}$ and $P_{1/2}$ will play an important role, and so on. In each case we may obtain some knowledge about π-hyperon interaction by eqs. (28) and (29).

(3) If there are some evidences of $S_K=0$ and $S_\Sigma=3/2$ in other experiments and, in addition, if the form of asymmetry mentioned above is $\cos\theta$, then it may be said that the parity of Σ-particle is the same as that of K^--meson.

Appendix

We here write down the terms of even powers for $\cos\theta$ in the expression for angular distribution (see § 4).

Case (I) : the case of $K=0^{+(-)}$ and hyperon$=3/2^{-(+)}$

$$\begin{aligned}
&(1/32k^2)\,(R_1{}^*R_1+R_2{}^*R_2)\Big(\{C_{s2}^*(1/2)\,C_{s2}(1/2)+C_{p1}^*(1/2)\,C_{p1}(1/2)\}\times(5-3\cos^2\theta)\\
&\quad+(2/5)\,C_{p1}^*(3/2)\,C_{p1}(3/2)\,\{5-3\cos^2\theta+12\,(1+3\cos^2\theta)\sin^2\theta'\}\\
&\quad+(18/5)\,C_{p3}^*(3/2)\,C_{p3}(3/2)\,\{2\sin^2\theta+(3-\cos^2\theta)\cos^2\theta'\}\\
&\quad+[\{C_{s2}^*(1/2)\,C_{p1}(1/2)+C_{s2}(1/2)\,C_{p1}^*(1/2)\}\\
&\quad-\sqrt{2/5}\,\{C_{s2}^*(1/2)\,C_{p1}(3/2)+C_{s2}(1/2)\,C_{p1}^*(3/2)\}\\
&\quad+3\sqrt{2/5}\,\{C_{s2}^*(1/2)\,C_{p3}(3/2)+C_{s2}(1/2)\,C_{p3}^*(3/2)\}]\times(5-3\cos^2)\cos\theta'\\
&\quad+\sqrt{2/5}\,\{C_{p1}^*(1/2)\,C_{p1}(3/2)+C_{p1}(1/2)\,C_{p1}^*(3/2)\}\\
&\qquad\qquad\times\{12\cos^2\theta-2-3\,(3\cos^2\theta+1)\cos^2\theta'\}\\
&\quad+3[\sqrt{2/5}\,\{C_{p1}^*(1/2)\,C_{p3}(3/2)+C_{p1}(1/2)\,C_{p3}^*(3/2)\}\\
&\quad-(2/5)\,\{C_{p1}^*(3/2)\,C_{p3}(3/2)+C_{p1}(3/2)\,C_{p3}^*(3/2)\}]\\
&\qquad\qquad\times\{-3\sin^2\theta+(8-6\cos^2\theta)\cos^2\theta'\}\Big).
\end{aligned}\tag{A·1}$$

Case (II) : the case of $K=0^{+(-)}$ and hyperon$=3/2^{+(-)}$

$$\begin{aligned}
&(1/32k^2)\,(R_1{}^*R_1+R_2{}^*R_2)\Big(\{C_{s1}^*(1/2)\,C_{s1}(1/2)+C_{p2}^*(1/2)\,C_{p2}(1/2)\\
&\quad+2C_{p0}^*(3/2)\,C_{p0}(3/2)\}\times(5-3\cos^2\theta)\\
&\quad+2C_{p2}^*(3/2)\,C_{p2}(3/2)\,\{2\,(1+3\cos^2\theta)+3\,(1-3\cos^2\theta)\cos^2\theta'\}\\
&\quad+[\{C_{s1}^*(1/2)\,C_{p2}(1/2)+C_{s1}(1/2)\,C_{p2}^*(1/2)\}\\
&\quad-\sqrt{2}\,\{C_{s1}^*(1/2)\,C_{p0}(3/2)+C_{s1}(1/2)\,C_{p0}^*(3/2)\}\\
&\quad+\sqrt{2}\,\{C_{s1}^*(1/2)\,C_{p2}(3/2)+C_{s1}(1/2)\,C_{p2}^*(3/2)\}]\times(5-3\cos^2\theta)\cos\theta'
\end{aligned}$$

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* S or P means the orbital angular momentum of K^- in the initial state, and the suffix shows the total angular momentum J for K^- plus nucleon system.

$$\begin{aligned}&+\sqrt{2}\,\{C_{p2}^{*}(1/2)\,C_{p0}(3/2)+C_{p2}(1/2)\,C_{p0}^{*}(3/2)\}\times(4-6\cos^{2}\theta-9\sin^{2}\theta\cos^{2}\theta')\\&+[2\,\{C_{p0}^{*}(3/2)\,C_{p2}(3/2)+C_{p0}(3/2)\,C_{p2}^{*}(3/2)\}\\&-\sqrt{2}\,\{C_{p2}^{*}(1/2)\,C_{p2}(3/2)+C_{p2}(1/2)\,C_{p2}^{*}(3/2)\}]\\&\qquad\times(1+3\cos^{2}\theta-6\cos^{2}\theta')\Big).\end{aligned}\tag{A·2}$$

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Criticism on the Assumptions in the Formal Scattering Theory

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We derive the conditions which give a well-defined meaning to the mathematical expressions appearing in the current formal scattering theory. These conditions remove some restrictions that are usually imposed on the spectra of H and H_0. By these conditions we can systematically discuss many problems, such as the equivalence between two kinds of S-matrices given by Heisenberg and Dyson respectively, or as the adiabatic theorem in the quantum field theory. The rearrangement collisions can be investigated from this view-point.

Introduction

By virtue of its compact form, the formal scattering theory1)−11) has been applied not only to the scattering problem itself but also to various other problems, such as the description of the nuclear phenomena12),−14) or the investigation of the renormalization procedure.15) Further application*) of the formal scattering theory may be expected to get more knowledge about eigenstates of the total Hamiltonian, which play an important part in the recent research16)−18) in the field theory. In many investigations so far made, several kinds of assumptions are adopted either explicitly or implicitly. Some of these seem to us quite reasonable but others do not. Neither the mutual relationship between these assumptions nor their necessity and sufficiency have been investigated. The main purpose of the present paper is to make a study on the problems stated above and to reduce some of these assumptions to more fundamental or more physically acceptable ones.

Stating in more detail, let us consider the application of the formal scattering theory to a system which allows the existence of bound states. In the current treatment of this nature, one usually assumes explicitly that the discrete eigenvalues of the total Hamiltonian, H, are isolated from the continuous eigenvalues of the free Hamiltonian, H_0, and at the same time assumes implicitly that the behaviour of the eigenfunctions of H is good enough to give a definite meaning to each mathematical expression. The first assumption allows us to exclude the virtual bound states from our consideration, at least when bound states do not exist in the initial or the final state. In the case in which actual bound states appear in the initial or final condition, one usually switches off in the remote past and future the part of the interaction that is not responsible for making the compound particle. In the quantum field theory, the above mentioned two assumptions seem to contradict the definition of the vacuum and the concept of the field.

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* See the foot-note in p. 60.

The vanishing of the interactions between several particles in the remote past and future is originally due to the spatial separation of particles from each other and not due to the vanishing of the coupling constant. In the time dependent perturbation method, however for convenience, one introduces a time-dependent coupling constant, say $ge^{-\lambda|t|}=g(t)$, in place of the actual coupling constant. From the physical view-point as stated above, this time dependent, $g(t)$, should be derived by taking into account the separation of wave packets from each other in the remote future or past.[19] In the stationary treatment one adopts, instead of the above $g(t)$, the boundary condition corresponding to a superposition of an incoming plane wave and a spherical outgoing-wave in solving the Schrödinger equation. The relationship between these two approaches was studied by many authors.

In the present paper we shall give a condition which guarantees the validity of the so-called adiabatic theorem (§ 1). From our argument we shall find that in the first place, our condition is also necessary in the stationary treatment of the scattering problem to give a solution which actually satisfies the above mentioned boundary condition (§ 2). In the second place, it will be seen that the assumption concerning the isolation of the discrete energy level of H from the continuous spectrum of H_0 is not always necessary. Finally it will be given that Heisenberg's S-matrix is equivalent to that of Dyson derived on the implicit assumption of the time-dependent coupling constant, $g(t)$, (§ 3). Further consideration will be made in § 4 on our condition concerning the energy-spectrum. It will be found that this condition covers almost all the physically acceptable cases with regerd to the structure of the energy-spectrum. Thanks to the present condition which gives a technique of applying the time-dependent treatment to the stationary problems, we can find a method of dealing with the complicated problems which have bound states in the initial or final conditions (§ 5). Since our condition seems too complicated to interpret, we shall reduce it in the Appendix to the more familiar but more fundamental one; namely to the assumption that any state vector of the total Hamiltonian can be expanded by means of a set of state vectors of the free Hamiltonian and vice versa. The former assumption has been made throughout the present article, but the latter in § 1. It is an open question to examine under what sort of interaction these assumptions are guaranteed.

§ 1. Adiabatic theorem

Let us consider a system of fields with a Hamiltonian $H=H_0+V$. Any eigenstate ϕ_a of the free Hamiltonian H_0 corresponding to an eigenvalue E_a is assumed to be expanded by means of the eigenstates of H.

The "modified"*) adiabatic theorem is written as

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* The adjective "modified" means that in the present paper we shall only discuss on (1·1) instead of $\lim U(ot)\phi_a=\psi_a{}^+$.

$$\lim_{t\to-\infty} U(ot)\,\Phi_i = \lim_{t\to-\infty}\int U(ot)\,\phi_a(\phi_a\Phi_i)\,da = \int \psi_a^+(\phi_a\Phi_i)\,da, \tag{1·1}$$

where $U(ot)$ is a transformation function in the interaction representation and ψ_a^+ is a solution of the following equation ;

$$\psi_a^+ = \phi_a + \frac{1}{E_a - H_0 + i\epsilon} V\psi_a^+. \tag{1·2*}$$

The function Φ_i means an arbitrary but well-behaved wave packet defined by

$$\Phi_i = \int (\phi_a\Phi_i)\,\phi_a da. \tag{1·3}$$

Following Hack's line of thought,[8)] ϕ_a is expanded as[**)]

$$\phi_a = \int dc\,(\psi_c^+\phi_a)\,\psi_c^+ + \sum_l (\psi_l\phi_a)\,\psi_l \tag{1·4}$$

where ψ_l is a solution of the equation

$$\psi_l = \frac{P}{E_l - H_0} V\psi_l\,. \tag{1·5}$$

What we are going to do is to give a condition which makes valid the equality of (1·1). The left-hand side of (1·1) is rewritten as

$$\lim_{t\to-\infty} U(ot)\,\Phi_i = \lim_{t\to-\infty}\int da\,\{\psi_a^+ + \rho_a^+(t) + \beta_a(t)\}\,(\phi_a\Phi_i) \tag{1·6}$$

where

$$\rho_a^+(t) = \int dc \frac{1}{E_c - E_a - i\epsilon}(\psi_c^+ V\phi_a)\,\psi_c^+ e^{i(E_c-E_a)t} \tag{1·7}$$

and

$$\beta_a(t) = \sum_l \psi_l \frac{P}{E_l - E_a}(\psi_l V\phi_a)\,e^{i(E_l-E_a)t}\,. \tag{1·8}$$

Therefore our task is to examine on which occasions the following equations hold.

$$\lim_{t\to-\infty}\int da\rho_a^+(t)\,(\phi_a\Phi_i) = 0, \tag{I}$$

$$\lim_{t\to-\infty}\int da\beta_a(t)\,(\phi_a\Phi_i) = 0\,. \tag{II}$$

Let us assume that

$(\psi_c^+ V\phi_a)$ *is continuous at* $E_a = E_c$ *(Condition A).*

Then the factor $(E_c - E_a - i\epsilon)$ in (I) can be replaced by $P/(E_c - E_a) + i\pi\delta(E_c - E_a)$ and the left-hand side of (I) gives

---

[*] ε is an infinitesimal, real, positive number.

[**] In this section we shall assume the commutability of the order between integration a and integration c or summation l.

$$\lim_{t\to-\infty}\int da\frac{P}{E_c-E_a}(\psi_c^+ V\phi_a)(\phi_a\Phi_i)e^{i(E_c-E_a)t}+i\pi\lim_{t\to-\infty}\int \delta(E_c-E_a)(\psi_c^- V\phi_a)(\phi_a\Phi_i)\,da. \tag{1·9}$$

Therefore we have the second condition

$$\lim_{t\to-\infty}\int da\frac{P}{E_c-E_a}(\psi_c^+ V\phi_a)(\phi_a\Phi_i)e^{i(E_c-E_a)t}=-i\pi\int_{E_c=E_a}(\psi_c^+ V\phi_a)(\phi_a\Phi_i)\,da^0 \tag{Condition B}$$

which, together with Condition *A*, is necessary and sufficient for the validity of (I). $\int da^0$ in the second term in Condition *B* stands for the integration over the quantum numbers except for the energy E_a. Unfortunately, the necessary and sufficient condition for making the integrand satisfy the relation *B* cannot be expressed in a compact form. We can, however, give some examples of this sufficient condition;[20] namely,

in addition to Condition A, the integrand should satisfy the relation

$$\int_\Omega dE_a\left|\int da^0(E_c-E_a+i\epsilon)^{-1}(\psi_c^+ V\phi_a)(\phi_a\Phi_i)\right|<M \tag{B′}$$

or, instead of the condition B′ for some infinite region of Ω, the condition that

$$\int (E_c-E_a+i\epsilon)^{-1}(\psi_c^+ V\phi_a)(\phi_a\Phi_i)\,da^0 \tag{B″}$$

should monotonously decrease to zero for $E\to\infty$ except for the harmonically oscilating terms contained in the above expression.

The domain Ω of the above integration means $[E_{\min},\ E_c-\mathcal{E}']$ and $[E_c+\mathcal{E}',\ \infty]$, where $\mathcal{E}'$ is an arbitrarily small but finite positive constant. These two conditions are wide enough for dealing with the practical problems, because we can apply, for example, the first condition to some finite domain of integration and at the same time the second one to the rest. It will be investigated in Appendix whether or not these complicated mathematical conditions can be realized in the actual systems.

According to a similar consideration one gets the following two conditions in order to prove the equation (1·1):

$$(\psi_l V\phi_a)=o((E_l-E_a)^\gamma)\quad \gamma>0\quad \textit{at}\quad E_a\sim E_l \tag{Condition C}$$

and

$$\lim_{t\to-\infty}\int da\frac{P}{E_l-E_a}(\psi_l V\phi_a)(\phi_a\Phi_i)e^{i(E_l-E_a)t}=-i\pi\int_{E_l=E_a}(\psi_l V\phi_a)(\phi_a\Phi_i)\,da^0 \tag{Condition D}$$

We can give some examples of a sufficient condition for this relation; namely,

in addition to the condition C, the integrand should satisfy the relation

$$\int_\Omega dE_a\left|\int da^0(E_l-E_a)^{-1}(\psi_l V\phi_a)(\phi_a\Phi_i)\right|<M \tag{D′}$$

or the condition that

$$\int da^0 (E_l - E_a)^{-1} (\psi_l V \phi_a)\,(\phi_a \Phi_i)\, da^0 \tag{D''}$$

should monotonously decrease as seen in B''.

The condition C becomes unnecessary if the discrete eigenvalue of H is assumed to be isolated from the continuous spectrum of H_0. The cases in which this assumption is not true will be discussed in detail in § 4.

§ 2. Boundary conditions on stationary solution

Let us consider a stationary solution for the scattering problem. In general, the solution has the expression

$$\psi_a^+ = \phi_a + \frac{1}{(E_a - H_0 + i\epsilon)} V \psi_a^+ = \phi_a + \sum C^a{}_b \phi_b = \phi_a + \int db \frac{1}{E_a - E_b + i\epsilon} (\phi_b V \psi_a^+)\, \phi_b \tag{2·1}$$

which corresponds to the boundary condition of the following type:

$$\psi_a^+ \rightarrow \textit{incoming plane wave} + \textit{scattered outgoing wave}. \tag{2·2}$$

However, it is an open question to examine under what kind of conditions the above type of solution really has such an asymptotic behaviour as that imposed in the beginning. In this section we shall give the condition of this nature and study the relationship of these conditions with those derived in § 1. In order to examine the asymptotic behaviour of the above solution, let us transform the above expression into that in the coordinate space and, for simplicity, consider the case of the two particle state. The scattered wave takes the form

$$e^{i\vec{k}_1{}^a \vec{x}_1 + i\vec{k}_2{}^a \vec{x}_1} \int db^1 k_2{}^2 dk_2 d\phi d\vec{k}_1 \left[\int_0^{(\pi/2)+0} C^a{}_b \sin\theta d\theta e^{ikr\sin\theta} + \int_{(\pi/2)+0}^{\pi} C^a{}_b \sin\theta d\theta e^{ikr\sin\theta} \right] \tag{2·3}$$

where $\vec{k}_1$ and $\vec{k}_2$ mean the momenta of the particles in b-state, and b' means the rest. The vector $\vec{r}$ stands for $\vec{x}_1 - \vec{x}_2$ and θ and ϕ are the polar angles of $\vec{k}_2$ with reference to $\vec{r}$. According to the discussion in § 1, let us divide the second term in the brackets in (2·3) into the principal integral and the term containing the δ-factor. The principal integral is cancelled by the δ-term if the following conditions are satisfied:

$(\phi_b V \psi_a^+)$ *is continuous at* $E_b = E_a$ *(Condition A_1)*

$$\lim_{r\to\infty} \int db \frac{P}{E_a - E_b} (\phi_b V \psi_a^+)\, e^{ikr} = -i\pi \int_{E_a = E_b} db^0 (\phi_b V \psi_a^+)\, e^{ik_0 r} : E_a = E_b(k_0) \quad (\textit{Condition } B_1)$$

A_1 is exactly identical with A. B_1 is equivalent to B apart from the factor of the wave packet amplitude $(\phi_a \Phi_i)$. The first integral in (2·3) actually represents the outgoing scattered wave when the upper limit of the θ-integration is replaced by $\pi/2 - 0$. This replacement is permissible, because we can expect the regularity of $(\phi_b V \psi_a^+)$ in the neighbourhood of $\theta = \pi/2$ from the physical view-point. The similar consideration can easily extend to the cases of many particle problems. The present conclusion can be stated as follows: the conditions A and B_1 are necessary and sufficient to assure that

the type of solution (2·1) *really has the asymptotic behaviour, incoming plane wave + spherical outgoing wave.* (III)

In a similar way we can get conditions to guarantee that

an eigenvector ψ_l having a discrete energy E_l represents a wave localized in some finite spatial region. (IV)

The neccessary and sufficient conditions are as follows:

$$(\phi_a V \psi_l) = o((E_a - E_l)^{\gamma}) \qquad \gamma > 0 \qquad at \quad E_l = E_a \qquad (Condition\ C_1)$$

$$\lim_{r \to \infty} \int da \frac{P}{E_l - E_a} (\phi_a V \psi_l) e^{ikr} = -i\pi \int_{E_a = E_l} da^0 (\phi_a V \psi_l) e^{ik_0 r}\,; \; E_l = E_a(k_0) \qquad (Condition\ D_1)$$

C_1 is exactly identical with C. D_1 is equivalent to D except for a contribution from the wave packet amplitude $(\phi_a \Phi_i)$. In this way we see that conditions for the adiabatic theorem are identical with those for stationary solutions apart from a contribution from a wave packet.

§ 3. Equivalence between an S-matrix given by the time independent treatment and that given by the time dependent treatment

In the current type of the S-matrix given by Dyson the following definitions are used;

$$(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\tau w} d\tau = \delta(w), \qquad (2\pi)^{-1} \int_{-\infty}^{0} e^{-i\tau w} d\tau = \delta_+(w) \tag{3·1}$$

which are tacitly based on the assumption of the adiabatic switching off and on of the coupling constant. It is our purpose in the present section to investigate the relationship between the above mentioned formal switch-off procedure and the physical one which usually appears implicitly in the form of the boundary condition for the stationary scattering problem. The comparison of Dyson's S-matrix, S_D, with that of Heisenberg, S_H, is useful for our present purpose. Let us introduce the following quantity,

$$\Omega(t) = e^{iH_0 t} \Omega e^{-iH_0 t} \tag{3·2}$$

where Ω is defined by

$$\Omega \phi_a = \phi_a + \frac{1}{E_a - H_0 + i\epsilon} V \Omega \phi_a \,. \tag{3·3}$$

$\Omega(t)$ satisfies the following conditionary equality as was shown by Ma[4)]

$$\Omega(-\infty) \doteq 1 \tag{3·4}$$

$$\Omega(+\infty) \doteq S_H \,. \tag{3·5}$$

By inserting (3·2) and (3·3) into the left-hand side of (3·4) and (3·5), the matrix-elements of the latter expressions can be written as

$$\lim_{t\to\mp\infty}\int(\Phi_i\phi_a)\,da\,(\phi_a\Omega(t)\,\phi_b)=\lim\int(\Phi_i\phi_a)\,da\,(\phi_a\Omega\phi_b)\,e^{i(E_a-E_b)t}$$

$$=(\Phi_i\phi_b)+\lim\int(\Phi_i\phi_a)\,da\frac{1}{E_b-E_a+i\epsilon}(\phi_aV\psi_b^+)\,e^{i(E_a-E_b)t}\,.$$

The right-hand side of the above expression agrees with the right-hand side of (3·4) and (3·5) if

$$\lim_{t\to\mp\infty}\int(\Phi_i\phi_a)\,da\frac{1}{E_b-E_a+i\epsilon}(\phi_aV\psi_b^+)\,e^{i(E_a-E_b)t}=\begin{cases}0\\-2\pi i\int\delta(E_a-E_b)(\phi_aV\psi_b^+)(\Phi_i\phi_a)da\end{cases}\tag{3·6}$$

holds, which is guaranteed by the conditions *A* and *B*.

In the second place let us derive the conditions under which $\Omega(t)$ satisfies the integral equation defining Dyson's $U(0,-\infty)$, that is,

$$\Omega(t)=1-i\int_{-\infty}^{t}V(t')\,\Omega(t')\,dt'.\tag{3·7}$$

Inserting (3·2) and (3·3) into the left-hand side of (3·7), we can rewrite (3·7) as

$$\int e^{i(E_a-E_b)t}\frac{1}{E_b-E_a+i\epsilon}(\Phi_i\phi_a)\,da\,(\phi_aV\Omega\phi_b)=-i\int(\Phi_i\phi_a)\,da\,(\phi_a\int_{-\infty}^{t}V(t')\,\Omega(t')\,dt'\phi_b)\tag{3·8}$$

where a complete set of states, $\{\Phi_i\}$, has been introduced. The integrand of the right-hand side of the above equation turns out

$$-ie^{i(E_a-E_b)t}\lim_{M\to\infty}\int_{-M}^{t}e^{i(E_a-E_b)\tau}d\tau\,(\phi_aV\Omega\phi_b)$$

$$\begin{aligned}&\to\infty && \text{for}\quad E_a=E_b\\ &=\lim_{M\to\infty}\frac{1}{E_b-E_a}(\phi_aV\psi_b^+)\,e^{i(E_a-E_b)t}(1-e^{-i(E_a-E_b)M}) && \text{for}\quad E_a\neq E_b\end{aligned}\tag{3·9}$$

by using the definitions (3·2) and (3·3). The second term in the parenthesis of the last expression vanishes after the integration with respect to *a* by virtue of the condition *B*. Also on account of the condition *A*, the integration over the vicinity of E_b becomes

$$\begin{aligned}\int_{E_b-\epsilon}^{E_b+\epsilon}(l.\ h.\ s.)\,dE_a&=-\int\frac{i\sin\epsilon(\tau-t)}{\tau-t}(\phi_aV\psi_b^+)\,(\Phi_i\phi_a)_{E_a=E_b}d\tau da^0\\&=-i\pi\int(\phi_aV\psi_b^+)\,(\Phi_i\phi_a)_{E_b=E_a}da^0.\end{aligned}\tag{3·10}$$

The result (3·10) combined with the second expression of (3·9) gives

$$\int e^{i(E_a-E_b)t}\left\{\frac{P}{E_b-E_a}-i\pi\delta(E_a-E_b)\right\}(\Phi_i\phi_a)\,da\,(\phi_aV\Omega\phi_b)$$

which exactly agrees with the left-hand side of (3·8) as we expected. The equivalence of S_D with S_H can be derived from the above result which is due to the assumption that the conditions A and B are satisfied with regard to the complete set, $\{\Phi_i\}$. From the last statement concerning $\{\Phi_i\}$, we see that the condition B in this case is nothing but the condition B_1.

The results so far obtained show that we can reveal the implicit relations between the three typical kinds of problems discussed in §§ 1,2 and 3 by introducing the conditions A, B_1, C and D_1. Among these conditions B_1 and D_1 (therefore B and D) will be derived in Appendix from the assumption that any eigenfunction of H can be expanded by means of the set of eigenfunctions of H_0. In § 4 it will be shown that the condition C becomes superfluous for practical problems.

§ 4 Consideration on the discrete eigenstates of *H*

In this section we shall examine the condition C in § 1, i.e.

$$(\phi_b V \psi_a) = o((E_b - E_a)^\gamma) \quad \gamma > 0 \quad \text{for} \quad E_a \sim E_b\,. \tag{C}$$

In the formal scattering theory which has been developed by many authors, it is usually assumed that the discrete energy spectra of H are not mixed with continuous energy spectra of H_0. If that is the case, the condition C becomes unnecessary. In the field theory, however, vacuum is considered as the lowest energy state of continuous energy spectra, because there may be massless particles. Therefore the above mentioned assumption must be unavailable for this case.

Let us expand discrete eigenstate ψ_b using the complete set $\{\phi_a\}$ as

$$\psi_b = \sum C_b^a \phi_a\,. \tag{4·1)*}$$

There are following two cases:

Case 1) The coefficient C_b^a regarded as a function of E_a vanishes in a small but finite neighbourhood of the point E_b.

Case 2) C_b^a has some non-vanishing value in the above neighbourhood.

In the first case, we have

$$(\phi_b V \psi_a) = 0 \qquad \text{for} \qquad E_b \sim E_a$$

which corresponds to a special case of (C). Many cases we meet in actual problems may be included in this case.

The second case will be classified into the following three cases.**)

Case 2a) In the expansion (4·1), there appear only the continuous levels of H_0 in the vicinity of E_b.

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* Here, suffixes a, b,⋯ represent both continuous and discrete energy states. ψ^+ and ψ respectively stand for continuous and discrete states.

** We cannot exclude the existence of the discrete level of H_0 from our discussion. The simple example is the system in the external field.

Case 2b) In (4·1) there appears only a single discrete level $E_a(=E_b)$ of H_0 in the vicinity of E_b.

Case 2c) In (4·1) there appear both the continuous levels and a discrete level $E_a(=E_b)$ of H_0.

Let us start with the discussion on the case 2a). In the integral equation

$$\psi_b = \frac{P}{E_b - H_0} V \psi_b, \tag{4·2}$$

the following conditions are necessary for giving a difinite meaning to the right-hand side of (4·2); namely,

$$\begin{aligned} &(\phi_b V \psi_a) = o((E_b - E_a)^{\gamma}) \qquad \gamma \geq 0 \\ \text{or} \qquad &\propto (E_b - E_a)^{-2n} \qquad (n: \text{positive integer}) \text{ at } E_a \sim E_b. \end{aligned} \tag{4·3}$$

The case of $\gamma > 0$ is nothing but the condition C. One of the examples is Lee's model[21)] of the renormalizable field with a particular type of the cut-off function, $f^2(\omega) = (c - m - \omega)^2 \omega^{-1} (\omega^2 - \mu^2)^{-1/2}$, where the constant c is defined by $(g/2\pi)^2 (c - m - \mu/\mu)^2 - (g/2\pi)^2 - 1 = 0$. This model has a discrete eigenvalue, $E(=C)$, of H. This pathological state describes a bound state of an N and a θ-particle because the conditions C and D are satisfied in this case.*)

An interesting case is that of $\gamma = 0$. In this case (4·2) can be rewritten as

$$\begin{aligned} \psi_b &= \frac{P}{E_b - H_0} V \psi_b \\ &= \frac{1}{E_b - H_0 + i\epsilon} V \psi_b + i\pi\delta(E_b - H_0) V \psi_b \\ &= \frac{1}{E_b - H_0 + i\epsilon} V \psi_b + \sum_{E_a = E_b} C_a \phi_a \qquad C_a \neq 0. \end{aligned} \tag{4·4}$$

Here it must be remarked that there should be at least a non-vanishing coefficient C_a, owing to the assumption $\gamma = 0$. Therefore ψ_b can be written as

$$\psi_b = \sum C_a \psi_a^{+} \tag{4·5}$$

and is not linearly independent of ψ. We can make a complete set, $\{\psi^{+}, \psi\}$, without using such a vector ψ_b. The existence of ψ_b given by (4·5) does not give any change to the conclusion (II) in § 1, but gives some modification to the conclusion (IV) in § 2. In the case of $\gamma = 0$, the standing wave is the asymptotic form of the solution of (4·4). One of this example is Lee's model with $f^2(\omega) = \omega(\omega^2 - \mu^2)^{-1/2}$. There is a discrete energy $E = m + \mu + \mu \exp(2\pi/g)^2$ in the above mentioned sub-space but this state vector ψ_b is linearly dependent of ψ^+ and written as

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* The appearance of this curious state seems to violate the conclusion of Källén and Pauli concerning the completeness of the sub-set of eigenfunctions of H, that is, the set $\{\psi_{N\theta}{}^{+}, \psi_V\}$. This situation, however, does not mean the contradiction of our result to their conclusion because their line of thought cannot be applicable to the present case owing to the divergence of the norm of $\psi^{+}{}_{N\theta}$.

$$\psi=C\int_E \psi^+_{N\theta}dk=C'\int_E \psi^-_{N\theta}dk=C''(\psi^+_{N\theta}-\psi^-_{N\theta})_E\,. \tag{4·6}$$

The case $(\phi_a V\psi_b)\propto(E_b-E_a)^{-2n}$ can be excluded from our discussion because of the impossibility of normalizing the state vector ψ_b.

In the case 2b), ψ_b can be written as

$$\psi_b=C\phi_b{}^{(d)}+\sum C_a\phi_a=C\phi_b{}^{(d)}+\psi'$$

where $\phi_b{}^{(d)}$ stands for the discrete state of H_0. By using the above expression, Schrödinger equation turns out

$$(E_b-H_0)\psi_b'=V\psi_b \tag{4·7}$$

from which we have

$$(\phi_b{}^{(d)}V\psi_b)=0. \tag{4·8}$$

In order to avoid ambiguities that are expected to occur in dealing with (4·7) and (4·8), we substitute the above two equations with

$$\psi_b=C\phi_b{}^{(d)}+\frac{1}{E_b-H_0+i\epsilon}V\psi_b, \tag{4·9}*$$

$$(\phi_b{}^{(d)}V\psi_b)=o(\epsilon^{1+\alpha})\quad \alpha>0. \tag{4·10}$$

(4·10) is necessary for giving a definite meaning to the equation (4·9) and is regarded as a condition for the renormalization of energy, because this equation can be rewritten as $(\phi_b(H-H_0)\psi)\to 0$. Therefore it is easily seen from this equation that this discrete energy E of H_0 agrees with the energy E_b. The solution of (4·9) and (4·10) gives a state of fields being localized in a finite spatial region owing to the condition (4·10). From this situation we see that the condition (4·10) is a substitute for the condition C. In fact we can prove that (4·10) also gives the validity of (II) in § 1. The mass renormalization of the V-particle in Lee's model can be performed in this fashion with the same result as the usual one. This method seems quite useful in studying the renormalization problem and in fact seems much simpler than the current method. The last case 2c) is concerned with a problem of decaying particles. There remain unknown facts as to renormalization.[22)23)] Equations which correspond to (4·9) and (4·10) are

$$\psi_b=C\phi_b{}^{(d)}+\int_{E_b} C_k\phi_k dk+\frac{1}{E_b-H_0+i\epsilon}V\psi_b, \tag{4·11}$$

$$(\phi_b{}^{(d)}V\psi_b)=o(\epsilon^{1+\alpha})\qquad \alpha>0, \tag{4·12}$$

$$(\phi_k V\psi_b)=o((E_k-E_b)^{\gamma})\quad \gamma\geq 0. \tag{4·13}$$

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* We are planning to investigate the application of the equations (4·9)—(4·13) in more detail.

In order to explain our line of thought, consider the following equations

$$\psi_b^{(d)}=\phi_b^{(d)}+\frac{1}{E_b-H_0+i\epsilon}V\psi_b^{(d)}$$

and

$$\psi_k^+=\phi_k+\frac{1}{E_b-H_0+i\epsilon}V\psi_k^+ \; . \qquad (1\cdot2)$$

The solution ψ_b of $(4\cdot11)$ can be represented as

$$\psi_b=C\psi_b^{(d)}+\int C_k\psi_k^+dk$$

with the condition $(4\cdot12)$. When γ is zero, at least some of C_k are not equal to zero from the similar consideration in $(4\cdot4)$. $(4\cdot12)$, in this case, is not the condition for the renormalization of energy but turns out to be a condition for a relative ratio of C and C_k. When γ is positive all C_k are zero, and $(4\cdot12)$ is an energy renormalization condition. As stated above, the case $\gamma>0$ can be excluded from the present argument because this case can be reduced to the case 2b). Therefore the problem we should examine is the asymptotic behaviour and the validity of (II) in the case $\gamma=0$. In the argument in § 1 if ψ_k^+ (§ 1) defined by $(1\cdot2)$ is replaced with

$$\psi_{(k)}^+=C\phi_b^{(d)}+\phi_k+\frac{1}{E_b-H_0+i\epsilon}V\psi_{(k)}^+ \, , \qquad (4\cdot14)$$

then the discussions concerning (II) can be followed without any further modification. It must be noted that the coefficient C in $(4\cdot14)$ is determined as a function of k by the condition $(4\cdot12)$. Finally it must be remarked that, in the present case, ψ_b represents a standing wave in the asymptotic region contrary to (IV).

§ 5. Scattering including bound states

In the present section we shall reexamine the scattering theory of a system including bound states in the initial and (or) final states from the view-point of the quantized field theory. For simplicity, let us consider a scattering of a proton by a deuteron. The initial deuteron state is assumed to be in a state $\sum c_\alpha^0\phi_\alpha$ where $\{\phi_\alpha\}$ is, of course, the set of the eigenstates of H_0. If the creation operator of a proton with a momentum $\vec{p}$ is denoted by $a_p{}^*$, some particular linear combination of $a_p{}^*\sum c_\alpha^0\phi_\alpha\equiv\sum c_\alpha\phi_\alpha$ will give the actual initial condition where a deuteron and a proton exist. The final state is similarly given by a superposition of ϕ_β, that is, $\sum d_\beta\phi_\beta$.

Here, let us introduce and "effective" hermitian interaction Hamiltonian X which is formally defined by

$$(H-E_i)\sum c_\alpha\phi_\alpha=X\sum c_\alpha\phi_\alpha \qquad (5\cdot1)$$

where E_i is the sum of the initial deuteron energy E_d and that of proton ω_p. Consider a complete set of state vectors, $\{\Psi_i\}$, whose first element Ψ_1 is $\sum c_\alpha\phi_\alpha$. Then the equation (5·1) defines only the first row and column of X. It will be seen that the following argument will not be changed no matter how the rest element of X may be chosen. Similarly we can define another one, Y, by using $\sum d_\beta\phi_\beta$;

$$(H-E_f)\sum d_\beta\phi_\beta = Y\sum d_\beta\phi_\beta\,. \tag{5·2}$$

By means of X and Y, the outgoing and the incoming solutions are defined by

$$\psi^+ = \sum c_\alpha\phi_\alpha + \frac{1}{E_i-(H-X)+i\epsilon}X\psi^+ \tag{5·3}$$

$$\psi^- = \sum d_\beta\phi_\beta + \frac{1}{E_f-(H-Y)-i\epsilon}Y\psi^-\,. \tag{5·4}$$

Once we can get the above type of equations, the current method19)24)—26) gives the S-matrix of the form

$$S_{if} = \left(\sum d_\beta\phi_\beta,\ \sum c_\alpha\phi_\alpha\right)_{E_i=E_f} - 2\pi i\delta(E_i-E_f)\left(\sum d_\beta\phi_\beta Y\psi^+\right) \tag{5·5}$$

X and Y are so complicated that it may be preferable to eliminate these operators from our expression (5·5). The elimination of Y is easily made by substituting $H-E_f$ for Y, whereas that of X from ψ^+ can be made after some tedious analysis of ψ^+. If we introduce $\chi^+ (=\psi^+-\sum c_\alpha\phi_\alpha)$, it satisfies the following equations,

$$\begin{aligned}\chi^+ &= \frac{1}{E_i-H_0+i\epsilon}\left(V\chi^+ + X\sum c_\alpha\phi_\alpha\right)\\ &= \frac{1}{E_i-H_0+i\epsilon}V\chi^+ + \frac{1}{E_i-H_0+i\epsilon}(H-E_i)\sum c_\alpha\phi_\alpha\\ &= \sum c'_\alpha\phi_\alpha + \frac{1}{E_i-H_0+i\epsilon}V\chi^+\,.\end{aligned} \tag{5·6}$$

The solution χ^+ of (5·6) can be written as

$$\chi^+ = \sum c'_\alpha\chi^+_\alpha \tag{5·7}$$

where χ^+_α is a solution of

$$\begin{aligned}\chi^+_\alpha &= \phi_\alpha + \frac{1}{E_i-H_0+i\epsilon}V\chi^+_\alpha\\ &= \phi_\alpha + \frac{1}{E_\alpha-H_0+i\epsilon}\,\frac{E_\alpha-H_0+i\epsilon}{E_i-H_0+i\epsilon}V\chi^+_\alpha\,.\end{aligned} \tag{5·8}$$

χ^+_α is the usual type of the outgoing solution except that we take $(E_\alpha-H_0+i\epsilon)\cdot(E_i-H_0+i\epsilon)^{-1}V$ as the potential instead of V. Therefore we see that the present complicated scattering problem including bound states has been reduced to solve a familiar one with a slight modification with regard to the potential.

For a while, let us restrict our discussion to the case in which the final state is composed of free particles. We must examine whether our final expression for χ^+ (scattered wave) actually have an outgoing behaviour in the asymptotic region; namely, whether $(\phi_f, \sum c_\alpha' V \chi_\alpha^- + (H-E_i)\sum c_\alpha \phi_\alpha)$ does or does not satisfy the conditions A and B_1 in §§ 1 and 2. It will easily be conjectured that the first term of the above quantity will have a quite similar behaviour to that in the usual case (case of no bound states), as is seen from (5·6) and (5·8). The second term, which seems to violate the condition B_1 on account of a positive power of H_0, can be written as

$$\begin{aligned}(H-E_i)\sum c_\alpha\phi_\alpha &= a_p^* H \sum c_\alpha{}^0\phi_\alpha + \omega_p a_p^* \sum c_\alpha{}^0\phi_\alpha - E_i \sum c_\alpha \phi_\alpha + [V, a_p{}^*]\sum c_\alpha{}^0\phi_\alpha \\ &= (E_d + \omega_p - E_i)\sum c_\alpha \phi_\alpha + [V, a_p^*]\sum c_\alpha{}^0\phi_\alpha. \end{aligned} \tag{5·9}$$

In (5·9), the first term vanishes because of the relation $E_i = E_d + \omega_p$, and the second term in general contains a negative power of the energy as a factor. Accordingly the condition A and B may be satisfied. The similar conjecture can be made for cases in which some bound states occur in the final state. In this case, "the outgoing scattered wave" must be interpreted as a state where at least the distance between a couple of particles becomes infinitely large. Our criterion for checking the behaviour of the solutions is unfortunately too complicated to deal with the practical problem exactly. It may be better, however, to examine by means of our conditions whether the result obtained in each stage of the calculation do or do not actually have such properties as we expected before the calculation.

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Appendix

The completeness of the set $\{\phi_a\}$ and the conditions B_1 and D_1

In what follows the completeness of the set $\{\phi_a\}$ is interpreted as follows:

Any state vector ψ *can be represented by* $\psi = \sum C_\alpha \phi_\alpha$

where the right-hand side is regarded to converge to the limit weakly and at the same time strongly in the sense of the theory of Hilbert-space. Namely, we mean

$$(\Phi_i \psi_l) = \int da (\Phi_i \phi_a)(\phi_a \psi_l) \qquad \text{(weak convergence)} \tag{W}$$

$$\|\psi - \int da (\phi_a \psi)\phi_a\| = 0 \qquad \text{(strong convergence)} \tag{S}$$

where $\{\Phi_i\}$ is any complete set of state vectors.

Suppose ψ^+ being represented by

$$\psi_i^+ = \int da (\phi_a \psi_i^+)\varphi_a, \tag{A·1}$$

then from (A·1) and (W) we have

$$(\Phi_i \psi_l^+) = (\Phi_i \phi_l) + \int da\, (\Phi_i \phi_a) \Big(\phi_a \frac{1}{E_l - E_a + i\epsilon} V \psi_l^+ \Big)$$

$$= (\Phi_i \phi_l) + \int da\, (\Phi_i \phi_a) \frac{1}{E_l - E_a + i\epsilon} R_{al} \qquad (\mathrm{A}\cdot 2)$$

by using the original equation for ψ^+. The convergence of the integral in (A·2) imposes on the integrand the following condition;

$$\int R_{al} (E_l - E_a + i\epsilon)^{-1} (\Phi_i \phi_a)\, da < M. \qquad (\mathrm{A}\cdot 3)$$

This restriction gives the validity of B'' for extremely large values of E_a, (for example $E_a >$ some large constant W). On the other hand the definition (S) gives

$$(\psi_l^+ - \int da\, (\phi_a \psi_l^+) \phi_a,\ \psi_{l'}^+ - \int da\, (\phi_a \psi_{l'}^+) \phi_a) = 0. \quad \text{for} \quad l = l'. \qquad (\mathrm{A}\cdot 4)$$

This equation can be written as

$$0 = \delta_{ll'} - \int da\, (\psi_l^+ \phi_a)\, (\phi_a \psi_{l'}^+)$$

$$= \delta_{ll'} - \int da \Big(\phi_l + \frac{1}{E_l - E_a + i\epsilon} V \psi_l^+ ,\ \phi_a \Big) \Big(\phi_a,\ \phi_{l'} + \frac{1}{E_{l'} - E_a + i\epsilon} V \psi_{l'}^+ \Big) \qquad (\mathrm{A}\cdot 5)$$

$$= -\frac{1}{E_l - E_{l'} - i\epsilon} R_{l'l}^* - \frac{1}{E_{l'} - E_l + i\varepsilon} R_{ll'} - \int da \frac{1}{E_l - E_a - i\epsilon} \cdot \frac{1}{E_{l'} - E_a + i\epsilon} R_{al}^* R_{al'} .$$

For the convergence of an integration with respect to *a* in the third term in the right-hand side of (A·5), the following relation must be satisfied,

$$\int_\Omega |R_{al}|^2 |E_l - E_a - i\epsilon|^{-2} da < M \qquad (\mathrm{A}\cdot 6)$$

where Ω is a similar region to that defined in § 1 and does not include $[E_l - \epsilon', E_l + \epsilon']$. This condition (A·6) gives the validity of B' for the value of E_a being in the region $E_a < W$. From the above two results we see that the condition B_1 can be reduced to the more fundamental but familiar postulates, (S) and (W). The validity of B can be derived from that of B_1 quite easily if the character of the wave packet appearing in B is not so bad as to destroy the previous discussions in § 1. The similar consideration is also applicable to the case of D by replacing ψ^+ in the foregoing argument with ψ.

Before concluding this appendix we shall refer to some examples where the conditions A, B_1 and D_1 are actually satisfied. The R-matrix for $N-\theta$ scattering in Lee's model satisfies the conditions A and B_1, and the state vector of V particle in that model also satisfies the condition D_1. Another example is the quantum electrodynamics and the meson theory in which the conditions A and B_1 are satisfied by the R-matrix (up to the second order) derived by the perturbation after the renormalization has been performed.

Reversing the argument, we can start with the assumption that any ϕ_a can be represented by a linear combination of the complete set $\{\psi^+, \psi\}$. In this case we can derive the following conditions,

$$\lim_{t\to-\infty}\int dc\frac{P}{E_c-E_a}(\psi_c^+V\phi_a)e^{i(E_c-E_a)t}=-i\pi\int_{E_a=E_c}dc^0(\psi_c^+V\phi_a). \qquad (\text{Condition } \bar{B}_1)$$

We can prove the modified adiabatic theorem using conditions A, $\bar{B}_1$ and D_1 instead of conditions A, B_1 and D_1. The condition B can be substituted with

$$\lim_{t\to-\infty}\int dc\frac{P}{E_c-E_a}(\phi_a V\psi_c^+)(\phi_c\Phi_i)e^{i(E_c-E_a)t}=-i\pi\int_{E_a=E_c}dc^0(\phi_a V\psi_c^+)(\phi_c\Phi_i) \qquad (\text{Condition } \bar{B})$$

for a particular wave packet Φ_i.

Then, we can prove the equality between S_H and S_D for an initial state Φ_i, using conditions A and $\bar{B}$ instead of A and B. The present conditions B and $\bar{B}$ do not guarantee the asymptotic behaviour of ψ^+.

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Cloudy Crystal Ball Model for Neutron Reactions at Higher Energies

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A systematic analysis using the cloudy crystal ball model of Feshbach, Porter and Weisskopf is carried out for the total cross sections and the non-elastic cross sections of neutron scattering by nuclei. The energy range considered extends from 3.5 to 14.0 Mev. Almost all the regions of mass number of the target nuclei are covered except for very light ones. The optical potentials assumed are of square well shape. The energy dependence of the potential strength, $V+iW$, and the mass number dependence of the range of potential, R, are carefully studied. It is found that the mass number dependence of the total cross sections at each of the fixed energies is well reproduced by the calculation with the suitable choice of the potential parameters. On the other hand, the calculation fails to give the correct magnitude of the non-elastic cross sections. The calculated values are considerably smaller than the experimental ones especially for lower incident energies.

The parameters which have been found to give the best fit to the experimental data are as follows:

$R=(0.5+1.35\ A^{1/3})\times 10^{-13}$ cm, $V=40$ Mev for all the neutron energies studied and W varies from 2 to 7 Mev as the incident energy increases from 3.5 to 14.0 Mev.

§ 1. Introduction

By using the so-called cloudy crystal ball model proposed by Feshbach, Porter and Weisskopf[1] the analyses of the nucleon scattering by nuclei have been carried out extensively by many authors.[2-19] From these analyses it became apparent that the cloudy crystal ball model would be valid for much wider range of incident nucleon energy than that covered by the original work of Feshbach, Porter and Weisskopf. It has been found, however, that one should permit the potential parameter, especialy its imaginary part, to vary with the energy of the incident nucleon. As has been pointed out by several authors,[20] the energy dependence of the imaginary part of the potential strength thus found is in semi-quantitative agreement with that predicted by the semi-classical calculation of the mean free path of the incident nucleon within the nuclear medium which is assumed to be described by degenerate Fermi gas.

It seems to the authors, however, that the systematic analysis of the nucleon scattering in higher energy region is still insufficient. The analyses carried out so far were mostly concerned with the angular distributions of the elastically scattered nucleons, at fixed, fairly well-defined energies, by the particular target nuclei which occupy scattered positions on the periodic table. In order to see the applicability of the model and to determine the energy dependence of the potential parameters, it seems necessary to carry out the systematic analysis of the cross sections. For that purpose it should cover as wide ranges of both incident energy and mass numbers of the target nuclei as possible, since the general trend of the variations of the cross sections with both the incident energy and the mass numbers of the target nuclei should primarily be accounted for by the cloudy crystal ball model. In particular it seems most suitable, at first, to study the total cross sections and the total reaction cross sections (non-elastic cross sections) of nucleon scattering, although the angular distribution of the elastic scattering might give some information useful for the further refinement of the model. It is, however, difficult, especially at low incident energies, to estimate the compound elastic contributions[1] to the elastic scattering and furthermore the experimental data are still insufficient at present. These situations make it difficult to carry out systematic analysis of the angular distribution of elastically scattered nucleons. On the other hand, the experimental data on the total cross sections are sufficiently provided[21] and the data on the non-elastic cross sections have recently been accumulated.[10,12,21,22] In view of above situations we have carried out an analysis of the total and non-elastic cross sections of neutrons in higher energy range for the target nuclei which cover almost all the region of the periodic table except for very light and very heavy ones. The energy range taken up in the present analysis is extended from 3.5 to 14.0 Mev. Actual calculation has been carried out for the energies of the incident neutrons of 3.5, 4.5, 5.5, 7.0, 9.0, 11.0, 12.7 and 14.0 Mev, using the complex potential of square well shape with various values of the potential parameters. Fixing the incident energy at one of those values, attempt has been made of finding the set of potential parameters which gives the best fit to the experimental *total* cross sections. In comparing the calculated values with the experimental data a particular emphasis has been put on the overall fitting throughout the wide range of mass numbers of the target nuclei rather than on the minute detail of the variation of the cross sections from one particular nucleus to another.

After finding, in this way, the appropriate values of the potential parameters for each of the incident neutron energies, the comparison has been made of the calculated compound nucleus formation cross sections resulting from these potentials with the experimental non-elastic cross sections. Theoretically, the former should exceed the latter by the contribution of the compound elastic cross sections,[1] the estimation of which cannot be unambiguous, but it depends, to some extent, on the model assumed and the parameters included therein which are indispensable for the numerical calculations. One of the plausible ways of the estimation will be to assume the statistical model of Hauser and Feshbach[2] and to carry out the similar calculation as what has been done by Oleksa.[2] This sort of calculation, however, becomes tedious as the incident nucleon energy increases.

In the case of the present analysis, however, it has incidentally been found that the calculated compound nucleus formation cross sections σ_c are considerably smaller than the experimental non-elastic cross sections σ_n for all the incident energies and the target nuclei investigated, contradicting to the theoretical expectations. Therefore, no calculation has been attempted for the compound elastic cross sections. It seems probable that this failure has been caused by the improper shape of the potential assumed.

The need of the diffuse edge for the optical potential in reproducing the experimental data has already been pointed out by many authors[1,6,7,10,11,13,16] and some suitable choice of potential shape has been proposed through the analysis of the angular distribution of the elastic scattering of nucleons.[6,13,16,17] It has also been noted by several authors[7,10,11,19] that the square well shape of the optical potential fails to give correct magnitude of the non-elastic cross sections for some particular incident energies and target nuclei. It is found, in the present analysis, that this tendency is of general nature regardless of either the kind of target nuclei or the incident energy. This shows a definite evidence for excluding the square well shape of the potential as far as the cloudy crystal ball model is retained. Probably a part of the discrepancy mentioned above might be ascribed to the failure of the cloudy crystal ball model itself, although such a discussion is beyond the scope of the present paper.

In § 2 the underlying assumptions of the present analysis and the procedures of the calculations are described. The results are compared with the experimental data.

In § 3 critical discussions are given on the results obtained and on the assumptions made in § 2.

§ 2. The calculation and the result

The calculations have been carried out of the total and the compound nucleus formation cross sections for the incident neutron energies of 3.5, 4.5, 5.5, 7.0, 9.0, 11.0, 12.7 and 14.0 Mev. The mass number dependence of the cross sections for each of these energies has been compared with the relevant experimental data. For some of the light nuclei which are still in the resonance region for the lower incident energies the suitable average over the cross sections within an energy interval is taken from the experimental data.

The calculations have been carried out by the exact partial wave analysis using the complex square well potential $(V+iW)$. For each of the incident neutron energies the various values for the potential parameters, i.e., V, W and its range, R, have been taken, so as to get the best fit result to the experimental data of the mass number dependence of the cross sections by trial and error.

Throughout the analysis the following assumptions have been taken as the working hypothesis:

(1) The potential parameters V and W are independent of the mass number of the target nucleus. They are, however, dependent on the incident neutron energy.

(2) The range of the potential, R, depends on the mass number, A, through the formula,

$$R=aA^{1/3}+b$$

where a and b are constants.

(3) R is assumed to be independent of the incident neutron energy.

We shall discuss the validity of these assumptions in the later section in some detail.

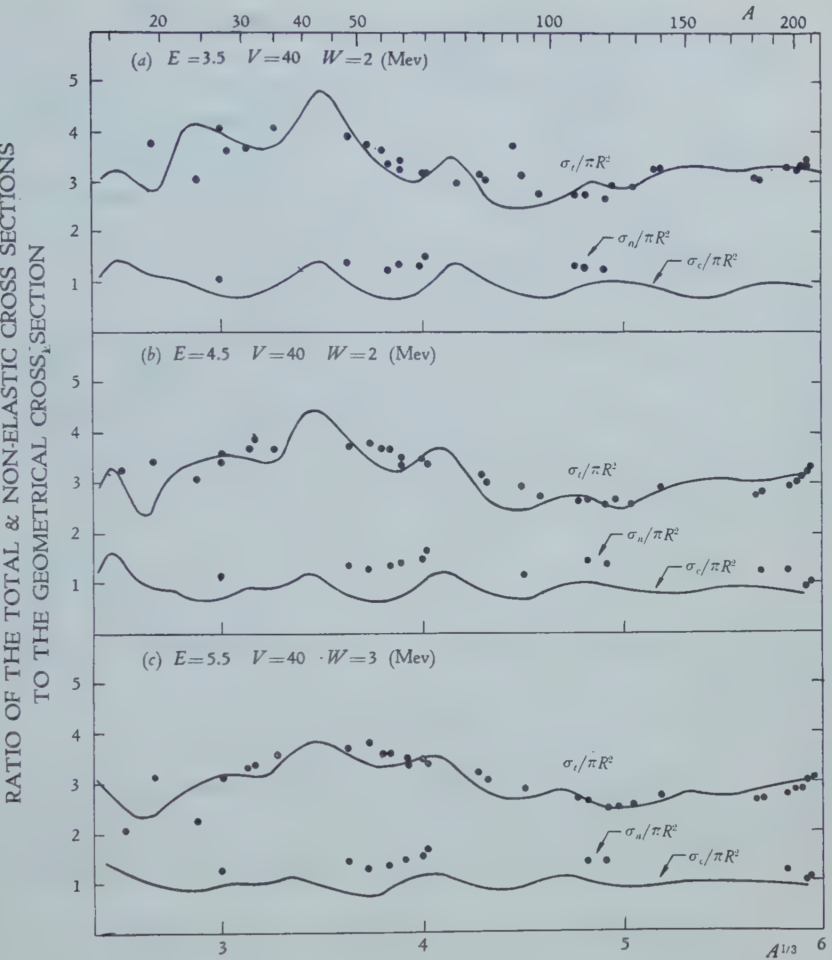

Fig. 1. The comparison of the experimental data with the best fit result obtained by the cloudy crystal ball model calculation for lower incident energies.

The solid curves represent the result of the calculations using the square well potential with the depth, $V+iW$, for $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ (above) and $\sigma_c/\pi R^2$ *vs* $A^{1/3}$ (below). The dots represent the experimental $\sigma_t/\pi R^2$ (above) and $\sigma_n/\pi R^2$ (below). The values of the parameters used in the calculations are indicated in the upper part of each figure. The incident energies are (a) $E=3.5$ Mev, (b) $E=4.5$ Mev and (c) $E=5.5$ Mev, respectively.

In Figs. 1 and 2 the best fit curves, $\sigma_t/\pi R^2$ *vs* $A^{1/3}$, are compared with the experimental data for each of the incident neutron energies mentioned above.

We shall discuss in some detail the comparison of the calculation with the experiment.

(1) $E=3.5$ Mev

The overall feature of the experimental results can be best reproduced by the calculation with the parameters $V=40$ Mev, $W=2$ Mev and $R=(1.35\,A^{1/3}+0.5)\times10^{-13}$ cm. There is also one-to-one correspondence between the experimental maxima in the $\sigma_t/\pi R^2$

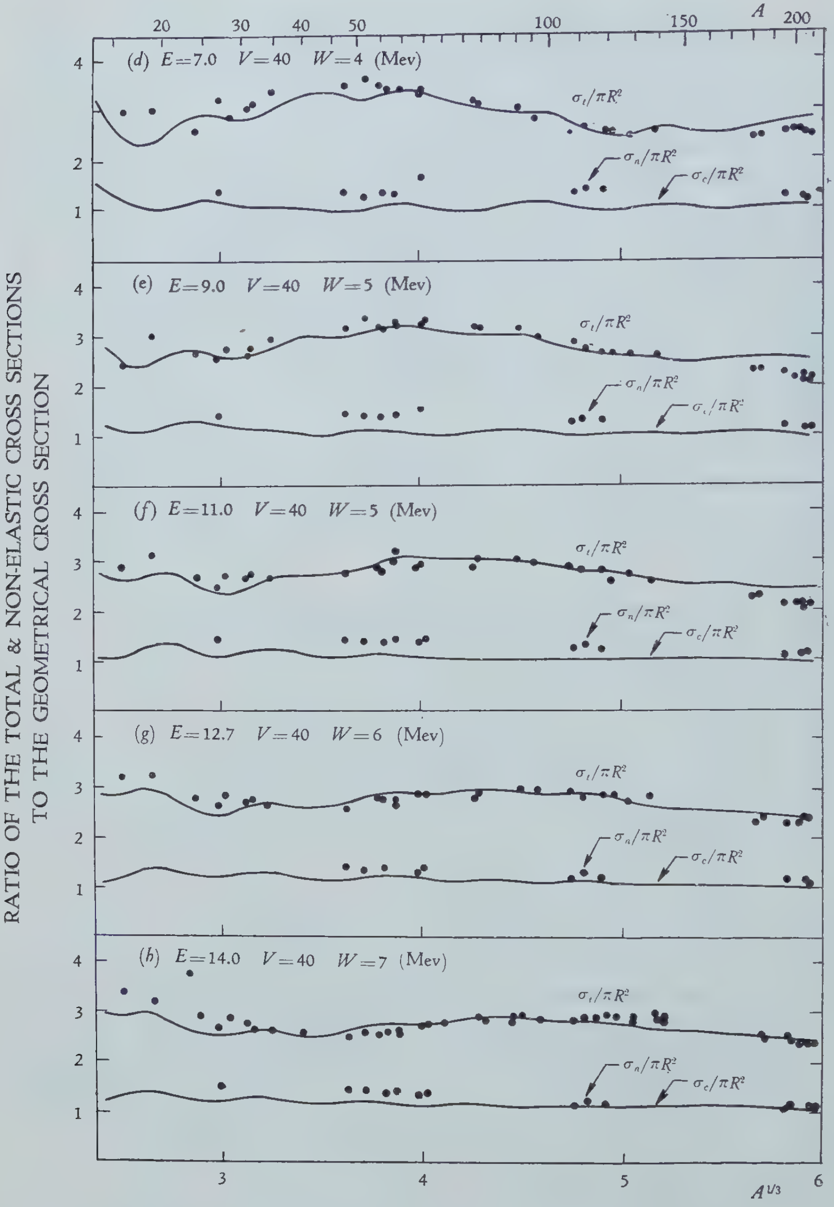

Fig. 2. The comparison of the experimental data with the best fit result obtained by the cloudy crystal ball model calculation for higher energies.

The solid curves represent the result of the calculations using the square well potential with the depth, $V+iW$, for $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ (above) and $\sigma_c/\pi R^2$ *vs* $A^{1/3}$ (below). The dots represent the experimental $\sigma_t/\pi R^2$ (above) and $\sigma_n/\pi R^2$ (below). The values of the parameters used in the calculations are indicated in the upper part of each figure. The incident energies are (d) $E=7.0$ Mev, (e) $E=9.0$ Mev, (f) $E=11.0$ Mev, (g) $E=12.7$ Mev and (h) $E=14.0$ Mev, respectively.

vs $A^{1/3}$ curve and the "shape resonance" peaks in the calculated one. The peak in small A region corresponds to the *p*-wave resonance. The broad peak around $A=40$ is accounted for by the *d*-wave resonance, and it is probable that the peak in the experimental curve around $A=80$ corresponds to the *f*-wave resonance which is situated at $A=70$, a little deviating from the experiment. This discrepancy cannot be removed by adjusting the potential parameters. In order to displace the position of the *f*-wave resonance to a larger value of A, the parameter V must be taken smaller, which, in turn, gives rise to a shifting towards a larger value of A of the position of the *d*-wave resonance mentioned above. Thus, it is clear that the discrepancy is inherent to the square well potential. The increase in $\sigma_t/\pi R^2$ as we proceed to the region $150>A>110$ corresponds to the *g*-wave resonance and the unresolved *d*- and *s*-wave resonances. The heaviest nuclei investigated, $A>180$, are located near the *f*-wave resonance. On account of the large value of R, however, the peaks resulting from these resonances are not so pronounced as in the case of light nuclei. Furthermore, the increase in the experimental $\sigma_t/\pi R^2$ with increasing A in this region is more rapid than that resulting from the mentioned resonance.

(2) $E=4.5$ Mev

The best fit parameters are $V=40$ Mev, $W=2$ Mev. The experimental data are still insufficient for checking the position of the most pronounced *d*-wave resonance at $A\sim40$. However, it can be seen that the broad peak covering the region $20<A<90$ can be accounted for by the successive *p*-, *d*- and *f*-wave resonances. A small maximum around $A=110$ and the gradual increase for $130<A<150$ may be attributed to the *g*-wave resonance and the unresolved *d*- and *s*- wave resonances, respectively. The rise for $A>180$ can be accounted for as the broad *f*-wave resonance. In this case also the rise in the experimental data is more rapid than that in the calculated one as is the case for $E=3.5$ Mev.

(3) $E=5.5$ Mev

The best fit parameters are $V=40$ Mev and $W=3$ Mev. Nearly the same thing as for 4.5 Mev can be said for the case of 5.5 Mev. Only difference noted is that resonance peaks are much more broadened than that for $E=4.5$ Mev. The cross sections for $20<A<90$ can be well reproduced by partially unresolved *p*-, *d*- and *f*-wave resonances and the rise of the cross sections for the heaviest nuclei is attributed to the broad *f*-wave resonance. It should be noted, however, that the experimental $\sigma_t/\pi R^2$ for this region is slightly smaller than the calculated ones, which tendency is common to the higher energies, as will be discussed later.

(4) Higher energies

Since there is no remarkable difference, in their general features, among the results obtained for $E=7.0$, 9.0, 11.0, 12.7 and 14.0 Mev, we shall discuss them as a whole under the same heading. The best fit parameters are $V=40$ Mev for all cases, and $W=4$, 5, 5, 6 and 7 Mev for $E=7.0$, 9.0, 11.0, 12.7 and 14.0 Mev, respectively. The broadening of the peaks is the more apparent, as the incident energy becomes higher. For the case of $E=12.7$ and 14.0 Mev the $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ curves are almost smooth and flat, except for very small value of $A^{1/3}$. This is due to the overlapping of the many partial wave resonances. The general trend of the experimental data is well reproduced by the above choice of the potential parameters. However, a closer examination shows that the general tendency common to all these cases, especially conspicuous for the cases of $E=7.0$, 9.0 and 11.0 Mev, is a slight overestimation of $\sigma_t/\pi R^2$ for the heaviest nuclei ($A>180$) in our calculation. This fact suggests that the assumptions made on the potential parameters might be incorrect. This point will be discussed in the next section.

The apparent discrepancy in the total cross sections for very light nuclei ($A<20$) shown in Figs. 1 and 2 will also be discussed in the next section.

The compound nucleus formation cross section σ_c has been calculated using the parameters which have given the best fit to the curves of $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ for each of the energies mentioned before. The results are shown in Figs. 1 and 2. As can be seen from these figures, the qualitative features of the experimental curves of $\sigma_n/\pi R^2$ *vs* $A^{1/3}$ can be reproduced by the calculated ones of $\sigma_c/\pi R^2$ *vs* $A^{1/3}$. Quantitatively, however, the calculated $\sigma_c/\pi R^2$ are systematically smaller than the experimental $\sigma_n/\pi R^2$. This behavior is especially conspicuous for lower incident energies. On account of the above situations no attempt has been made of the estimation of the compound elastic scattering cross section as has been mentioned in § 1, the discussions of which will be given in the next section.

§ 3. Discussions

As can be seen from the figures in § 2, the agreement between the calculated and the experimental total cross sections is in general satisfactory in view of the approximate nature of the model. Therefore, we may conclude that the cloudy crystal ball model of the square well type can approximately reproduce the overall features of the total cross

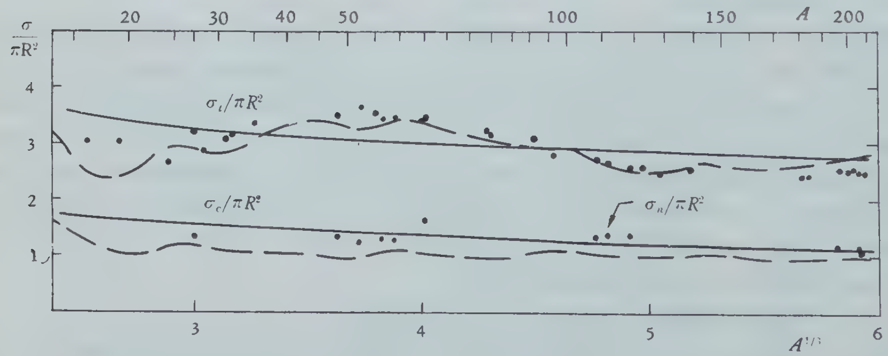

Fig. 3. The result of the continuum theory for $E=7.0$ Mev.

Solid curves represent the calculated $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ (above) and $\sigma_c/\pi R^2$ *vs* $A^{1/3}$ (below) using the continuum theory with the parameter of $K=1.16\times10^{13}$ cm^{-1}. Broken curves represent the best fit result obtained by the cloudy crystal ball model, the one shown in Fig. 2-(d). The experimental data on $\sigma_t/\pi R^2$ (above) and $\sigma_n/\pi R^2$ (below) are represented by dots.

sections for neutron collision with nuclei in their absolute magnitude as well as in their mass number dependence, if appropriate values are chosen for the potential parameters.

On the other hand, the failure of the continuum theory[25)] in reproducing the characteristic behavior of $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ is apparent in Fig. 3.

$\sigma_t/\pi R^2$ calculated with the continuum theory decreases monotonously with increasing A, in disagreement with the experimental data which show characteristic diffraction-like pattern in the figure. It should be noted, however, that the reasonable absolute magnitude is obtained for the total cross sections by the continuum theory.

As to the compound nucleus formation cross sections, σ_c, the agreement is poor as is seen in Figs. 1 and 2. The calculated values of σ_c are by several tens per cent smaller than the experimental non-elastic cross sections, σ_n, for low incident energies. The discrepancy becomes smaller for higher incident energies and the calculation gives nearly the correct absolute magnitude for the incident energies higher than, say, 12 Mev. However, even in these cases the experimental σ_n are systematically a little larger than the calculated σ_c. Since the relation between σ_c and σ_n is given by

$$\sigma_c=\sigma_n+\sigma_{ce},$$

where σ_{ce} is the compound elastic cross section, we have $\sigma_c\geq\sigma_n$ where the equality approximately holds for high incident energy, σ_{ce} being expected to be very small. Thus, the result obtained in § 2 is clearly in contradiction to the theoretical expectation.

The fact that the square well complex potential is unable to give the correct absolute magnitude of the compound nucleus formation cross sections has already been pointed out by several authors[7),10),11),19)] through their analyses for particular nuclei and incident energies. The discrepancy has been attributed to the strong reflection of the incident wave due to the abrupt change in its wave number at the boundary of the potential. It can be seen from the present analysis that this situation is of general nature, not depending on the particular choice of the incident energy or the target nucleus.

At first sight one might suppose that the calculated σ_c can be as large as one wishes, if a sufficiently large value is chosen for the imaginary part of the potential, W. This is however, not the case for the following reason. It is true that the σ_c is an increasing function of W, as far as W is smaller than or comparable to the energy of the incident neutrons. But, if W becomes very large, it makes the wave number of the incident wave very large inside the potential and the smooth continuation of the wave function at the boundary of the potential becomes impossible unless the amplitude is very small there. Thus, the scattering reduces to that by a hard sphere and the cross section, σ_c, tends to vanish in the limit of infinitely large W,*

$$\lim_{W\to\infty}\sigma_c=0.$$

Hence the quantity σ_c has a certain upper limit as a function of W. In Fig. 4 $\sigma_c/\pi R^2$

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* It might be worth mentioning here that the cloudy crystal ball model does *not* reduce to the continuum theory even in the limit of infinitely large W

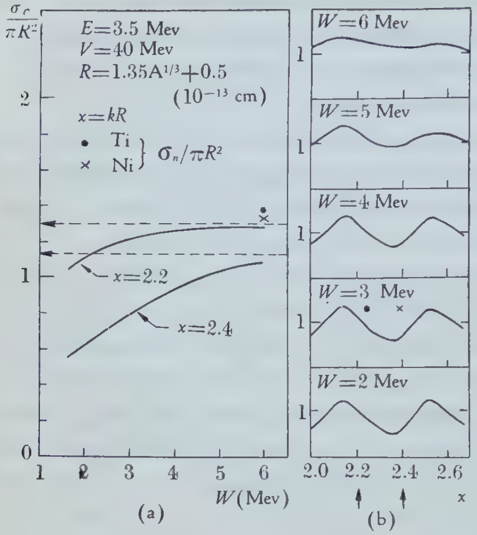

Fig. 4. (a) Dependence of $\sigma_c/\pi R^2$ on W.

Curves are for $x=2.2$ (above) and $x=2.4$ (below) at $E=3.5$ Mev, where $x=kR$, k being the wave number of the incident neutrons. V is taken to be 40 Mev in both cases. The former corresponds nearly to a maximum and the latter a minimum in the $\sigma_c/\pi R^2$ *vs* x curve as are shown in (b). These correspond to the target nuclei, Ti and Ni, respectively. The probable upper-limits of $\sigma_c/\pi R^2$ for these values of x's at 3.5 Mev are indicated by arrows in the figure.

(b) $\sigma_c/\pi R^2$ *vs* x curves for $E=3.5$ Mev for various values of W and $V=40$ Mev.

The values of W are indicated in the upper part of each figure. The values of x which are taken in (a) are indicated by arrows.

is plotted against W for $E=3.5$ and $x=kR=2.2$ and $x=2.4$, k being the incident wave number. It can be seen from the figure that the tangent of the slope of the curve, $(d/dW)\cdot(\sigma_c/\pi R^2)$ decreases as W increases and it might be reasonably supposed that the maximum values of $\sigma_e/\pi R^2$ for these E and x's lie somewhere around the greatest values indicated in the figure. These maximum values of 1.30 and 1.12 are small compared with the experimental ones, 1.38 and 1.33, for the target nuclei, Ti and Ni, respectively, and the incident energy, $E=3.5$ Mev. Moreover, the increase in W results in the flattening out of the curve, $\sigma_t/\pi R^2$ *vs* $A^{1/3}$, and the agreement between the calculation and the experiment is destroyed.

In this connection, it might be noted that the continuum theory gives better agreement with the experimental data than the present calculation as far as the absolute magnitude of σ_c is concerned, as is seen from Fig. 3. On the other hand, it fails to reproduce the characteristic fluctuations in the $\sigma_c/\pi R^2$ *vs* $A^{1/3}$ curves, just as in the case of $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ curves.

The only way of removing the contradiction about σ_e mentioned above is to choose the parameter R larger, as far as the square well is adopted, so that the experimental values of $\sigma_n/\pi R^2$ become smaller than the calculated values of $\sigma_c/\pi R^2$. But this procedure is again untenable, because then the unreasonably large value of R is required, which results in the serious discrepancy between the experimental and the calculated total cross sections. Thus, we conclude that the cloudy crystal ball model with the square well can hardly reproduce both the total and the compound nucleus formation cross sections consistently, the calculated value of the latter being too small.

The calculated ratios of the elastic scattering cross section, σ_{sc}, to the compound nucleus formation cross section, σ_c, are plotted in Fig. 5 and are compared with the experimental ratios of σ_{sc} to σ_n. The experimental values of σ_{sc} can be obtained either by integrating the angular distribution of the elastic scattering over angles or by subtracting σ_n from σ_t. The values shown in Fig. 5 are those obtained by the latter method.

The theoretical estimation of σ_{sc} is difficult because of the reason mentioned in § 1. At the higher incident energies, however, the following approximations are expected to hold,

$$\sigma_n \sim \sigma_c,$$

and hence we have

$$\sigma_{sc}^{th} \sim \sigma_t - \sigma_c$$

where σ_{sc}^{th} indicates the theoretical estimation of the elastic scattering cross section. More generally,

$$\sigma_n \leq \sigma_c$$

and so,

$$(\sigma_{sc}^{th}/\sigma_c) \leq (\sigma_{sc}/\sigma_n).$$

The underestimation of σ_n mentioned in connection with Figs. 1 and 2 and consequently the overestimation of σ_{sc} are even more apparent in Fig. 5, although the theory can reproduce the qualitative features of the experimental σ_{sc}/σ_n *vs* $A^{1/3}$ curves. This fact has already been noted in the case of the 14.0 Mev neutron reactions.7)

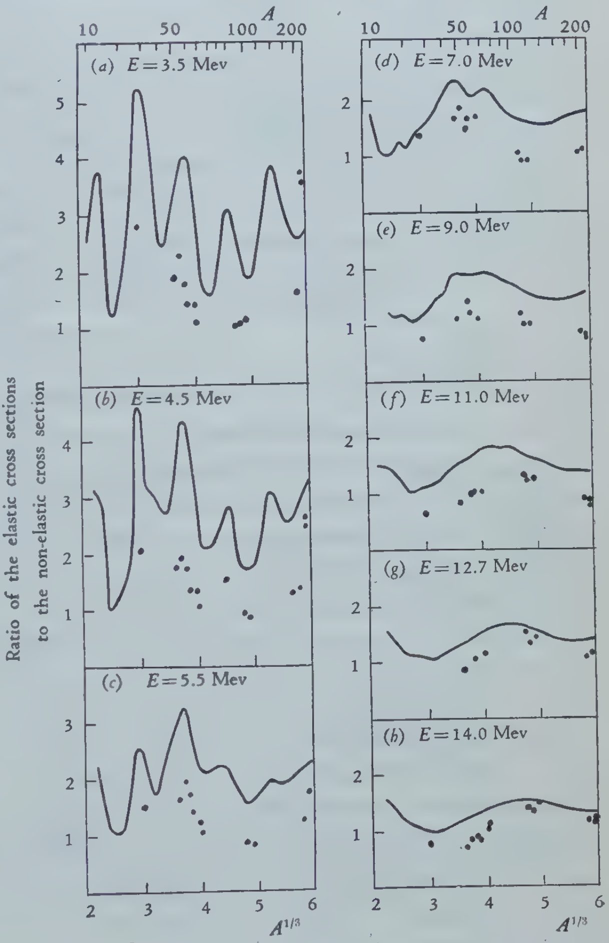

Fig. 5. Ratio of the elastic scattering cross section to the non-elastic cross section, σ_{sc}/σ_n *vs* $A^{1/3}$.

Solid curves represent the theoretical values of σ_{sc}/σ_c in which σ_{sc} is approximated by $\sigma_t - \sigma_c$. Dots represent the experimental values of the ratio σ_{sc}/σ_n. Parameters used in the calculations are the same as those indicated in Figs. 1 and 2.

In order to illustrate the dependence of the calculated cross sections on the potential parameters, some of the examples of the cross sections *vs* x curves are shown in Fig. 6 for several sets of the potential parameters for the case of $E=7.0$ Mev. As is seen in the figure the general tendency of the parameter dependence of the calculated results is such that the increase in V results in the shifting of the whole pattern towards the direction of small x and the increase in W results in the flattening out of the pattern without changing the positions of the peaks and the dips in the curves. As can be seen from the figures the calculated values are mildly dependent on these parameters. This mild dependence of the calculated cross sections on the potential parameters is common to all the incident energies investigated, so that the best fit parameters can not be determined very sharply. The uncertainty of the determination of the parameters V and W is

$$|\Delta V| \lesssim 2 \text{ Mev}$$

and $$|\Delta W| \lesssim 1 \text{ Mev},$$

respectively.

The energy dependence of the parameters, V and W, determined through the present analysis is shown in Table 1.

Table 1. Energy dependence of the potential parameters, V and W.

For each value of the incident energy, E, in the first column, the values of V and W determined through the present analysis are shown in the second and the third columns respectively. The uncertainty of the determination of these parameters is $|\Delta V| \lesssim 2$ Mev and $|\Delta W| \lesssim 1$ Mev respectively (see text).

E (Mev)	V (Mev)	W (Mev)
3.5	40	2
4.5	40	2
5.5	40	3
7.0	40	4
9.0	40	5
11.0	40	5
12.7	40	6
14.0	40	7

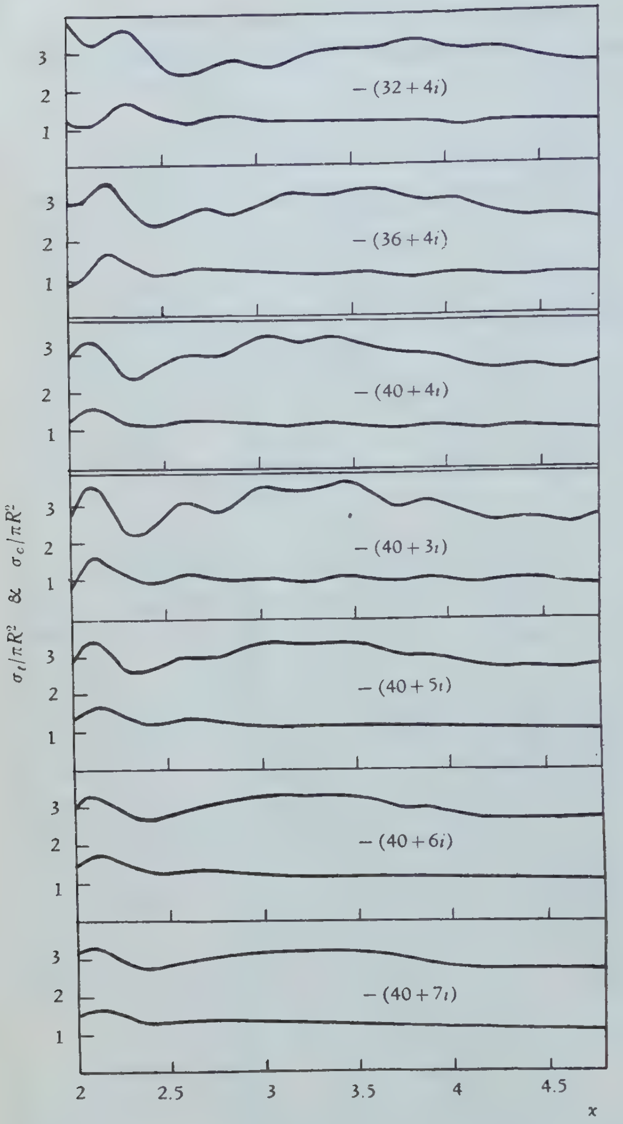

Fig. 6. Dependence of the calculated $\sigma_t/\pi R^2$ and $\sigma_c/\pi R^2$ on the potential parameters, V and W, for the case of $E=7.0$ Mev.
$R=(0.5+1.35\,A^{1/3})\times 10^{-13}$ cm is assumed for all parts of the figure. The potential parameters, $-(V+iW)$, are indicated in each part of the figure.

It is remarkable that the real part, V, of the potential is constant, within the accuracy of the present analysis, for energy range investigated. In Fig. 7 W is plotted against the incident energy, E, and is compared with the value calculated from the mean free path of the incident nucleon in the nuclear matter which is described by the Fermi gas model.[20]

As can be seen from the figure, the values of W calculated in this way are in semi-quantitative agreement with the ones obtained by the analysis. We see also that the calculated values based on the "$1/E$"-law for the free nucleon-nucleon cross sections are slightly in better agreement with that determined by the analysis than the value obtained from the energy independent free nucleon-nucleon cross sections.

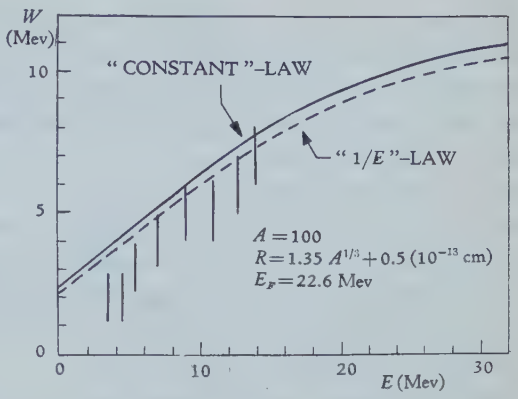

Fig. 7. Energy dependence of W for $A=100$.

Solid and broken curves represent the theoretical value obtained by the semi-classical mean free path calculation using the energy independent and "$1/E$"– free nucleon-nucleon cross sections, respectively. The energy of the incident neutron inside the nucleus is given by $E_1=E+E_F+B$ where $E_F=22.6$ Mev and $B=8.6$ Mev. Also plotted are the values of W determined by the present analysis.

At this point we shall discuss on the assumptions made in the analysis mentioned in § 2. The potential depths, V and W, were assumed to be independent of the mass number of the target nucleus, A. Generally speaking, however, the possibility of the mass number dependence of these parameters might be expected, even if we disregard the effect of the shell structure of the individual nuclei on these parameters.

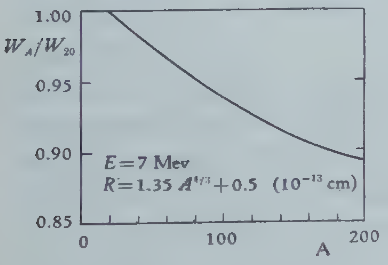

Fig. 8. Mass number dependence of W for $E=7.0$ Mev.

The ratio of W for mass number A to that for $A=20$ is plotted against A. $R=(0.5+1.35\ A^{1/5})\times10^{-13}$ cm is assumed. The calculation is based upon the semi-classical mean free path theory of W. The energy of the incident neutron inside the nucleus is given by $E_1=7.0+E_{FN}(A)+B(A)$ Mev. where $E_{FN}(A)$ is the Fermi energy and $B(A)$ the binding energy of neutrons in a nucleus with the mass number A.

In fact, the imaginary part, W, of the potential for the neutron scattering, for example, is expected to become smaller for heavier nuclei than for lighter ones, if the semi-classical mean free path theory of W[20)] is adopted. Because the neutron excess in heavier nuclei results in (1) the enhanced effect of the Pauli principle on the elementary collision process and (2) the reduction of the average nucleon-nucleon cross section in the nucleus, since the free neutron-neutron cross section is by a factor of about 3 smaller than the free neutron-proton cross-section. The mass number dependence of W calculated on these lines is illustrated in Fig. 8 for $E=7.0$ Mev.*

Unfortunately, however, the uncertainty of the determination of the potential parameters is comparable with the amount of reduction in W of 10% for

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* In the calculations it is assumed that the volumes occupied by the neutrons and the protons are the me, though it might not always be the case, as, for example, has been pointed out by Johnson and Teller.[26)]

$A=200$ nucleus compared with $A=20$ nucleus, so that we can hardly draw any definite conclusion about the existence of the mass number dependence of W in the present analysis. For this purpose it would be highly desirable to make more careful study, experimental and theoretical, of both the non-elastic cross sections and the angular distributions of the elastic scattering.

In contrast to the mild dependence of the calculated cross sections on the potential parameters, V and W, they are sensitive to the variation of the potential range, R. In view of this situation we shall examine in detail the assumption (2) in § 2 with regard to the relation between the range R and the mass number. On fitting the calculated values of $\sigma_t/\pi R^2$ to the experimental ones for the medium weight nuclei the slight systematic deviations are found in the region of heavy nuclei as has been mentioned in § 2, as far as we adhere to the same formula of $R=aA^{1/3}+b$ for all the region of A. The above discrepancy can be remedied if we assume different formulae for R *vs* $A^{1/3}$ for different region of mass numbers.[12] In fact, if we assume $R=(1.25\,A^{1/3}+0.8)\times10^{-13}$ cm for heavy nuclei with $A>180$, the discrepancy mentioned above can be considerably improved. Even in that case, however, it is difficult to reproduce accurately the large variation with energy in the shape of the $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ curves only by adjusting R, which probably represents the difficulty caused by the improper shape of the potential.

For light nuclei the experimental data show considerable fluctuations from nucleus to nucleus, although the general trend of the mass number dependence of the cross sections can be reproduced by the calculation. One of the possible interpretations of this situation is to attribute the fluctuations of the cross sections to that of the potential ranges. In fact the calculations for the light nuclei with properly modified R give much better agreement with the experiment than the original ones as is illustrated in Fig. 9. In the figure the

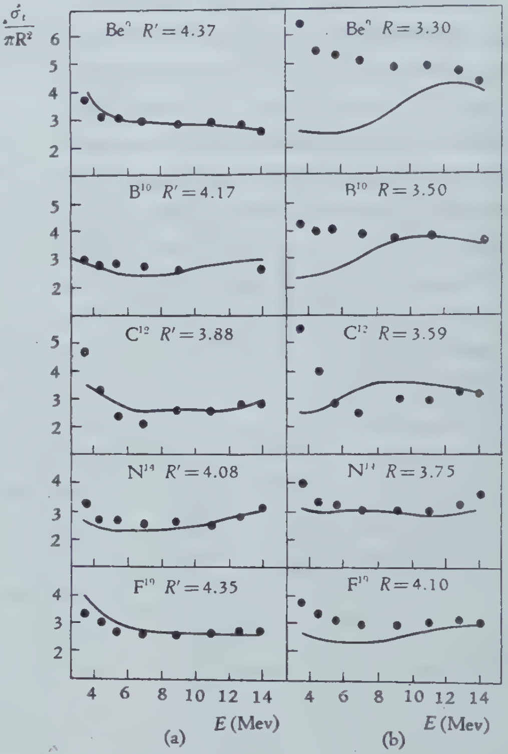

Fig. 9. Fluctuation of R's for light nuclei, Be^9, B^{10}, C^{12}, N^{14} and F^{19}.

The solid curves in (a) and (b) represent the result of the calculations using the modified R's (indicated as R's in the figure in units of 10^{-13} cm) and using the original values of R given by $R=(0.5+1.35\,A^{1/3})\times10^{-13}$ cm, respectively.

The dots represent the experimental σ_t divided by $\pi R'^2$, in (a) and by πR^2 in (b). The potential depths, V and W, used in the calculations are the same as those shown in Fig. 1 and 2.

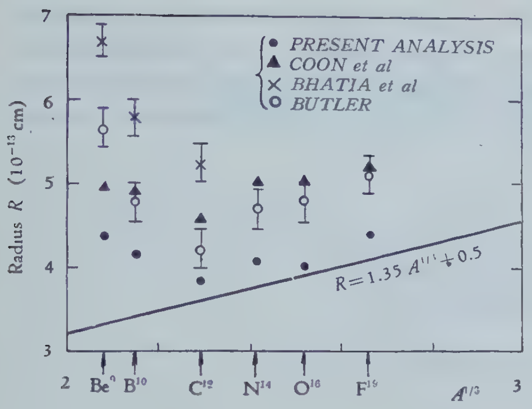

Fig. 10. Comparison of the values of nuclear radii for light nuclei taken from various sources of information.

The values of nuclear radii (or interaction ranges) obtained from various sources are plotted against $A^{1/3}$ for light nuclei. Crosses and open circles represent the values derived from the analysis of the deuteron stripping reaction in terms of the theories of Bhatia *et al.* and of Butler, respectively. Triangles represent the values derived by Coon *et al.* from the analysis of 15 Mev neutron total cross sections. Dots are the values obtained from the present analysis.

experimental energy dependence of $\sigma_t/\pi R^2$ is compared with the theoretical one for some of the light nuclei, (a) taking into account the fluctuation of R's for individual nuclei and (b) assuming $R=(0.5+1.35\,A^{1/3})\times 10^{-13}$ cm. The values of R's thus determined and the ones given by the above formula are indicated in upper part of each figure.

It may be interesting to compare the values of R's obtained in this way with those obtained from other sources. In Fig. 10 they are plotted against $A^{1/3}$ and are compared with the ones obtained from the deuteron stripping reactions[27] and the ones obtained from the 15 Mev neutron total cross sections[28] assuming the equation,

$$\sigma_t=2\pi R^2.$$

All these data show the similar trend of the fluctuations of R's: particularly large value of R for Be^9, a conspicuous minimum at C^{12}, small value of R for O^{16} and the gradual approach to the $R=aA^{1/3}+b$ curve. It should be noted, however, that the above formula for the total cross sections is inaccurate for the light nuclei, from the point of view of the cloudy crystal ball model, even in the case of the rather high incident energy of 15 Mev.

Finally, it may be concluded that the cloudy crystal ball model is applicable to the energy range, 3 to 14 Mev of neutrons, considered in the present paper. However, the square well shape of the potential, is essentially to be modified because of the fact that it fails to give the correct magnitude of the compound nucleus formation cross sections and that the fit of the theoretical $\sigma_t/\pi R^2$ *vs* $A^{1/3}$ curves to the experimental ones is not quite satisfactory in every detail. The same type of analysis as the present one using the modified shape of the potential may be desirable. In this connection it is of special interest to work the potential shape which is known to reproduce the angular distribution of the elastic scattering of nucleons such as the one presented by Woods and Saxon.[6]

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Charge Distribution of Pions produced in High Energy Nuclear Collisions

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The charge distribution of pions produced in a high energy collision of two nucleons is calculated by means of a statistical model including excited nucleons. The results are compared with experimental data. The qualitative agreement is obtained for the ratio of (single π^-) to ($\pi^-+\pi^0$) to ($\pi^++\pi^-$) in an n–p collision, but not for the ratio of (π^+) to (π^-) in a $p-$Be collision.

§ 1. Introduction

The application of the hypothesis of charge independence to nucleon-nucleon collisions has been discussed by several authors. Using his statistical method,[1] Fermi calculated the charge distribution of the produced pions by means of the statistical model,[2] in which only direct production of pions by the incident nucleons was considered. But Fermi's simple statistical theory[3] fails to explain the experimental data. Peaslee[4] and Belinfante[5] discussed independently the consequence of the hypothesis that the produced pions are the decay products of nucleons excited by the collision.

In this paper we apply Fermi's method to a statistical model by which we can treat direct production of excited nucleons as well as of pions.

We shall consider all possible processes creating pions. For example, the processes for production of two pions are

$$0 \quad n+n\rightarrow n+n+\pi+\pi,$$

$$\mathrm{I} \quad n+n\rightarrow n^*+n+\pi\rightarrow n+n+\pi+\pi,$$

$$\mathrm{II} \quad n+n\rightarrow n^*+n^*\rightarrow n+n+\pi+\pi,$$

where n and n^* denote a nucleon and an excited nucleon, respectively. π denotes a pion and the charge sign of these particles are omitted.

The process 0 represents direct production of two pions associated with emission of two nucleons. The process I represents direct production of a pion, a nucleon and an excited nucleon which disintegrates into a pion and a nucleon. The process II is direct production of two excited nucleons accompanied with no pions. The contribution of these processes to pion production is determined by following two factors: the statistical weight in momentum space and the statistical weight in isotopic spin space based on the hypothesis of charge independence.

§ 2. Isotopic spin space weights

We assume that the excited nucleon has isotopic spin 3/2. The statistical weights in isotopic spin space are obtained immediately from the well-known Clebsch-Gordan coefficients, as was shown by the several authors mentioned above.

Table I. The Clebsch-Gordan coefficients for a proton-proton system ($T=1$).

	no excited nucleon	one excited nucleon	two excited nucleons
(pp)	1	0	0
$(pp0)$	1/2	1/6	0
$(pn+)$	3/2	5/6	0
$(pp+-)$	6/5	42/60	1/5
$(pp00)$	2/5	14/60	8/45
$(pn+0)$	9/5	58/60	26/45
$(nn++)$	3/5	6/60	2/45
$(pp+0-)$	144/60	504/360	66/90
$(pp000)$	18/60	10/60	8/90
$(pn++-)$	90/30	648/360	114/90
$(pn+00)$	126/60	444/360	68/90
$(nn++0)$	72/60	144/360	14/90

Table II. The Clebsch-Gordan coefficients for a neutron-proton system. The resultant isotopic spin $T=1$.

	no excited nucleon	one excited nucleon	two excited nucleons
(pn)	1	0	0
$(pp-)$	1/2	1/6	0
$(nn+)$	1/2	1/6	0
$(pn0)$	1	2/3	0
$(pp0-)$	4/5	18/60	1/45
$(pn+-)$	9/5	68/60	41/45
$(pn00)$	3/5	16/60	2/45
$(nn+0)$	4/5	18/60	1/45
$(pp+--)$	1.2	72/120	20/90
$(pp00-)$	0.9	156/360	22/90
$(pn+0-)$	4.2	912/360	158/90
$(nn++-)$	1.2	72/120	20/90
$(nn+00)$	0.9	156/360	22/90
$(pn000)$	0.6	18/45	28/90

The results are given in Table I for p-p collisions which produce two nucleons and up to three pions. The first column gives the different types of particles which may appear in the final state ; the symbols p and n denote a proton and a neutron respectively. The sign symbols $+$, 0, and $-$, indicate positive, neutral and negative charged pions, respectively.

Table III. The Clebsch-Gordan coefficients for a neutron-proton system. The resultant isotopic spin $T=0$.

	no excited nucleon	one excited nucleon	two excited nucleons
(pn)	1	0	0
$(pp-)$	1/3	0	0
$(pn0)$	1/3	0	0
$(nn+)$	1/3	0	0
$(pp0-)$	1/3	1/9	1/9
$(pn+-)$	1	5/9	5/9
$(nn+0)$	1/3	1/9	1/9
$(pn00)$	1/3	2/9	2/9
$(pp+--)$	0.6	48/180	11/135
$(pp00-)$	0.4	32/180	9/135
$(pn+0-)$	1.8	184/180	93/135
$(pn000)$	0.2	16/180	2/135
$(nn+00)$	0.4	32/180	9/135
$(nn++-)$	0.6	48/180	11/135

Table II gives data on n-p collisions which take place in the state of resultant isotopic spin equal to one, and Table III for n-p collisions in the zero-spin state.

§ 3. Phase space weights and relative probabilities

We assume that the production of the excited nucleon and its decay are independent processes, and treat the excited nucleon as a particle having the rest mass $1.17\,Mc^2$ $(=Mc^2+0.17\,Mc^2)$. Mc^2 is the nucleon rest mass and $0.17\,Mc^2$ indicates the excitation energy of the nucleon.

Then Fermi's statistical method becomes applicable to our processes. Fermi's formula for n-pion production by nucleon-nucleon collisions are[1),6)]

$$S_0(n)=\frac{1}{n!}\left\{\frac{251}{\omega}(\omega-2)^3\right\}^n\Big/\frac{3}{2}\times\frac{5}{2}\times\cdots\times\frac{6n+1}{2},$$

(no excited nucleon) (1)

$$S_1(n)=2\times\frac{1}{(n-1)!}\times1.12\times\left\{\frac{251}{\omega}(\omega-2.17)^3\right\}^{n-1}\Big/\frac{3}{2}\times\frac{5}{2}\times\cdots\times\frac{6(n-1)+1}{2},$$

(one excited nucleon) (2)

$$S_2(n)=\frac{1}{(n-2)!}\times1.265\times\left\{\frac{251}{\omega}(\omega-2.34)^3\right\}^{n-2}\Big/\frac{3}{2}\times\frac{5}{2}\times\cdots\times\frac{6(n-2)+1}{2}, \quad (3)$$

(two excited nucleons)

where ω represents in units of Mc^2 the total energy of colliding particles in the center of gravity system. In these calculations the interaction volume of colliding particles was taken to be, following Fermi,

$$(2/\omega)\,(4/3)\,\pi R^3, \quad R=\hbar/\mu c=1.4\times 10^{-13}\text{cm}.$$

The relative probability for an event will be calculated in the following way: For p-p collisions, it is given by the linear combination of the corresponding phase space

Table IV. The statistical weights for multiple pion production by proton-proton collisions at the bombarding energies 1.25, 1.75, 2.2 and 3.0 Bev.

E Bev.	1.25	1.75	2.2	3.0
ω	2.58	2.78	2.94	3.22
pp	9.40	4.32	2.32	0.90
$pp0$	10.46	8.66	7.13	5.17
$pn+$	38.37	29.20	23.12	16.16
$pp+-$	10.66	13.93	14.71	14.99
$pp00$	4.96	5.14	5.19	5.08
$pn+0$	18.88	21.15	21.60	21.64
$nn++$	2.10	3.86	3.32	3.98
$pp+0-$	1.23	3.76	5.88	9.54
$pp000$	0.15	0.46	0.71	1.15
$pn++-$	1.97	5.89	8.86	9.85
$pn+00$	1.22	3.69	5.63	8.90
$nn++0$	0.29	0.93	1.51	2.65

Table V. The statistical weights for multiple pion production by neutron-proton collisions.

Bev.	1.25	1.75	2.2	3.0
ω	2.58	2.78	2.94	3.22
pn	14.42	6.65	3.63	1.41
$pp-$	9.93	9.21	8.02	6.11
$nn+$	9.93	9.21	8.02	6.11
$pn0$	18.92	15.13	12.45	9.18
$pp0-$	4.91	6.07	6.59	7.20
$pn+-$	25.93	26.52	26.50	25.96
$pn00$	7.31	8.07	8.26	8.27
$nn0+$	4.91	6.07	6.59	7.20
$pp+--$	0.37	1.22	2.03	3.63
$pp00-$	0.32	1.04	1.65	2.80
$pn+0-$	2.48	7.09	11.43	13.54
$pn000$	0.28	0.85	1.31	2.16
$nn+00$	0.32	1.04	1.65	2.80
$nn++-$	0.37	1.22	2.03	3.63

factors $S(n)$, with the coefficients listed in Table I. For example, the relative probability for the reaction $(pp)\rightarrow(pp+-)$ is given by

$$6/5\cdot S_0(2)+42/60\cdot S_1(2)+1/5\cdot S_2(2), \qquad (4)$$

apart from the normalization constant.

Table IV illustrates various possible products which may emerge from a proton-proton collision. The first line represents in units of Bev the kinetic energy of the bombarding particle in the laboratory system. The second line represents in units of Mc^2 the corresponding total energy in the center of gravity system. The following lines give the relative probabilities for the events listed in the first column.

For a neutron-proton collision, we assume that the interaction cross section is the same for both isotopic spin states, $T=0$ and $T=1$. Table V gives data on neutron-proton collisions.

In Table IV and V, each column is normalized to unity. If we assume,[2)] however, that the interaction cross sections are equal to each other for *p-p* and *n-p* collisions, then the data listed in the corresponding column in Table IV and V indicate the relative probabilities for *p-p* and *n-p* collisions, apart from the normalization constant 1/2. We will adopt this assumption.

§ 4. Discussion of the results

All the physically significant quantities can be calculated immediately from Table IV and V. Table VI gives the ratio of (single π^-) to $(\pi^-+\pi^0)$ to $(\pi^++\pi^-)$ in neutron-proton collisions at 1.25, 1.75, 2.2, and 3.0 Bev. The second line gives the calculated ratio, not containing 3-pion productions. The third line is the experimental data observed by Fowler et al.[7)]

Table VI. The ratio of (single π^-) to $(\pi^-+\pi^0)$ to $(\pi^++\pi^-)$ in *n-p* collisions at 1.25—3.0 Bev.

	1.25 Bev	1.75 Bev	2.2 Bev	3.0 Bev
Cal.	2.06 : 1.0 : 5.28	1.52 : 1.0 : 4.37	1.22 : 1.0 : 4.02	0.85 : 1.0 : 3.61
Exp.	(0.8 ± 0.3) : (1.0 ± 0.35) : (3.2 ± 0.7)			

Table VI shows the qualitative agreement between the calculation and the experimental data.

Table VII. The ratio of single- to double- to triple- pion events with 3-prongs in *n–p* collisions

	1.25 Bev	1.75 Bev	2.2 Bev	3.0 Bev
Cal.	35 : 108 : 11	27 : 97 : 30	22 : 91 : 41	16 : 86 : 52

Table VIII. The ratio 1-prong : 3-prong : 5-prong stars produced in *n–p* collisions.

	1.25 Bev	1.75 Bev	2.2 Bev	3.0 Bev
Cal.	75 : 79 : 0.6	67 : 85 : 2	61 : 90 : 3	56 : 92 : 6

Table VII gives the ratio of single- to double- to triple-pion events with 3-prongs in neutron-proton collisions. Table VIII gives the ratio of 1-prong to 3-prong to 5-prong stars in n-p collisions. In these tables, each column is normalized to 154, the observed number of 3-prong stars. The elastic scattering, which is observed as one-prong stars, is not taken into account. Table VII shows the large probability for 3-pion events with 3-prongs while 3-pion event with 3- or 5- prongs has not been observed.

Table IX. The ratio π^+/π^- produced from protons on beryllium.

	1.25 Bev	1.75 Bev	2.2 Bev	3.0 Bev
Cal.	2.10	1.82	1.71	1.76
Exp.[8]	~6		~1.8	

Table IX gives the ratio of positive to negative pions produced by the proton bombardment of beryllium. This table shows considerable descrepancies at low energies.

The discrepancies with the experiment may indicate that conservation of angular momentum, which we have disregarded, makes large contribution to pion production in nucleon collisions.

In the above calculation, we have neglected the possibilities of four pion production by nucleon collisions. The formulas (1)–(3) lead to the result that this probability for n-p collisions is about 0.0% at 1.25 Bev, 0.6% at 1.75 Bev, 2.8% at 2.2 Bev, and 9.1% at 3.0 Bev, if we set the isotopic spin space weights approximately equal to 10 for the resultant isotopic spin 1 and approximately equal to 5 for its spin 0. Consequently four pion production gives no contribution to pion production at energies up to 2.2 Bev.

§ 5. Conclusions

We have applied Fermi's statistical method to a statistical model including excited nucleons. The calculated ratio of (single π^-) to ($\pi^-+\pi^0$) to ($\pi^++\pi^-$) in n-p collisions is in rough agreement with the experiment. The calculated probability for 3-pion production is more than 30 times higher than the observed one. The calculated ratio π^+/π^- in p-Be collisions does not agree with the experiment.

Further agreement would be obtained by taking into account the conservation of angular momentum.

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Letters to the Editor

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$E2+M1$ Mixed Transitions in Heavy Odd A Nuclei

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February 22, 1957

Many experimental data accumulated in the field of nuclear spectroscopy show the existence of $E2+M1$ mixed nuclear transitions. And these mixing ratios are obtained from the experiment of the angular correlation, the measurement of the angular distribution, the internal conversion coefficient and from the conversion ratio etc. In most cases, the value of the mixing ratio obtained from one method coincide not always with the value of that obtained from the other method for the same nucleus. However, as we can find a certain appreciable variation between the mixing ratios for two nuclei differing in mass number, we can get some information on the nuclear structure and the transition mechanism from these data of the mixing ratio. The available data are tabulated in Table 1, which contains only the transitions decaying from the first excited state to the ground state in nuclei. $E2/M1$ means the intensity ratio. L, ang. cor., ang. dist. and pol. dir. respectively indicate L conversion ratio, the angular correlation, the angular distribution in nuclear h.f.s. alignment and the polarization-directional measurement. Other notations are to be understood as usual ones. Asterisks mean that the data are corrected for the finite nuclear size effect.13) 15) As the data are not sufficient and have some spread on each nucleus, it may be premature to draw a definite conclusion. But we believe it is worth noting the following several points.

1) The mixing ratio increases smoothly with increasing mass number in general.

2) The mixing ratio for the isotope having $(N+2)$ neutrons is always smaller than that for the isotope having N neutrons. Though the value for Tl^{205} is not always well established, Te, Au and Tl present good examples.

3) Mixing ratios for Te^{121}, Eu^{153}, Ir^{191}, and Tl^{203} are appreciably larger than that for the neighbouring nuclei.

We might consider the existence of the difference between the mixing ratios for two isotopes of Te, Gd, Ir or Au to be reliable from the fact that these data are obtained respectively from the same experimenters. According to the extreme single particle shell model assignment, more than half transitions listed in Table 1 should be obviously l-forbidden $M1$ transitions and the others should have the negligible order of $E2$ transition matrix element to $M1$ one. For l-forbidden $M1$ transitions, Jensen and Mayer19) pointed out the fact that the spin orbit coupling effect in the single particle model for proton produces non-

Table 1 $E2+M1$ mixed transitions in some heavy odd A nuclei

Nucleus	Energy (kev)	Spin, Parity	$E2/M1$	Method	Reference
${}_{50}\mathrm{Sn}_{67}^{117}$	162	3/2+—1/2+	0.0015	ang. cor.	(1)
${}_{52}\mathrm{Te}_{69}^{121}$	214	3/2+—1/2+	0.056	ang. cor.	(2)
			0.055	ang. cor.	(3)
${}_{52}\mathrm{Te}_{71}^{123}$	159	3/2+—1/2+	0.013	ang. cor.	(2)
			0.012	ang. cor.	(3)
			0.009	ang. cor.	(3)
${}_{57}\mathrm{La}_{82}^{139}$	166	5/2+—7/2+	0.04	ang. dist.	(4)
${}_{61}\mathrm{Pm}_{86}^{147}$	92	7/2+—5/2+	0.03	ang. dist.	(4)
${}_{63}\mathrm{Eu}_{90}^{153}$	84	3/2+—5/2+	2.5	K/L	(5)
${}_{64}\mathrm{Gd}_{91}^{155}$	60	(5/2) −3/2	0.052	K/L	(25)
${}_{61}\mathrm{Gd}_{93}^{157}$	55	(5/2) −3/2	0.042	K/L	(25)
${}_{67}\mathrm{Ho}_{98}^{165}$	96	5/2+—7/2+	0.09	K/L	(5)
${}_{71}\mathrm{Lu}_{104}^{175}$	114	9/2+—7/2+	0.05	K/L	(5)
	113.6		0.30	α_k L	(6)
	113.81		0.18	α_k αL	(7)
${}_{73}\mathrm{Ta}_{108}^{181}$	135	9/2+—7/2+	0.25	ang. cor.	(8)
	137		0.075	K/L	(9)
	136.25		0.111	$\alpha_K{}^*$ L^*	(10)
${}_{75}\mathrm{Re}_{112}^{187}$	134.2	7/2+—5/2+	0.111	K/L^*	(11) (14)
	135		0.12	K/L	(5)
${}_{77}\mathrm{Ir}_{114}^{191}$	129	5/2− 3/2	1.16	K/L	(5)
${}_{77}\mathrm{Ir}_{116}^{193}$	139	(5/2) −3/2	0.28	K/L	(5)
${}_{79}\mathrm{Au}_{116}^{195}$	61.4	1/2+—3/2+	0.20	L	(12)
${}_{79}\mathrm{Au}_{118}^{197}$	77.4	1/2+—3/2+	0.14	L	(12)
${}_{81}\mathrm{Tl}_{122}^{203}$	279	3/2+—1/2+	3.0	α_K α_L	(13)
			2.25	pol. dir.	(14)
			1.5	$(\alpha_K$ $K/L)^*$	(15)
			1.38	$\alpha_K{}^*$ $\alpha_{LI}{}^*$ $\alpha_{LII}{}^*$	(16)
${}_{81}\mathrm{Tl}_{124}^{205}$	205	1/2+	(0.27)	α_T	(17)

vanishing matrix element and using this assumption deWaard and Gerholm[20] explained the experimental half life of l-forbidden $M1$ transition for proton. However, the experimental half life for l-forbidden type odd neutron nuclei[21] also suggests the existence of the other important effects.[22] [23] From the standpoint of the collective model, it seems that the tendency in case 1) corresponds to that of the mixing ratios affected by the core deformation effect[23] due to the particle surface coupling scheme.[21] But it would be necessary to further our study about the quantitative consideration on this and the other two points theoretically as well as experimentally.

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On the Effective Coupling Strength in the Static Pion Theory

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It is the purpose of this note to reexamine the static theory of the pion-nucleon scattering in terms of the " effective coupling strength ". In this model, the scattering of pion is caused only by the flip of σ- and τ-spin of the nucleon. These spin operators never change their character even if they receive the reaction of the pion cloud around them, while the strength of the coupling of pion with these spins (which is a scalar quantity with respect to the rotation of these spin spaces) is changed as the result of the reaction of the pion cloud. This is the peculiarity of the static model.

In Chew and Low's theory[1] some part of reaction of the pion cloud is taken into account by introducing the renormalized coupling constant. But there remain some effects of the pion cloud on the nucleon which are not exhausted by the renormalization of coupling constant.

Let us take an example for definiteness. The Chew-Low equation for the charged scalar field is as follows :

$$h_\alpha(\omega)=\frac{\lambda_\alpha f^2}{\omega}+\frac{1}{\pi}\int_\mu^\infty k' v(k')^2 d\omega'$$
$$\times\left[\frac{|h_\alpha(\omega')|^2}{\omega'-\omega-i\epsilon}+\sum_\beta\frac{A_{\alpha\beta}|h_\beta(\omega')|^2}{\omega'+\omega}\right],$$

$$h_\alpha(\omega_q)=\frac{e^{i\delta_\alpha(q)}\sin\delta_\alpha(q)}{q v(q)^2},$$

$$\lambda_\alpha=\begin{bmatrix}+1\\-1\end{bmatrix},\quad A_{\alpha\beta}=\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

where $\alpha=1,\ 2$ stand for the states in which the isospin of the nucleon is parallel and anti-parallel to the isospin of the pion, respectively. Needless to say, α is an exact quantum number in this model.

One may raise a question, then, why two states with different quantum numbers are mixed with each other through the last term of the equation (the so-called crossing-term).

This equation is easily solved by observing the following property of the matrix $A_{\alpha\beta}$, i.e.,

$$A_{\alpha\beta}=\delta_{\alpha\beta}+\lambda_\alpha S_\beta,\quad S_\beta=(-1,\ 1).$$

Substituting this expression in the above equation, we can rewrite it as follows:

$$h_\alpha(\omega)=\frac{\lambda_\alpha f(\omega)^2}{\omega}+\frac{1}{\pi}\int_\mu^\infty k'v(k')^2d\omega'$$

$$\times\left[\frac{|h_\alpha(\omega')|^2}{\omega'-\omega-i\epsilon}+\frac{|h_\alpha(\omega')|^2}{\omega'+\omega}\right],$$

$$f(\omega)^2=f^2+\frac{\omega}{\pi}\int_\mu^\infty k'v(k')^2d\omega'$$

$$\times\sum_\beta\frac{S_\beta|h_\beta(\omega')|^2}{\omega'+\omega}.$$

The states with different quantum numbers are no more mixed with each other in this equation, while the coupling strength becomes dependent on energy. (We call it the effective coupling strength.) This energy dependence is due to that part of the reaction of the pion cloud which cannot be taken into account by the renormalization of *coupling constant.* Of course, the value of the effective coupling strength at $\omega=0$, $f(0)^2$, is equal to f^2; this is in accordance with the definition of the renormalized coupling constant. Such an introduction of the "effective coupling strength" and the consequent separation of the states with different quantum numbers are always possible for any kind of static models.

For example in the case of symmetrical scalar field, we are to put

$$\lambda_\alpha=\begin{bmatrix}-2\\1\end{bmatrix},\quad A_{\alpha\beta}=\begin{bmatrix}-1/3&4/3\\2/3&1/3\end{bmatrix},$$

instead of the previous definitions. Here α takes 1 or 3, which means twice of the total isospin of the system. Now, using the following S_α, $S_\alpha=(2/3,\ -2/3)$, we can rewrite $A_{\alpha\beta}$ in the same way as before. Thus we can similarly introduce the "effective coupling strength" in this case, too.

Further, in the case of the symmetrical pseudoscalar field, the Chew-Low equation is obtained by replacing $v(k)$ by $kv(k)$ in the above equation and using the following values for λ_α and $A_{\alpha\beta}$:

$$\lambda_\alpha=\frac{1}{3}\begin{bmatrix}-8\\-2\\4\end{bmatrix},$$

$$A_{\alpha\beta}=\frac{1}{9}\begin{bmatrix}1&-8&16\\-2&7&4\\4&4&1\end{bmatrix}.$$

Here α takes 1, 2 or 3, which means $(1/2,\ 1/2)-$, $(1/2,\ 3/2)-$ or $(3/2,\ 3/2)-$ state, respectively. (One of the figures in the parenthesis stands for the total isospin and the other the total angular momentum.)

In this cases, defining

$$S_\alpha=[1/3,\ 1/3,\ -2/3],$$

we can go on quite similarly.

It is to be noticed that the effective coupling strength gets smaller by the contribution of the states with attractive scattering phase shift and larger by that of the repulsive states.

Finally we shall the actual expression of the effective coupling strength for the case of Lee and Serber's example[2] (a charged scalar field without cutoff) :

$$\frac{f(\omega)^2}{f^2} \fallingdotseq 1-\frac{\omega\omega_0^2}{\pi}\left[\frac{\pi}{2}\frac{1}{\omega_0^2\omega}\right.$$

$$-\frac{2(\omega^2-1)^{1/2}}{\omega(\omega^2-\omega_0^2)}\tan^{-1}\sqrt{\frac{\omega-1}{\omega+1}}$$

$$-\frac{1}{\omega-\omega_0}\frac{\sqrt{1-\omega_0^2}}{\omega_0^2}\tan^{-1}\sqrt{\frac{1-\omega_0}{1+\omega_0}}$$

$$\left.-\frac{1}{\omega+\omega_0}\frac{\sqrt{1-\omega_0^2}}{\omega_0^2}\tan^{-1}\sqrt{\frac{1+\omega_0}{1-\omega_0}}\right],$$

$$\omega_0=2f^2/(1+f^1),$$

where the pion mass is taken equal to unity.

The numerical values of $f(\omega)^2/f^2$ for $f^2=0.134$ are 0.98, 0.99 and 0.93 at $\omega=1,5$ and ∞, respectively. In this example, $f(\omega)^2$ is weakly dependent on energies, but in some other cases it strongly depends on them.

In fact, $f(\omega)^2$ for symmetrical pseudoscalar field shows a strong energy dependence,

$$f(\omega)^2-f^2=-0.0104 \quad \text{at } \omega=1,$$

$$=-0.0337 \quad \text{at } \omega=\infty,$$

which are obtained by making use of the recent experimental values[3],

$$\delta_1\sim\delta_2\sim 0$$

and $$\tan\delta_3\sim 0.134\,p^3/\omega(2.1-\omega).$$

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2) Castillejo, Dalitz and Dyson ; Phys. Rev. **101** (1956), 453.
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Numerical Discussions on *P*-wave Meson-Nucleon Scattering

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April 30, 1957

Previously, Fujino and one of us (J. O.)[1] investigated the Chew-Low equation[2] by the iteration method, gave the Feynman-graphical interpretation to so-called " one-meson approximation " and presented a recipe for direct calculation of the renormalized values of arbitrary Feynman diagrams in the static model.

Using this method, we examined numerically all diagrams of meson-nucleon scattering to the sixth order in coupling constant and obtained some interesting results. (We took $\omega_0=6\mu$, where ω_0 is the cut-off energy and μ is the meson mass.) Here, we report the main results briefly.

(i) Among the sixth order diagrams, those which are corrected only in propagators (Figs. *c*, *d* and *e*) are of specially large values (10～100 times larger than the others), as Chew[3] anticipated previously. Moreover, for this type of diagrams, the perturbation method is quite unsuitable even for the fairly small usual value of coupling constant ($f^2=0.08$), and some different methods must be used.

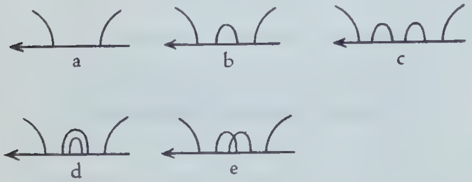

Fig. 1

(ii) If the renormalized coupling constant is not sufficiently small, one-meson approximation is not a good one. Only when the sixth order terms are far less important than the lower-order terms, that approximation is effective.

(iii) The sum of the values of all sixth-order non-crossing diagrams (the diagrams in which the initial meson line and the final one are not crossing) are not very large compared with that of all sixth-order crossing diagrams. As Wyld4) discussed, the non-crossing diagrams of some order are calculated in the Tamm-Dancoff approximation of the order lower by f^2, but the crossing ones are not. Therefore, the T—D approximation of even the fourth-order is not suitable in this case.

The numerical results are as follows: At the threshold.

$$\delta_1=\eta^3\{-3f^2/[1+5.73f^2+(3.64+8.40^*)f^4]+0.333f^2-0.18f^4+(-36.0+40.4^*)f^6\},$$

$$\delta_2=\eta^3\{-0.667f^2+0.43f^4+(17.5+14.4^*)f^6\},$$

$$\delta_3=\eta^3\{1.333f^2+6.81f^4+(42.3+39.0^*)f^6\};$$

at $\omega=2\mu$ (corresponding to 171 Mev meson kinetic energy in Lab. system),

$$\tan\delta_1=\eta^3_{\omega=2}\{-3f^2/[2+30.0f^2+(50.0+36.9^*)f^4]+0.167f^2-1.92f^4+(-61.4+36.3^*)f^6\},$$

$$\tan\delta_2=\eta^3_{\omega=2}\{-0.333f^2+0.76f^4+(10.7+13.4^*)f^6\},$$

$$\tan\delta_3=\eta^3_{\omega=2}\{0.667f^2+7.25f^4+(100.6+40.9^*)f^6\}.$$

These were obtained by calculating all diagrams to the sixth order by means of the method of reference 1. As mentioned above, we cannot use the perturbation method for the diagrams like Figs. *b*–*e*. Inserting Chew's closed form[illegible]) of the propagator corrected to the fourth order into the simplest non-crossing diagram (Fig. *a*), we took this type of diagrams into account. In the equations, the figures with the star are due to "two-meson states" and do not appear in one-meson approximation. $\eta=p/\mu$, p is the meson momentum in C. M. system and δ_1, δ_2 and δ_3 is the phase shift in (1/2, 1/2)—state, (1/2, 3/2)— or (3/2, 1/2)—state, (3/2, 3/2)—state, respectively. (The first number in brackets means the value of isospin and the second the value of angular momentum.)

If we take $f^2=0.08$, $\delta_1\sim-4.2°\,\eta^3$, $\delta_2\sim-1.9°\,\eta^3$ and $\delta_3\sim11.0°\,\eta^3$ at the threshold and $\delta_1\sim-17°$, $\delta_2\sim-3°$ and $\delta_3\sim43°$ at $\omega=2\mu$. The ratios of the sum of all crossing diagrams to that of non-crossing ones in the sixth order are -0.52, 2.53 and 0.07 at the threshold and -0.15, 2.02 and 0.07 at $\omega=2\mu$ respectively for three states. δ_2 is as small as expected from experiments, but

δ_1 is not so small. Also this result tells us very instructively that the perturbation method is quite unsuitable for such a resonating phase shifts δ_3 even in the case of fairly small coupling constant.

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Stationary Fusion Reactions at Low Temperatures

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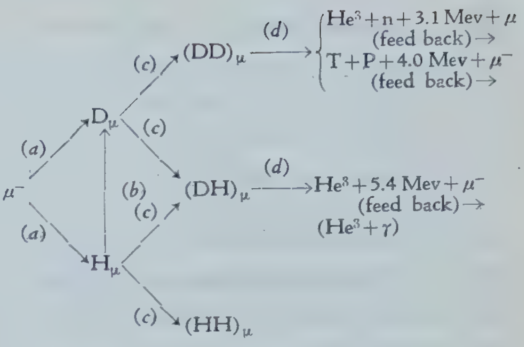
Recently Alvarez et al.[1] have reported a successful experiment to fuse a proton and a deuteron in a bubble-chamber making use of μ^--meson as a catalizer. Here we inquire about the relationship between the stationary energy out-put and the concentration of deuterons in liquid hydrogen, considering a sufficiently large number of μ^--mesons supplied in a sufficient number of atoms, H and D, in the liquid hydrogen. The dissipation of the mesons comes on the one hand from the natural decay, and on the other hand from the formation of the mesic molecular ions H_2^+. Fig. 1 shows the mechanism of the reactions.



μ^-
(a)
(a)
D_μ
H_μ
(b)
(c)
(c)
(c)
(c)
$(DD)_\mu$
$(DH)_\mu$
$(HH)_\mu$
(d)
(d)
$He^3+n+3.1\,Mev+\mu$ (feed back)→
$T+P+4.0\,Mev+\mu^-$ (feed back)→
$He^3+5.4\,Mev+\mu^-$ (feed back)→
$(He^3+\gamma)$

Fig. 1

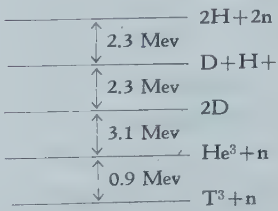


Fig. 2

Denoting the number densities of the elements which take part in the above reactions by $n(\mu)$, $n(H)$, ⋯ respectively, we have, for example,

$$\begin{aligned} n(\mu) = & -p(\mu H)\,n(\mu) - p(\mu D)\,n(\mu) \\ & + p((DH)_\mu)\,n((DH)_\mu) \\ & + p((DD)_\mu)\,n((DD)_\mu) - \lambda n(\mu) + \alpha, \end{aligned} \tag{1}$$

similar to the above expression for other processes. Here, for example, $\langle \sigma(\mu H)\, g(\mu H)\rangle_{AV} n(H)$ is written briefly as $p(\mu H)$ where $\sigma(\mu H)$ implies the capture cross section of μ^- by H, and $g(\mu H)$ stands for the relative velocity of μ^- to H, while α is the number of the mesons supplied per second to sustain the stationary state. On solving the simultaneous equations for the stationary state, we can set the left-hand side of (1) to vanish. In fact, if we put

$$n(\mathrm{H})=(1-x)n,$$

$$n(\mathrm{D})=xn,$$

$$p(\mu\mathrm{D})=p'(\mu\mathrm{D})x,$$

$$p(\mu\mathrm{H})=p'(\mu\mathrm{H})(1-x),$$

where x stands for the concentration of D and n for the number density of hydrogen atoms, the number of occurrence of the $d-h$ and $d-d$ reactions per unit time are given respectively by

$$\left.\begin{aligned}
&p((\mathrm{DH})_\mu)n((\mathrm{DH})_\mu)\\
&\quad=\frac{p((\mathrm{DH})_\mu)}{p((\mathrm{DH})_\mu)+\lambda}\\
&\quad\cdot\frac{(1-x)x(Ex+F)}{(Ax+B)(Cx+D)}\cdot n(\mu),\\
&p((\mathrm{DD})_\mu)n((\mathrm{DD})_\mu)\\
&\quad=\frac{p((\mathrm{DD})_\mu)}{p((\mathrm{DD})_\mu)+\lambda}\\
&\quad\cdot\frac{x^2(E'x+F')}{(Ax+B)(Cx+D)}\cdot n(\mu)
\end{aligned}\right.$$

where

$$\begin{aligned}
A&\equiv p'(\mathrm{H}_\mu\mathrm{D})+p'(\{\mathrm{DH}_\mu\})\\
&\quad-p'(\mathrm{H}_\mu\mathrm{H}),\\
B&\equiv p'(\mathrm{H}_\mu\mathrm{H})+\lambda,\\
C&\equiv p'(\mathrm{D}_\mu\mathrm{D})-p'(\mathrm{D}_\mu\mathrm{H}),\\
D&\equiv p'(\mathrm{D}_\mu\mathrm{H})+\lambda,\\
E&\equiv p'(\mathrm{D}_\mu\mathrm{H})[Ap'(\mu\mathrm{D})\\
&\quad-p'(\mu\mathrm{H})p'(\{\mathrm{DH}_\mu\})]\\
&\quad+p'(\mathrm{H}_\mu\mathrm{D})p'(\mu\mathrm{H})C,\\
F&\equiv Bp'(\mu\mathrm{D})p'(\mathrm{D}_\mu\mathrm{H})\\
&\quad+p'(\mathrm{D}_\mu\mathrm{H})p'(\mu\mathrm{H})p'(\{\mathrm{DH}_\mu\})\\
&\quad+p'(\mathrm{H}_\mu\mathrm{D})p'(\mu\mathrm{H})+D,\\
E'&\equiv p'(\mathrm{DD}_\mu)[p'(\mu\mathrm{D})A\\
&\quad-p'(\mu\mathrm{H})p'(\{\mathrm{DH}_\mu\})],\\
F'&\equiv p'(\mathrm{DD}_\mu)[p'(\mu\mathrm{H})p'(\{\mathrm{DH}_\mu\})\\
&\quad+p'(\mu\mathrm{D})B].
\end{aligned}\qquad(2)$$

From (1) the number of the mesons supplied should be

$$\begin{aligned}
\alpha&=p(\mathrm{H}_\mu\mathrm{H})n(\mathrm{H}_\mu)+\lambda[n(\mu)+n(\mathrm{H}_\mu)\\
&\quad+n(\mathrm{D}_\mu)+n((\mathrm{DH})_\mu)+n((\mathrm{DD})_\mu)].
\end{aligned}\qquad(3)$$

Then the ratio (2)/(3) plotted in Fig. 3 shows the number of occurrence of the $d-h$ and $d-d$ reactions per unit time and per meson supplied. The numerical values of p''s used for plotting Fig. 3 are listed in Table 1.

Table 1

(a)	$p'(\mu\mathrm{D})=1.1\times10^{12}$, $p'(\mu\mathrm{H})=1.3\times10^{12}$
(b)	$p'(\{\mathrm{DH}_\mu\})=1.3\times10^{9}$
(c)	$p'(\mathrm{D}_\mu\mathrm{D})=1.1\times10^{5}$, $p'(\mathrm{D}_\mu\mathrm{H})=1.7\times10^{5}=p'(\mathrm{DH}_\mu)$, $p'(\mathrm{H}_\mu\mathrm{H})=2.2\times10^{5}$
(d)	$p'((\mathrm{DH})_\mu)\simeq10^{6}\sim10^{8}$, $p'((\mathrm{DD})_\mu)\simeq10^{8}$

Now let us note briefly the way to obtain the values in the Table. Process (a). A slowing-down process has been investigated by Wightman[2)] and Wheeler.[3)] Process (b). As pointed out by Yamaguchi,[4)] the increase in the phase volume of the relative motion of D_μ and H, after a collision of H_μ and D, favours to exchange μ^- between H and D. The cross section for the exchange reaction is thus given by σ (exchange) $=(E'/E)^{1/2}\,\sigma$ (elastic) where σ's imply the cross sections of the collision

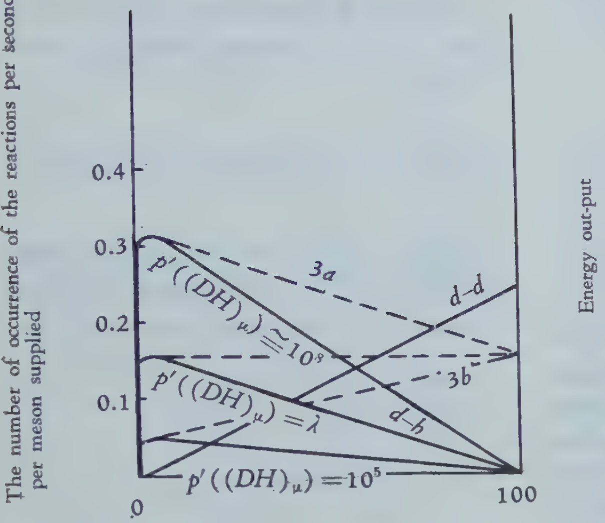


Fig. 3. D-concentration

between H_μ and D, and E and E' stand for the translational energies of D relative to H before and after the reaction, respectively. Process (c). The formations of the mesic molecular ions occur by means of the Frank-Condon principle. We have calculated the cross section for the molecular formation: $\sigma=2\pi^2\lambda\!\!\!^{-2}\Gamma/D$, where $\lambda\!\!\!^{-}$ is the de Broglie wave length of the relative motions of the colliding mesic atoms and protons or deuterons, and Γ is the level width of the $2p$-state of the molecular ion, and D is its level spacing.[5] Process (d). As for the transition probability of the nuclear reaction, $(DH)_\mu \rightarrow He^3+5.4$ Mev$+\mu$, the $0-0$ transition probability[5] is applicable. No exact evaluation of $p'((DH)_\mu)$ has been obtained for very low energies. Some resonating capture might occur in this region. The value of $p'((DH)_\mu)$ of order 10^5 will be the lower bound.

The energy out-put shown by the dotted curves in Fig. 3 is obtained from the mean values of the numbers of occurrences of the two nuclear reactions weighted by the respective Q-values 5.4 Mev and 3.6 Mev. The curve $3a$ corresponds to the case $p'((DD)_\mu)\cong p'((DH)_\mu)\gg\lambda$ and the curve $3b$ corresponds to $p'((DH)_\mu)=1.0\times10^5$. The former has a maximum at a comparatively low concentration, while the latter reaches maximum at the 100% concentration.

At the present stage, the lack of our knowledge about the $d-d$ reaction at extremely low energies and the rough estimations of the values of p's prevent us to go further beyond qualitative conclusion mentioned above. We should expect the more exact data concerning those processes, especially the molecular formation and the $d-d$ reaction at low energies.

The authors thank Professor Husimi for calling their attention to this problem. They are much indebted to Professor Yamaguchi for his suggestions.

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Some Remarks on the Goldhaber Model for Strange Particles

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In the efforts of accounting for the meaning of the "strangeness" quantum number, many authors have proposed various ideas. One direction in these researches, e.g. theory of d'Espagnat and Prentki,[1] lies in that the strangeness conservation is due to some transformation properties of the space in which the behaviours of elementary particles are described. In the other type of treatments elementary particles are considered as the objects composed of a few of more fundamental components, i.e., they have some internal structures. In Goldhaber's theory,[2] these basic components are nucleons and $\bar{\vartheta}$-mesons, and the state of resultant isotopic spin $I=1$ corresponds to Σ-particles and $I=0$ to a Λ^0-particle. If these components should preserve their individual characteristics even in bound states, as is the case in atomic nuclei, we might see a Pais-Piccioni type effect[3),4)] in the time variation of the relative intensity of Λ^0 because of the different lifetimes of $\bar{\vartheta}^0$ and ϑ^--mesons. Now the charge function of Λ^0- and Σ^0-particle is $\psi(\Lambda^0)=(1/\sqrt{2})[\phi_1(p,\ \vartheta^-)-\phi_2(n,\ \bar{\vartheta}^0)]$ and $\psi(\Sigma^0)=(1/\sqrt{2})[\phi_1(p,\ \vartheta^-)+\phi_2(n,\ \bar{\vartheta}^0)]$, respectively. We write also the wave function of the system at time t after the production of Λ^0-particles as $\Psi(t)=\alpha_1(t)\phi_1+\alpha_2(t)\phi_2$. It changes with time according to

$$\left.\frac{d\Psi(t)}{dt}\right|_{\text{phase}}=-i\omega_1\alpha_1(t)\phi_1-i\omega_2\alpha_2(t)\phi_2,$$

where ω_1 and ω_2 are the phase changes which result from the change of kinetic energies of $\bar{\vartheta}$-mesons in bound states, and

$$\left.\frac{d\Psi(t)}{dt}\right|_{\substack{\text{decay \&}\\ \text{creation}}}=-\frac{1}{2\tau_1}\alpha_1\phi_1$$
$$-\frac{1}{2\tau_2}\alpha_2\phi_2-\frac{1}{2T_2}\alpha_2\phi_2$$
$$\times[n+\bar{\vartheta}^0\rightarrow\Lambda^0+\pi^0]$$
$$-\frac{1}{2T}\frac{\alpha_1+\alpha_2}{\sqrt{2}}\psi(\Sigma^0)[\Sigma^0\rightarrow\Lambda+\gamma].$$

In the above equation, τ_1, τ_2 and T is the mean decay lifetime of bounded ϑ^-, $\bar{\vartheta}^0$-meson and of Σ^0-particle, respectively. The third term expresses the reaction induced by $\bar{\vartheta}^0$-mesons in the bubble chamber and its lifetime of absorption is T_2. The recontribution to Λ^0-amplitude from the third and the fourth term is not considered here, although it is anticipated that these contributions would be appreciable. And moreover we must take into consideration the process

$$\left.\frac{d\Psi(t)}{dt}\right|_{\text{transition}}=-\frac{1}{2T_1}\alpha_1\phi_1$$
$$\times[p+\vartheta^-\rightarrow n+\bar{\vartheta}^0].$$

Comparing each coefficient of ϕ_1 and ϕ_2, we obtain following system of equations

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$$\left\{\begin{aligned} \frac{d\alpha_1(t)}{dt} &= -\lambda_1{}^0\alpha_1(t) - \frac{1}{4T}\alpha_2(t), \\ \frac{d\alpha_2(t)}{dt} &= -\frac{1}{4T}\alpha_1(t) - \lambda_2{}^0\sigma_2(t) \ ; \end{aligned}\right.$$

$$\lambda^0{}_i \equiv i\omega_i + \frac{1}{2\tau_i} + \frac{1}{2T_i} + \frac{1}{4T} \ (i=1,2).$$

As a result, for the relative intensity of Λ^0-particles we have (with the initial condition $\alpha_2(t=0)=-\alpha_1(t=0)$)

$$\frac{I(t)}{I(0)} = \frac{|\Lambda^0\text{—component in } \Psi(t)|^2}{|\Lambda^0\text{—component in } \Psi(0)|^2}$$

$$= \frac{e^{-(1+\beta+B_1+B_2+B)t/2}}{4[(1+R_1{}^2-R_2{}^2)^2+4R_1{}^2R_2{}^2]}$$

$$\times\Big[\{[(1-R_1)^2-R_2{}^2]^2$$

$$+4R_2{}^2(1-R_1)^2\}e^{-\delta_1 t/\tau_2}$$

$$+\{[(1+R_1)^2-R_2{}^2]^2$$

$$+4R_2{}^2(1+R_1)^2\}e^{+\delta_1 t/\tau_2}$$

$$+2\{[(1-R_1)^2-R_2{}^2][(1+R_1)^2-R_2{}^2]$$

$$-4R_2{}^2(1-R_1{}^2)\}\cos(\delta_2 t/\tau_2)$$

$$-4\{R_2(1-R_1)[(1+R_1)^2-R_2{}^2]$$

$$+R_2(1+R_1)[(1-R_1)^2-R_2{}^2]\}$$

$$\times\sin(\delta_2 t/\tau_2)\Big];$$

$$R=(1/2T)[(\lambda_1{}^0-\lambda_2{}^0)$$

$$+\sqrt{(\lambda_1{}^0-\lambda_2{}^0)^2+(1/4T)^2}]^{-1}=R_1+R_2,$$

$$\delta_1+i\delta_2$$

$$=\frac{1}{2}\sqrt{(2i\gamma+\beta-1+B_1+B_2)^2+B^2},$$

$$\beta=\tau_2/\tau, \quad B_1=\tau_2/T_1,$$

$$B_2=\tau_2/T_2, \quad B=\tau_2/T,$$

$$\gamma=\tau_2\Delta\omega=\tau_2(\omega_1-\omega_2).$$

In the case of $B \gg 1$ ($T\sim10^{-20}$ sec[5]), $\delta_1\sim B/2$, $\delta_2=0$, $R_1=0$, $R_2=0$,

$$I(t)/I(0)\sim e^{-(1+\beta+B_1+B_2)t/2\tau_2}$$

$$\sim e^{-t/(1.2\tau_2)}.$$

Here we take $B_1\sim B_2$[6]~0.3 (for Xenon[4]) and values of free particles for the ratio β, but it is very plausible, as in the β-disintegration of nuclei, that the lifetimes in bound states differ appreciably from those in free states.

The author wishes to express his thanks to Dr. Y. Ohnuki for his illuminating discussions and to Prof. S. Sakata for his interest in this work.

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The Indirect (*p*, *d*) Process

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In 300 Mev-proton bombardment on

various target, Hess and Moyer[1)] observed high energy deuterons ejected at large angles where direct pick-up deuterons were improbable. For these deuterons, the dependence of the cross section on the mass number of the target nucleus suggests an Indirect Pick-up Process, induced by secondary nucleons which are produced by one or more collisions of the primary proton in the same nucleus. With some assumptions about the pick-up probability of the secondary nucleons, the indirect process hypothesis is shown by the above authors to give results consistent with the experimental data on energy spectrum shape and angular dependence of the deuterons. In the present note, we shall calculate the effective pick-up probability considering the attenuation effects of produced deuterons and have obtained the absolute magnitude of indirect pick-up cross sections.

Along the direction of secondary nucleon beam, the effective pick-up probability is given by,

$$P_{\text{eff}}=\int_0^d \frac{dx}{l_p} e^{-(d-x)/l_d} \qquad (1)$$

where l_p is the mean free path of a secondary nucleon allowing for the forward pick-up process only and is calculated by the well-known direct pick-up theory, l_d the mean free path of produced deuteron which corresponds to thickness of pick-up layer, d the path length of a secondary nucleon travelling through the nucleus, being approximately of the order of nuclear radius. For $d/l_d \gg 1$, we have

$$P_{\text{eff}}=l_d/l_p. \qquad (2)$$

In view of the strongly forward ejection of directly picked-up deuterons, we can make use of the total cross section σ_t for calculating the pick-up mean free path l_p in Eq. (2), which approximate procedure is justified when the energy spectrum of the secondary nucleons is slowly dependent on the scattering angle in shape and magnitude. The error in our approximation in the final results is estimated to be smaller than 10%. By the use of the cross section of the direct pick-up process given by Chew and Goldberger,[2)] we have

$$\left.\begin{aligned}
\sigma_t &= \frac{24\pi A^2(\beta^2-\alpha^2)^2}{a^2k^2}\cdot\left(\frac{\sqrt{4\pi}}{a}\right)^3 \\
&\quad\times[f(t_-)-f(t_+)] \\
&\quad\times\exp\left\{\frac{1}{a^2}\left(2\beta^2+k^2-\frac{K^2}{2}\right)\right\}, \\
f(t_\pm) &= \frac{e^{-t_\pm}}{t_\pm}-\int_{t_\pm}^{\infty}\frac{e^{-t}}{t}dt, \\
t_\pm &= \frac{2}{a^2}\left[\beta^2+\left(k\pm\frac{K}{2}\right)^2\right], \\
\frac{\hbar^2a^2}{2m} &= 16\ \text{Mev}
\end{aligned}\right\} \qquad (3)$$

where we have assumed 16 Mev-Gaussian for the momentum distribution of a picked-up nucleon, which has been successful in high energy processes.[3)] The wave numbers k and K of the secondary nucleons and the produced deuteron within the nucleus should be increased, corresponding to the increase of kinetic energy in nuclear matter, i. e. 30 Mev. Furthermore, the energy conservation law has been evaluated in the average by the Fermi Gas Model.

The deuteron produced inside the nucleus will be scattered or disintegrated before it can escape from the nuclear surface. As our knowledge of these cross sections is scanty, we shall assume that the deuteron disappears from its own direction, if either of the constituents of deuteron collides with the nucleons of the nucleus. Then, assuming the Fermi Gas Model for the nucleus in addition to the free nucleon scattering data,[4] we can get the attenuation mean free path of the deuteron, l_d.

The numerical values of P_{eff} are plotted in Fig. 1. Using the obtained P_{eff} and the observed spectra of the secondary nucleons, we have finally calculated the indirect pick-up cross sections in a similar way to that of Hess and Moyer. The calculated energy spectra at three angles for a carbon target are shown in Fig. 2 and seem to be consistent with the experimental data. The absolute magnitudes of differential cross sections for various targets are listed in Table I. The agreement with experimental values is also satisfactory in general. For heavier elements, however, the observed values are slightly larger than the calculated ones. This behaviour may be interpreted as a neutron excess effect in the nuclear surface layer. Considering the approximate character of the theory, we shall not discuss the effect in more detail. Furthermore, an indirect effect on the (p, d) process induced by a 95 Mev-proton can be estimated. Using the energy spectra of scattered protons,[5] the indirect cross sections in carbon are estimated to become at least 10% and 30% of measured (p, d) cross sections,[6] at 30° and 42°, respectively. Such an effect may reduce the higher momentum components in the momentum distribution of target nucleons obtained by Selove.[6] The authors wish to thank Prof. T. Muto for his encouragement and valuable discussions.

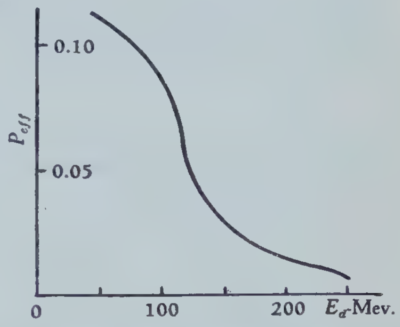

Fig. 1. P_{eff}; plotted in produced deuteron energy.

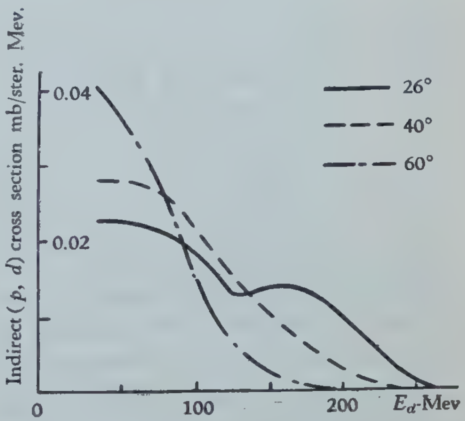

Fig. 2. Deuteron energy spectra; carbon target.

Table. I. Differential cross sections, $E_\alpha \geqq 50$ Mev, (mb/ster).

element	angle	theoretical	experimental
C	26°	2.5	1.72±0.41
	40°	2.5	1.90±0.35
	60°	1.8	1.42±0.30
Al	26°	4.2	3.54±0.70
	40°	4.2	4.69±0.48
Cu	26°	6.8	6.44±1.32
	40°	7.2	7.86±1.04
Cd	40°	10.	11.22±1.20
Pb	26°	7.0	13.9 ±2.86
	40°	12.	18.0 ±2.14
U	40°	13.	16.80±1.68

1) Hess and Moyer, Phys. Rev. **101** (1956), 337.
2) Chew and Goldberger, Phys. Rev. **77** (1950), 470.
3) Brueckner, Eden and Francis, Phys. Rev. **98** (1955), 1445.
4) Chen, Leavitt and Shapiro, Phys. Rev. **103** (1956), 211.
5) Strauch and Titus, Phys. Rev. **103** (1956), 200.
6) Selove, Phys. Rev. **101** (1956), 231.

Magnetic Dipole Sum Rules for Photodisintegration of the Deuteron*

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May 18, 1957

In a recent paper, Levinger[1] has studied the photodisintegration of the deuteron and among other matters has given the value of the bremsstrahlung weighted cross section $(\sigma_b = \int (\sigma/w)\, dw)$ for magnetic dipole transitions using the effective range theory matrix elements. We shall in this note re-derive Levinger's result for central forces from first principles; and show that it does not change on introducing tensor forces.

For magnetic dipole transitions in a deuteron[2] neglecting the contribution of exchange moments,

$$\sigma_b = \frac{2\pi^2 e^2 \hbar}{Mc} \sum_n \frac{f_{on}}{(E_n - E_0)}$$

$$= \frac{2\pi^2 e^2 \hbar}{Mc} \frac{1}{2Mc^2\hbar^2} \sum_n |(\boldsymbol{\nu} \cdot \boldsymbol{M})_{on}|^2 \quad (1)$$

where $\boldsymbol{\nu}$ is the unit vector normal to the direction of polarization and to the direction of propagation of the photon. The vector $\boldsymbol{M}$ is the magnetic moment operator of the deuteron given by

$$\boldsymbol{M} = \hbar \left[\frac{1}{2} \boldsymbol{L} + (\mu_p + \mu_n) \boldsymbol{S} + \frac{1}{2} (\mu_p - \mu_n)(\sigma_p - \sigma_n) \right]. \quad (2)$$

Only the last term contributes in transition ${}^3S_1 \to {}^1S_0$. The spin matrix element, when averaged over all initial states of polarization of the deuteron, gives $4/3$ and the space part gives $\sum_n (1)_{on} (1)_{no} = (1)_{oo} = 1$. Combining all these results, we get

$$\sigma_b = \left(\frac{\pi^2 e^2}{3\hbar c} \right) \left(\frac{\hbar}{Mc} \right)^2 (\mu_p - \mu_n)^2,$$

which agrees with Levinger.[1]

On including the 'S' and 'D' states both, in the ground state wave function for the deuteron, it is found that the matrix elements for all other transitions vanish except for two namely: ${}^3S_1 \to {}^1S_0$ and ${}^3D_1 \to {}^1D_2$. Since the 'D' transition is not large in itself and since it does not interfere with the 'S' transition in the total cross section, we expect the D contribution to be fairly small. On carrying out the calculation in a way outlined above, it is found that is not changed by introducing tensor forces.[2]

We have also calculated $\sigma_{\text{int}} = \int \sigma dw$ for M–1 transitions for central spin dependent forces using Hulthen wave function. For E–1 transitions attractive forces that do not commute with position (Majorana and Heisenberg) increase σ_{int}. For M–1 transitions we find[3] an analogous increase in σ_{int} due to Heisenberg and

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* Supported by Research Corporation.

Bartlett forces that do not commute with the magnetic moment operator $\boldsymbol{M}$:

$$\sigma_{\text{int}} = \int \sigma dw = \frac{\pi^2 e^2 \hbar}{Mc} \cdot \frac{(y+z)}{2Mc}$$
$$\times \{[[v(r)p^H, (\boldsymbol{\nu}\cdot(\mu_p\sigma_p+\mu_n\sigma_n))],$$
$$(\boldsymbol{\nu}\cdot(\mu_p\sigma_p+\mu_n\sigma_n))]\}, \tag{4}$$

where y and z are fractions of Heisenberg and Bartlett forces respectively. On carrying out the calculations, we find

$$\sigma_{\text{int}} = \int \sigma dw = \frac{2\pi^2}{3}\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{Mc}\right)^2$$
$$\times (y+z)(\mu_p-\mu_n)^2 \frac{\hbar^2}{M}$$
$$\times \frac{2\gamma}{(1-\gamma r_0)}(\beta^2-\gamma^2)$$
$$\times \left[\frac{1}{(\gamma+\beta)} - \frac{1}{2\beta}\right], \tag{5}$$

where we have used $\gamma = (M\epsilon/\hbar^2)^{1/2}$ and $\beta = 5.83\gamma$ as determined from the effective range theory using $r_0(-\epsilon, -\epsilon) = 1.79 \times 10^{-13}$ cm. Using $y+z=0.2$ (Inglis or Rosenfeld mixture) we get $\sigma_{\text{int}} = 1.55$ Mev-mb which is in fairly good agreement with an approximate calculation of Levinger in which he obtains 1 Mev-mb.

1) J. S. Levinger, **97** (1955), 970.
2) R. G. Sachs "Nuclear Theory" Addison-Wesley Publishing Co., Inc. Cambridge (1953).
3) M. L. Rustgi, Doctoral Dissertation, Jan. 1957, Louisiana State University.

Addenda

On the Asymptotic Conditions in Quantum Field Theory

Kazuhiko Nishijima

Prog. Theor. Phys. **17** (1957), 765

(1) The Theorem IV was proved in the text assuming the *PT* invariance of the theory, where *PT* is the product of space reflection and time reversal. This theorem holds more generally if one only assumes the *CPT* invariance of the theory. As is well known, this invariance holds for all known theories that are invariant against proper Lorentz transformations. (Lüders-Pauli theorem)

(2) In the text the formal solution of the equation (6·10) was given by (6·18), the first term of this is, however, not generally symmetric in the variables $y, x_1, x_2, \cdots, x_n$, and in order to get a symmetric special solution it is necessary to take further terms in (6. 13) than the first term in expressing the function $h(u, v)$.

The author is indebted to Dr. W. Zimmermann on these points.

ERRATA

Quantum Field Theory of Unstable Particles

Huzihiro Araki, Yasuo Munakata, Masaaki Kawaguchi and Tetsuo Gotô

Prog. Theor. Phys. **17** (1957), 419

Page	line	Original	Corrected
423	11	(2.1)	(2.12)
434	3	....../2π,	....../$[2\pi(m_s-E_k-i\gamma_s/2)]$
434	7	The term......drop it.	to be omitted
434	15	$[k_{m_s}R-im_sR(v_N+v_0)^{-1}{}_{m_s}]$	$[k_{m_s}R-m_sR(v_N+v_\theta)^{-1}{}_{m_s}]$
434	26	......$\|I\rangle 2\pi$.	......$\|I\rangle/2\pi$.
435	28	(4.22)	(4.33)

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Inelastic Scattering of Nucleons by Light Nuclei

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(Received April 18, 1957)

Inelastic scattering of nucleons by light nuclei is analysed by adopting the direct interaction model. The interaction between an incident nucleon and the target nucleus is taken into account by using the distorted wave method. As the interaction which causes inelastic excitation of the target nucleus, the two-body interaction between the incident nucleon and the nucleons which are most loosely bound to the target nucleus is assumed. Calculations are made for C^{12} and Mg^{24} nuclei as examples. It is shown that the results are very sensitive to the parameters that are involved in our calculation. Although the fine fit between the calculated results and the experimental data is not intended here, but some general features of them are compared in some detail. Some unknown factors which have not been taken into our consideration are also discussed.

§ 1. Introduction

In the last few years, experimental data on inelastic scattering of proton which excites the target nucleus to a definite level have gradually been accumulated. In many cases, the angular distribution of scattered proton shows 90° asymmetry and is appreciably variant with slight change of the incident energy of proton. Measurements of the absolute cross section are also being performed but with larger errors so far.

The compound nucleus theory[1] with the statistical assumption cannot explain these general features of experimental data, but gives rather flat and exactly 90° symmetric angular distribution.

Many authors[2)~10)] are now attempting to explain these data by other models (mainly by the direct interaction model). Particularly, calculations using the distorted wave method[3] have been performed by many workers. Among them Hayakawa and Yoshida[4] and Brink[5] assumed interactions between the incident nucleon and surface oscillation of the target nucleus. Lately, Yoshida[6] has published a more exact treatment of this approximation.

The authors (with T. Sasakawa)[7] studied the angular distribution of inelastically scattered protons from the first excited state of Fe^{56}, assuming the two-body interaction between an incident nucleon and particles which are most loosely bound to the target nucleus. Agreement of the theory with the experimental data was fairly well. Important parameters contained in this model are the size of Hartree potential (assumed as square well) acting on the incident and emerging nucleons, and the strength and shape of the two-body interaction. In the course of investigation, however, they found that the results were very sensitive to the parameters involved in the formulation. Therefore, the sole calculation which is performed for a fixed incident energy will not give a significant conclusion by itself. More extensive comparison of general features of the theory with those of the experimental data should be performed over a wide range. Parameters which are involved in the theory need to be examined in their general effects. In this paper, we shall consider about this point in some detail.

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Feshbach and Lamarch[8] have also calculated inelastic scattering of protons from C^{12} nucleus. Their results about the angular distribution agree with ours in its general features, but the absolute cross section is somewhat different. That is caused by the different choice of the two-body interactions between an incident nucleon and nucleons in the target nucleus: They assumed the delta-type interaction between the incident neutron and the neutrons in the last shell, but did not consider the interaction with protons. Moreover, they represented the contribution from the whole neutron shells by that of only one neutron. While we assume a sum of the Yukawa-type interactions and each nucleon (proton or neutron) in the last shell of the target nucleus can be excited. Although the difference in the shape of interactions, does not allow the exact comparison of the strengths of two interactions a qualitative agreement is obtained by dividing our result by the number of nucleons in the last shell of the target nucleus.

From their standpoint that the "direct process" model may not explain the rapid change of the experimental angular distribution, Eisberg and Hintz[9] proposed an attempt to ascribe these data to possible correlations in the phases of levels excited in the compound nucleus.

Other attempts have been proposed to analyse experimental data taking into account the effects of the compound nucleus formation process as well as the effects of the direct process.[10]

Section 2 will be devoted to the general considerations and our standpoint will be explained. In section 3, the explicit form of the interaction will be given, and formula for the differential cross section are derived. Our calculation is, so to speak, based upon the perturbation theory, so that the strength of the two-body interaction which is the cause of inelastic scattering should not be too large. This will be examined in section 4 by comparing the calculated cross section with the experimental absolute cross section. It seems that the order of magnitude of the interaction is permissibly small for the perturbation expansion, but the exact justification of its validity is not performed. Results of the calculation will be discussed and compared with experimental data of C^{12} and Mg^{24}.

In a comparison of the theory with the experiments, we should note that dependence of the angular distribution on parameters is too sensitive to draw any quantitative conclusions. Since available experimental data are not sufficiently accumulated, it is too early to aim at perfect agreement of the theory with particular experimental data. We endeavor in this paper to discuss the general features, but not detailed features of inelastic scattering. Finally, some remarks will be given in section 5.

§ 2. General considerations

Some modes of nuclear reactions (for example, (p, p'), (α, α'), (d, p) etc.) show irreguralities in their energy dependences of cross sections which may be considered as the "resonance" phenomena. An energy distance of adjacent "resonances" is usually large. Although the experimental angular distribution of emerging particles shows some change with a slight variation of the incident energy, the change itself is not so abrupt as to be attributed to the contribution of each level with the same weight, when the reaction goes through the compound nucleus in which great many levels are excited. That is, the interferences between these states would make the angular distribution change more rapidly than is seen in experiments. On the other hand, this rapid variation, possibly due to the compound nucleus effect may be flattened by some experimental conditions, for example, by the energy spread in the incident energy.

According to this consideration, we may reasonably think that there is some "gross structure". It may be said that for the case of inelastic scattering there is a kind of the "gross structure" which is analogous to the "gross structure"[11),12)] in elastic scattering. As we confine ourselves to discuss such a "gross structure", it is reasonable to consider that some few levels of the compound nucleus contribute more strongly than others to give rise to the "gross behavior" of inelastic scattering. With only these strongly contributing levels, the gross behavior will approximately be described and contributions of other levels may be smeared out or become small resonance effects which will be only a correction for the "gross structure".

In view of the success of the optical potential model in the study of elastic scattering and of the shell model in the study of nuclear structure, it seems to be a fairly good approximation to describe the "gross structure" of inelastic scattering in terms of the single particle picture. Then, in the case of inelastic scattering of protons, an incident proton collides with a target nucleus to form a compound system but the proton still maintains its particle nature considerably even in the compound system, namely, the interaction between the nucleus and the proton must be rather weak.

If the interaction is strong inside the nucleus and the mean free path of the incident nucleon is much shorter than the nuclear radius, the above mentioned statement could not be true. But in this case, it is also possible to consider the contribution from outside of the nucleus as the "direct" process while the contribution from the inside of the nucleus as the compound formation process, and the interference effect between them should be taken into account.[13)] However the contribution from the outside which is calculated with the two-body interaction of reasonable strength always amounts only to $2\sim5\%$ of the experimental value (see § 4). Therefore, this model will not give such a remarkable 90° asymmetry as is seen in the experiment.

In this paper, we assume that the single particle nature of an incident nucleon is still maintained even in the inside of the compound system. There are two methods to analyze the experimental data by this assumption: One is to make use of the many level formula in which contributions of a few levels are distinguishably stressed with its single particle character.11) Another one is the analysis along the direct interaction model in which the incident particle is assumed to interact directly with the most loosely bound nucleons in the target nucleus. The incident nucleon excites the target nucleus and is scattered inelastically with a different kinetic energy which is lower than the incident energy just by the amount of the excitation energy of the target nucleus.

The cross section of the transition from a initial state, a, to a final state, b, is given by

$$\sigma_{ab}=(2\pi/\hbar)|T_{ab}|^2\rho_b v_a^{-1}. \tag{1}$$

ρ_b is density of the final state b, v_a represents velocity of an incident particle in the initial state a. If one represents total interactions between the target nucleus with A nucleons and the incident nucleon 0 by

$$\sum_{i=1}^{A} V(i,\ 0)\rightarrow U+V,$$

where U is a function only of the incident (or scattered) nucleon coordinate and will be treated exactly, V is the residual part of the interaction and is assumed to be so small that it will be treated as a perturbation. Then, T_{ab} can be written as follows :

$$\begin{aligned}
T_{ab}&=\langle\phi_b|U+V|\Psi_a^{(+)}\rangle=\langle\phi_b|U|\Phi_a^{(+)}\rangle+\langle\Phi_b^{(-)}|V|\Psi_a^{(+)}\rangle\\
&=\langle\phi_b|U|\Phi_a^{(+)}\rangle & \text{(2–a)}\\
&+\langle\Phi_b^{(-)}|V|\Phi_a^{(+)}\rangle & \text{(2–b)}\\
&+\langle\Phi_b^{(-)}|V\cdot(E_a-H+i\eta)^{-1}\cdot V|\Phi_a^{(+)}\rangle. & \text{(2–c)}
\end{aligned}$$

Here, $\Psi^{(+)}$ is an outgoing function of the total system, $\Phi^{(\pm)}$ is an outgoing or incoming wave function distorted by U times a wave function of the target nucleus, and ϕ is a solution of the Schrödinger equation without an interaction, and is a plane wave function times a wave function of the target nucleus.

Division of the interaction $\sum_{i=1}^{A} v(i,\ 0)$ into U and V is somewhat arbitrary. One may define U, for the moment, as

$$\langle G_n(1,\ \cdots,\ A)|\sum_{i=1}^{A} V(i,\ 0)|G_n(1,\ \cdots,\ A)\rangle\rightarrow U, \tag{3}$$

where G_n $(n=0,\ 1,\ \cdots)$ are eigenfunctions of the target nucleus. Then V have only non-diagonal matrix elements with respect to the states of the target nucleus. We further assume, for the sake of simplicity, that

$$\langle G_0|\sum_{i=1}^{A} v(i,\ 0)|G_0\rangle\simeq\langle G_1|\sum_{i=1}^{A} v(i,\ 0)|G_1\rangle, \tag{3'}$$

where 0 and 1 mean the ground and first excited states of the target nucleus.

For inelastic scattering, (2–a) vanishes and (2–b, c) contribute appreciably. If the interaction V is sufficiently small, (2–b) would be the main term of the " gross structure ".

(2–c) does not vanish for elastic scattering as well as for inelastic scattering. It would give rise to a kind of "fine structure" in the both cases. In the region where the incident particle is sufficiently apart from the target nucleus, the interaction V may vanish. One can divide the configuration space into two regions: the outer region in which V vanishes and the inner region. One can define in the inner region a complete set of orthogonal functions which are solutions of the Schrödinger equation for the total Hamiltonian with suitable boundary conditions given on the surface of the boundary[12]:

$$HX_\lambda = E_\lambda X_\lambda .$$

Using this set of functions one can rewrite (2–c) as

$$(2\text{–c}) = \sum_\lambda \frac{\langle \Phi_b^{(-)}|V|X_\lambda\rangle\langle X_\lambda|V|\Phi_a^{(+)}\rangle}{E_a - E_\lambda + i\eta}.$$

The dependence of the numerator on the incident energy is given through distorted wave functions $\Phi_a^{(+)}$ and $\Phi_b^{(-)}$. When the energy of the incident particle is in the immediate vicinity of its resonance on the distorting potential U, $\Phi^{(\pm)}$ will have a large amplitude.* The numerator of (2–c) does not show so sensitive energy dependence except the "single particle resonances". If the numerator of (2–c) is sufficiently small, the contribution of (2–c) will appear only as sharp isolated resonances which take place when $E_a = E_\lambda$. As the density of compound levels E_λ is very large for such high excitation of about 15∼30 Mev, (2–c) may be averaged over the energy spread of the incident energy E_a. By this operation, the contribution of (2–c) becomes a smooth function of the energy and may be represented by a potential function. This potential may contain an imaginary part because of the tangential behavior of (2–c). For the case of elastic scattering, this potential together with the potential U may become the "optical potential".

In the case of inelastic scattering, the additional complex term which comes from (2–c) may be appreciably different from that of elastic case. But the quantitative discussion is very difficult because of the scanty knowledge of the compound system, hence it is not performed in this paper. Some quantitative discussions on this point have been given by Yoshida.[6] We shall examine the contribution from (2–c) in its general tendencies using U_{complex} instead of U_{real}. We think, at the present stage, it is important to analyze inelastic data phenomenologically assuming U_{complex} as a flexible parameter.

A precise form of the interaction V cannot be known any more. We may assume V as a sum of the two-body interaction between the incident particle and the most loosely bound nucleons in the target nucleus,

$$V = \sum_{i=1}^{\mu} v(i,\ 0),$$

where μ is the number of nucleons which are bound in the last shell of the target nucleus. If V is a large quantity, the single praticle description of inelastic scattering will lose a part of its significance and the compound nucleus model will be important. If the interaction V is sufficiently small, general features of the "gross structure" in inelastic scattering will be clarified by studying simply the general tendency of the first term of the distorted Born approximation.

---

* The phase shift of the resonating wave is $(2n+1)\pi/2$, its amplitude at the nuclear surface is proportional to $n_l(kR)$ of eq. (8), so that it is very large for light nuclei with small nuclear radius and for high angular momentum wave.

Because of our poor knowledge of the distorting potential and very little accumulation of experimental data, one is obliged to compare the results of the calculation with experimental data only in its general features. In other words, we should analyze what features could be explained by our treatment, or how the calculated results depend on the parameters involved in our calculation. Only after these examinations, we can proceed to aim at a perfect agreement of the theory with experimental data.

§ 3. Formulation

We assume that the interaction V which is the direct origin of inelastic scattering process is the sum of two-body interactions between the incident particle and nucleons which are most loosely bound to the target nucleus:

$$V(0, 1, \cdots, \mu) = \sum_{i=1}^{\mu} v(0, i).$$

For the sake of simplicity, we assume $v(0, i)$ is the charge independent and non-exchange type two-body interaction. In the following calculation, we take it to be the delta-type interaction or the Yukawa-type interaction with the range of the Compton wave length of meson, $1.4\,y$*:

$$v(0, i) = v \exp(-a|\boldsymbol{r}_0 - \boldsymbol{r}_i|)/a|\boldsymbol{r}_0 - \boldsymbol{r}_i|. \tag{4}$$

The range of the interaction will be an important parameter and its effect will be examined in § 4, ii). The strength of the interaction V may be expressed by the constant v (Mev) in (4) and we will estimate it from the experimental data comparing with the calculated result.

To avoid unnecessary complication, we examine inelastic scattering in which the target nucleus in its ground state with spin 0 (such as the even-even nucleus) is excited to its first excited state of spin 2. For the sake of simplicity, we neglect the Coulomb interaction throughout the calculation. Therefore, our results for the low incident energy $E_p \leq 5$ Mev are not correct, but the qualitative results may not be changed by inclusion of the Coulomb effect.

We adopt the single particle shell model for the target nucleus, and assume the configuration of the target nucleus with spin $0+$ as

$$\text{ground state}: \quad (\textit{Closed shell})\,(j_1^{\mu}(0+))\,(j_1'^{\nu}(0+)),$$

and the first excited state of the nucleus with spin $2+$ as

---

* We use a unit of length, the Yukawa (1 *Yukawa* = $1\,y = 10^{-13}$ cm).

$$\text{first excited state}: \quad (\textit{Closed shell})\,\{\alpha\,(j_1^{\mu-1}j_2(2+))\,(j_1'^{\nu}(0+)) + \beta\,(j_1^{\mu}(0+))\,(j_1'^{\nu-1}j_2'(2+))\}.$$

μ and ν respectively are the numbers of protons and neutrons in the last shell of the target nucleus. In some light nuclei, e. g. C^{12} and Mg^{24}, in which protons and neutrons have the same shell structure, if we make a simplifying assumption that the energies of above two excited configurations (which are represented by the first and second terms of the above equation) are equal, we can naturally take $|\alpha|=|\beta|=1/\sqrt{2}$. But, as the matrix element between the ground state and $\alpha=-\beta=1/\sqrt{2}$ state vanishes, we take $\alpha=\beta=1/\sqrt{2}$ for the first excited state.

An incident particle with the wave vector $\boldsymbol{k}$ collides with the target nucleus and excite the nucleus to its first excited state of spin 2, then it emerges with the wave vector $\boldsymbol{k}'$. The initial state wave function is

$$|\text{Target}\rangle|\boldsymbol{k}^{(+)}\rangle,$$

here, $|\text{Target}\rangle$ is the wave function of the target nucleus and is

$$|\text{Target}\rangle=|\text{Closed shell}\rangle|j_1^{\mu}(0)\cdot j_1'^{\nu}(0)\rangle,$$

$|\boldsymbol{k}^{(-)}\rangle$ is the wave function of the incident proton distorted by the potential U and its asymptotic form is

$$r\cdot|\boldsymbol{k}^{(+)}\rangle\rightarrow r\exp(ikz)+f(\theta)\exp(ikr)\quad(r\rightarrow\infty).$$

Following the notation in the text of Blatt and Weisskopf,[15)]

$$r\cdot|\boldsymbol{k}^{(+)}\rangle=\begin{cases}\pi^{1/2}\sum\limits_{l}i^{l+1}(2l+1)^{1/2}k^{-1}[u_l^{(-)}(r)-\eta_l u_l^{(+)}(r)]Y_{l,0}(\theta)\chi_{m_s}\\ \quad=(4\pi)^{1/2}\sum\limits_{l}i^{l}(2l+1)^{1/2}e^{i\delta_l}r[\cos\delta_l\cdot j_l(kr)-\sin\delta_l\cdot n_l(kr)]Y_{l,0}(\theta)\chi_{m_s}\\ \qquad\qquad\text{for } r\geq R,\end{cases}$$

$$=(4\pi)^{1/2}\sum i^{l}(2l+1)^{1/2}e^{i\delta_l}rA_l j_l(Kr)Y_{l,0}(\theta)\chi_{m_s}\quad\text{for } r\leq R. \qquad (5)$$

Here, $Y_{l,0}(\theta)$ is the spherical harmonics, χ_{m_s} the spin wave function of the incident particle, $K^2=k^2+(2m/\hbar^2)\cdot U(1+i\zeta)$, and $\eta_l=e^{2i\delta_l}$.

The wave function of the final state is

$$|\text{Residual}\rangle|\boldsymbol{k}'\rangle.$$

$|\text{Residual}\rangle$ is the wave function of the residual nucleus and is

$$|\text{Residual}\rangle=|\text{Closed shell}\rangle\cdot\{\alpha|(j_1^{\mu-1}j_2(2+))\,(j_1'^{\nu}(0+))\rangle + \beta|(j_1^{\mu}(0+))\,(j_1'^{\nu-1}j_2'(2+))\rangle\}.$$

(Where we set $\mu=\nu$ and $\alpha=\beta=1/\sqrt{2}$.) The wave function of the emerging nucleon is

$$r|\boldsymbol{k}'^{(-)}\rangle=\begin{cases}2\pi\sum_l i^{l+1}k'^{-1}[u_l^{(+)}(r)-\eta_l'^{*}u_l^{(-)}(r)]Y_{l,m}^{*}(\Theta,\Phi)\,Y_{lm}(\theta\phi)\chi_{m_s'}\\ \quad=4\pi\sum_l i^l e^{i\delta_l'^{*}}r[\cos\delta_l'^{*}\cdot j_l(k'r)-\sin\delta_l'^{*}\cdot n_l(k'r)]Y_{l,m}^{*}(\Theta,\Phi)\,Y_{l,m}(\theta,\phi)\chi_{m_s'}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } r\geq R,\end{cases}$$

$$=4\pi\sum_l i^l e^{i\delta_l'^{*}}r\,B_l^{*}j_l^{*}(K'r)\,Y_{l,m}^{*}(\Theta,\Phi)\,Y_{l,m}(\theta,\phi)\chi_{m_s'}\quad\text{for } r\leq R,$$

$$K'^2=k'^2+\left(\frac{2m}{\hbar^2}\right)U(1+i\zeta),\quad \eta_l'=e^{2i\delta_l'}.$$

After the substitution of above expressions in (2–b) and the summation over the z-component of the spin of the total system is performed, the matrix element of transition becomes

$$\begin{aligned}|T_{ab}|^2&=\sum_M|\langle\Phi_b^{(-)}|V|\Phi_a^{(+)}\rangle|^2\\ &=(2\mu)\,\{(l_1l_2 0 0|l_1l_2 2 0)\,W(l_1j_1l_2j_2;\tfrac{1}{2}2)\}^2(2j_1+1)\,(2l_1+1)\,(2l_2+1)\\ &\times\sum_L P_L(\cos\Theta)\cdot\Big[\sum_{\substack{l>l''\\ l'>l'''}}(-)^{l'+l'''-L}(ll''00|ll''L0)\,(l'l'''00|l'l'''L0)\,W(ll'l''l''';2L)\\ &\qquad\times 2\,\mathcal{R}e\{e^{i(\delta_l+\delta'_{l'}-\delta^{*}_{l''}-\delta'^{*}_{l'''})}\mathcal{R}_{l'l}\mathcal{R}^{*}_{l'''l''}\}\\ &\qquad+\sum_{l.l'}(-)^{l+l'-L}(ll00|llL0)\,(l'l'00|l'l'L0)\,W(ll'll';2L)\\ &\qquad\times e^{-2Im(\delta_l+\delta_{l'})}|\mathcal{R}_{l'l}|^2\Big],\end{aligned}\tag{7–a}$$

where

$$\mathcal{R}_{l'l}=(1/\sqrt{5})\cdot i^{l+l'}(2l+1)\,(2l'+1)\,(ll'00|ll'20)\,R_{l'l},\tag{7–b}$$

$$R_{l'l}=(kk')^{-1}\int_0^\infty dr_1\int_0^\infty dr_2r_2^2v_2(r_1,r_2)f_{l'}'^{*}(r_1)\,f_l(r_1)\,g_{l_2}(\alpha'r_2)\,g_{l_1}(\alpha r_2),\tag{7–c}$$

f_l is the radial part of wave function of the l-th partial wave which is distorted by the potential U and its form is

$$f_l(r)=\begin{cases}kr[\cos\delta_l\cdot j_l(kr)-\sin\delta_l\cdot n_l(kr)] & \text{for } r\geq R,\\ kr\,A_lj_l(Kr) & \text{for } r\leq R,\end{cases}\tag{8}$$

$f_l'^{*}(r)$ is obtained by changing $k\to k'$, $K\to K'$, $\delta_l\to\delta_l'$ and $A_l\to B_l$ in $f_l(r)$. g_{l_1} and g_{l_2} are the radial part of the bound state wave functions of nucleons in the ground state and that in the first excited state, respectively, and they are normalized as $\int_0^\infty g_l^2(\alpha r)r^2dr=1$. They are determined to give the observed binding energies of last nucleons. $v_2(r_1,r_2)$ is the coefficient of the $Y_{2,m}(\theta,\phi)$ in the expansion of the interaction $v(\boldsymbol{r}_1-\boldsymbol{r}_2)$

$$v(\boldsymbol{r}_1-\boldsymbol{r}_2)=\sum_{l,m}v_l(r_1,r_2)\,Y_{l,m}^{*}(\theta_1,\phi_1)\,Y_{l,m}(\theta_2,\phi_2).$$

From (1), the differential cross section is given by

$$\sigma_{ab}(\theta)=(4\pi)^{-2}(2M/\hbar^2)^2(k'/k)\,|T_{ab}|^2.$$

§ 4. Results of evaluation and comparison with experiment

We calculate the angular distribution of inelastically scattered protons from C^{12} and Mg^{24} nucleus as example. The reason for our taking up such nuclei is as follows: First, in a light nucleus the spacings of a few lower excited levels of nucleus are comparatively large, so that the single particle description of nucleus may be valid in good approximation, especially in C^{12}. Secondly, we can proceed with the partial wave analysis with high accuracy for such light nuclei because impinging waves with high angular momentum are not affected by the target nucleus. Thirdly, for these nuclei, experimental measurements16)~24) have been performed with some different incident energies of protons. In the present analysis, we have paid particular attention to the energy dependence of the cross section and the variation of angular distribution.

We adopt a distorting potential U for C^{12} nucleus as follows,

$$U(r)=\begin{cases}-45(1+i\zeta)\ \text{Mev} & r<R\\ 0 & r>R\end{cases}\qquad R=1.45\times A^{1/3}y, \tag{9}$$

where ζ runs from 0 to 0.2, but the real part of this potential is fixed. It gives a single particle resonance at the incident energy of 10 Mev for the incident wave with angular momentum $l=3$. In this case we take $j_1=j_1'=3/2$, $j_2=j_2'=1/2$, $l_1=l_2=1$, $l_1'=l_2'=1$ in the expression (7–a).

The effect of the variation of the real potential is examined for Mg^{24} nucleus by adopting a pure real potential of

$$U(r)=\begin{cases}-U & r<R\\ 0 & r>R\end{cases}\qquad R=1.35\times A^{1/3}y,\ U=30\sim50\ \text{Mev}. \tag{9'}$$

Although, the imaginary part of the distorting potential is not added in this case, the result for C^{12} nucleus permit us to derive similar qualitative conclusion for Mg^{24}.

i) *Energy dependence of the total cross section*

The total cross section for inelastic scattering is given by 4π times the coefficient of $P_0(\cos\theta)$ in (7–a). We examine its energy dependence for Mg^{24} using the real distorting potential. The results are shown in Fig. 1. A peak at $E_p=10$ Mev for $U=30$ Mev is the effect of the single particle resonance of the incident and scattered protons. In this case, $l=3$ partial wave is resonated. When the depth of the distorting potential increases gradually, the peak due to the single particle resonance of $l=3$ wave becomes broader and shifts toward the low energy side. The peak due to the resonance of $l=4$ wave which lies in higher energy region moves also toward the low energy side. In this graph, the energy dependence of the total cross section calculated with 50 Mev distorting potential is also shown. The peak at 13 Mev is due to the single particle resonance of $l=4$. The peak is sharper than the case of $l=3$ partial wave resonance.

The resonance peak of the higher angular momentum wave is always more striking and sharper than the lower wave resonance.

If we adopt the distorting potential of depth 40 Mev, no such remarkable peak of the cross section does appear in the energy range from 5 Mev to 20 Mev.

Experiments on Mg^{24} nucleus are performed with several incident energies. Inelastic data of 9 Mev protons show somewhat unusual large cross section. As to this point, some question may arise as to the estimation of the absolute value of cross sections because the occasional diversity is possibly expected in each estimation by respective authors. However, as the absolute cross sections obtained for other nucleus also show such steep energy dependence, it may be expected that the energy dependence mentioned above is due to a single particle resonance of the incident or scattered particles.

The strong resonance peak which results from the real distorting potential will be much influenced by addition of the imaginary part of the potential which is thought to be the energy average of the contribution of the second term in the perturbation expansion. This is done for C^{12} nucleus. The results are shown in Fig. 2. As the imaginary part increases, the resonance peak that is given by the real part of the distorting potential becomes broader and weaker preserving the general tendency in its energy dependence.

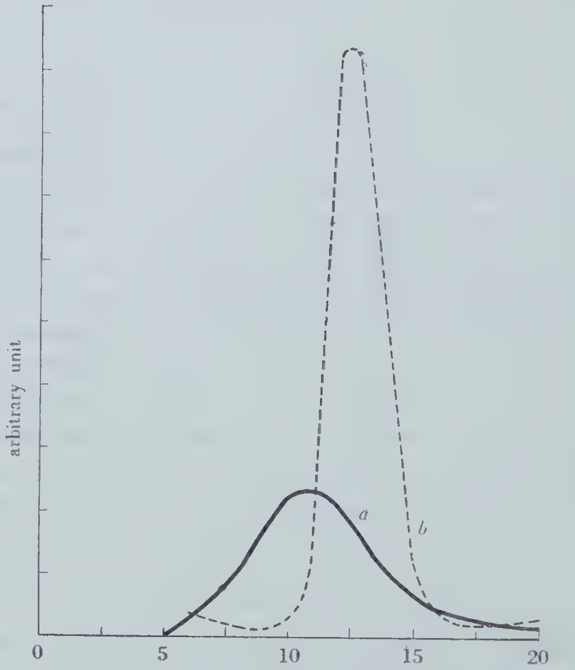


Fig. 1. Energy dependence of the total cross section for inelastically scattered protons from Mg^{24}. A square well distorting potential is assumed. For interaction between an incident nucleon and the target nucleus, the delta-type interaction is assumed.
Curve a: calculated with a real distorting potential $U=30$ Mev, $R=1.35\,A^{1/3}\,y$.
Curve b: a real distorting potential $U=50$ Mev, $R=1.35\,A^{1/3}\,y$. The peak of curve a at $E_p=11$ Mev is caused by single particle resonance of $l=3$ wave. It moves toward the low energy side as the depth of the potential increases. The peak of curve b is the result of $l=4$ wave resonance.

In the case of C^{12}, the experimental cross section may be rather a smooth function of the energy, and then we may consider that this is the consequence of the broad single particle resonance, but from the present data we cannot draw any decisive arguments.

The results for C^{12} which are shown in Fig. 2 are normalized to the experimental value by adjusting the strength of the two-body interaction v. v is estimated to be

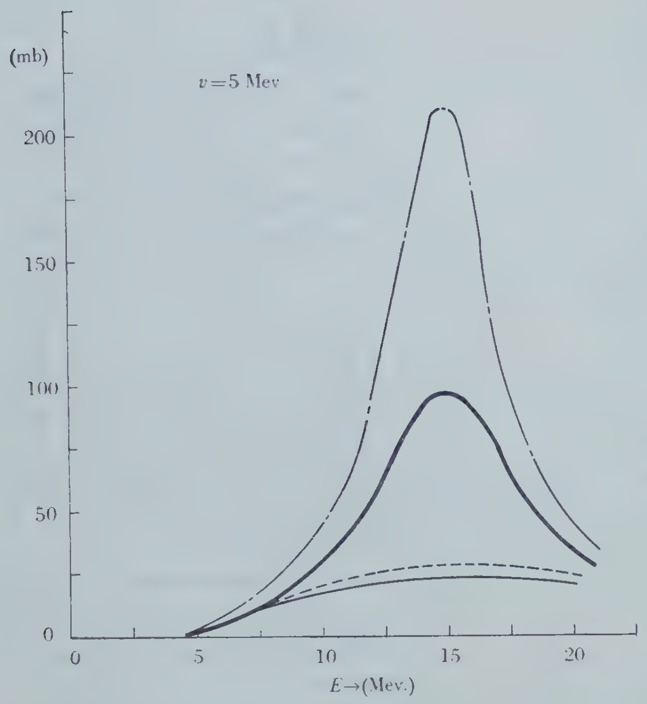


Fig. 2. Energy dependence of the total cross section for inelastically scattered protons from C^{12}. A complex potential (8) is assumed for distorting potential. For interaction between an incident nucleon and the target nucleus, the Yukawa-type interaction is assumed.
——·—— Calculated with a real distorting potential $\zeta=0$
▬▬▬ Calculated with a complex distorting potential $\zeta=0.03$
- - - - - Estimated with a complex potential $\zeta=0.1$
———— $\zeta=0.2$
The peak of curve *a* is broadened by adding the imaginary potential in the distorting potential. Strength of interaction is normalized to the experimental data at $E_p=15$ Mev for $\zeta=0.03$

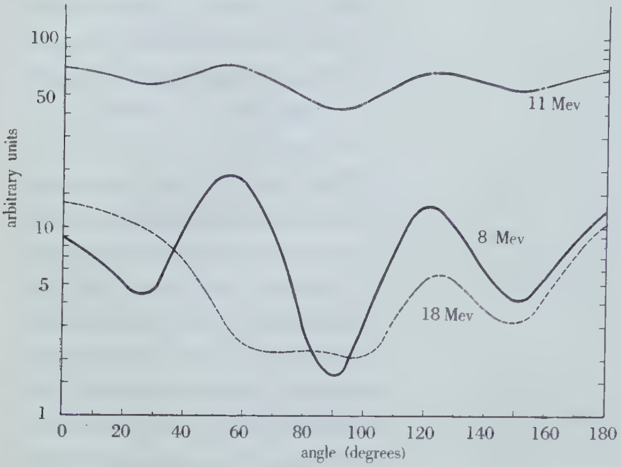
about 10 Mev. This is small in comparison with the strength of the two-body potential which is necessary to explain the scattering data in the free space. This fact is just compatible with our treatment of perturbation expansion, because our treatment is based on the anticipation that the considerable part of two-body potential in the free space should be included in the distorting potential, leaving only small fraction as the two-body interaction in nuclear reactions.

ii) *Angular distribution*

The angular distribution for protons calculated with the real distorting potential depends on the depth of the potential quite strikingly. Even if the imaginary part is added to the real potential, the situation does not change. In Fig. 3, calculated angular distributions for Mg^{24} with the real distorting potential are shown. The variation of the angular distribution with the incident energy is remarkable but it changes continuously and shows not so rapid fluctuation.

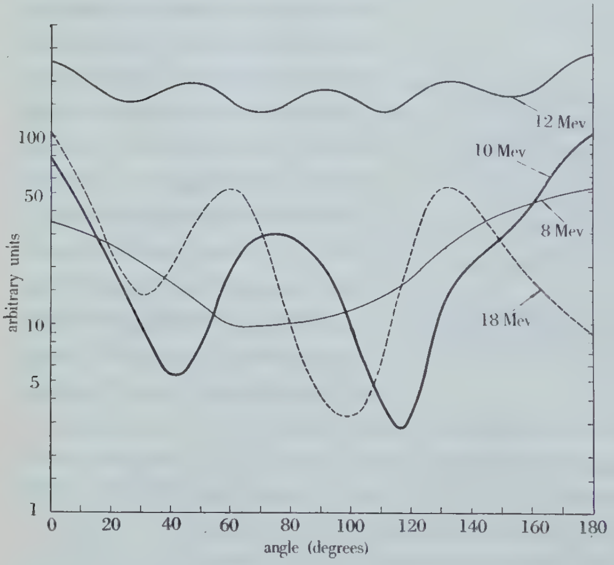
As is seen in Fig. 3*a*, *b*, 90° symmetric angular distribution will be obtained when a certain partial wave incident or scattered is in the " single particle resonances ". The 90° symmetry of Fig. 3*a*, $E_p=11$ Mev is the result of the single particle resonance for $l=3$ wave, and that of Fig. 3*b*, $E_p=10$ Mev is the one for $l=4$ wave.

In the energy region where such sharp resonance phenomena do not happen, many partial waves will contribute together and the interference effects between these waves become important, so that the resultant angular distribution will be 90° asymmetric. In Fig. 3*a*, the calculated angular distribution for the incident energy $E_p=8$ Mev is shown. When $E_p>11$ Mev, the contribution from $l=3$ wave becomes weaker and the interference effect between $l=3$ and $l=2$ waves become appreciable, so that a forward-peaked scatter-



a: Calculation is made with a square well potential $U=30$ Mev, $R=1.35\ A^{1/3}y$. For two-body interaction the delta-type potential is assumed.

90° symmetric distribution of $E_p=11$ Mev is the effect of $l=3$ wave resonance.



b: Calculated with $U=50$ Mev, $R=1.35\ A^{1/3}y$. The delta-type potential is assumed. The 90° symmetric dashed curve is calculated at $E_p=12$ Mev just the resonance energy of $l=4$ wave.

It can be seen from *a* and *b* that the angular distribution is much affected by changing the depth of the distorting potential.

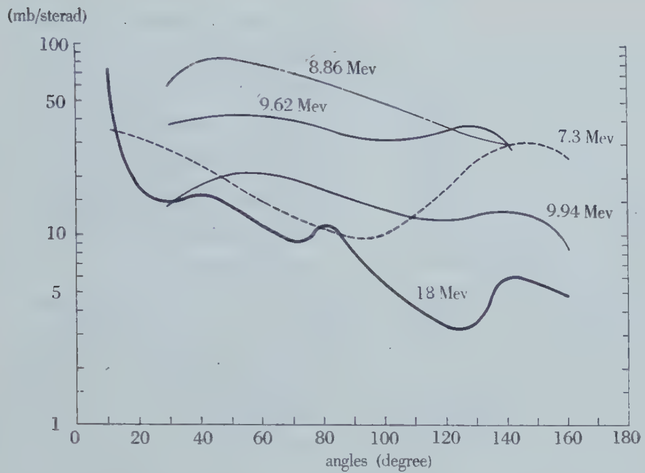
Fig. 3. Angular distribution for inelastically scattered protons from Mg^{24}.

ing is obtained as is shown for $E_p=18$ Mev.

In Fig. 3*b*, we show the results calculated on the distorting potential of $U=50$ Mev. The effect of the change of the potential depth will be seen by comparison of Fig. 3*a* with Fig. 3*b*. In the latter case, the single particle resonance of $l=4$ wave is dominant. The remarkable diffraction pattern of $P_8(\cos\theta)$ appears in the angular distribution calculated with the distorting potential of $U=50$ Mev, which is the consequence of the $l=4$ wave resonance.

The experimental angular distribution taken from some published data are also shown in Fig. 3*c*. The fair agreement is not intended here. The general feature of the experimental data of Mg^{24} shows that the angular distribution becomes forward-peaked as the incident energy of the proton increases. Calculated angular distributions show also such tendency, but definite conclusion would not be obtained since the results are too sensitive to the adopted parameters in the distorting potential.

Calculations with the complex distorting potential are made for C^{12} nucleus. The rusults are shown in Figs. 4*a*, *b*. The magnitude of the imaginary part of the potential is varied from $\zeta=0$ to 0.2. Calculated angular distributions are shown for $\zeta=0$ and $\zeta=0.03$ in Fig. 4*a* and



c: Experimental angular distributions for inelastically scattered protons from Mg^{24} with excitation of 1.38 Mev level, dashed line is arbitrarily normalized.

7.3 Mev: Taken from reference 16), 18.0 Mev: from 20), 8.86 and 9.62 Mev: from 18), 9.94 Mev: from 17).

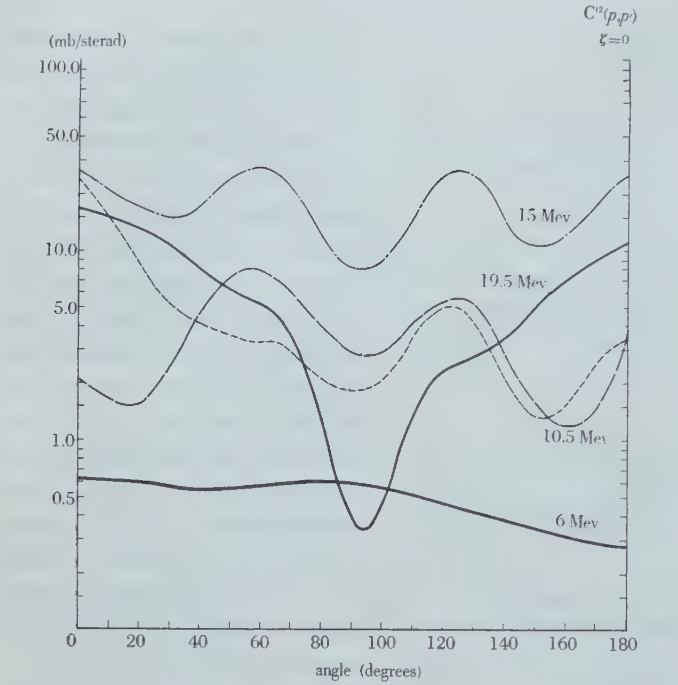
Fig. 3. Angular distribution for inelastically scattered protons from Mg^{24}.

Fig. 4*b*, respectively. The effect of the imaginary part is very large for the total cross section and the angular distribution is also influenced. Some dominant single particle effects are weakened and sharp variation of the cross section will be smeared out. 90° symmetry which is the consequence of the single particle resonance of an incident or scattered wave due to the real distorting potential will be lost by addition of the imaginary part of the potential. In the case of C^{12} nucleus, calculation in a wide range of energy shows to some extent such 90° asymmetric angular distribution as is seen in the experimental data. Although the imaginary part of potential affects the general features of the angular distribution in several points, the addition of the small imaginary potential of 0.03 does not seriously change its general tendency. Comparison of the results with the experimental data, which are shown in Fig. 4*c*, seems to be rather a good fit in its general features. From this result of comparison, we may proceed to a further attempt which aims at more precise agreement between the theory and the experimental data for C^{12}.

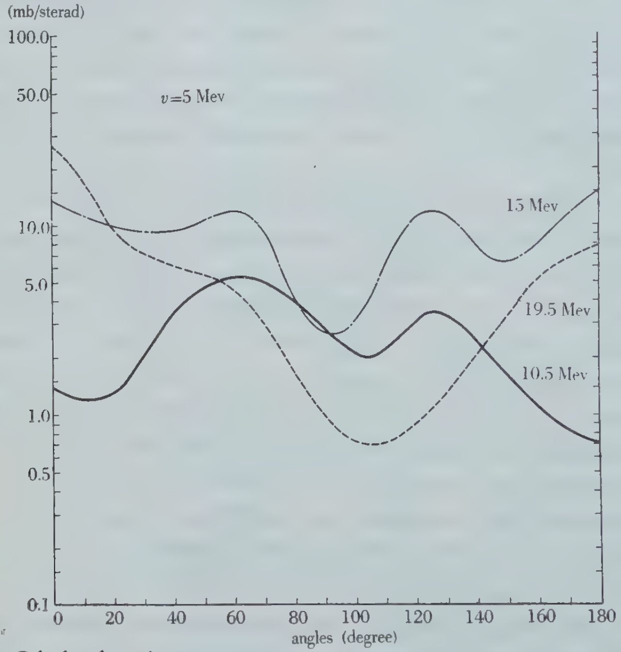
There is an other parameter used in our calculation: the range of the two-body interaction of the incident nucleon with nucleons in the target nucleus. The contribution which comes from the outside $(r>R)$ of the nucleus becomes larger in the case of the finite range interaction than in the case of the delta-type interaction. In the later case, the contribution from the outside of the nucleus is only about 1% of the calculated cross section, but in the case of the Yukawa-type interaction with the range of the meson Compton wave-length, it increases by a factor five or so. Sometimes, the variation of the range of the two-body interaction affects the angular distribution considerably as is shown in Fig. 4*a*. In this case, the interference term which is appreciable in the delta-type interaction is much reduced by adopting the interaction with the finite range.

iii) *Further comparison of the results on* C^{12} *and* Mg^{24} *with experimental data*

Experimental angular distributions for C^{12} taken from some published data (Fig. 4*c*) show about 90° symmetry in a wide range of the incident proton energy about 10∼15 Mev. Their minima are at about 90°∼100° and maxima near 40° and 150°. By Peelle's data, the angular distribution for high energy protons $E_p>15$ Mev become



a: Calculated with a real square well potential $\zeta=0$ in $U=45\ (1+i\zeta)$ Mev, $R=1.45\ A^{1/3}\,y$. The Yukawa-type interaction is assumed for the two-body interaction. The dashed line which is calculated with the delta-type interaction is also shown for comparison.



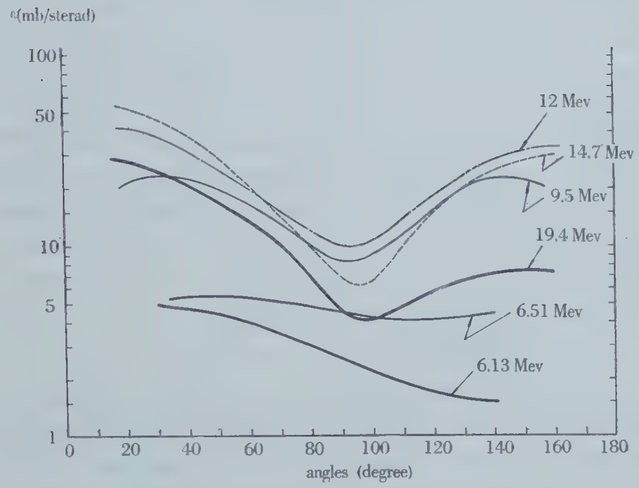
b: Calculated with a complex well potential $\zeta=0.03$. Other parameters are equally corresponding to *a*.

Fig. 4. Angular distribution for inelastically scattered protons from C^{12}.

forward peaked. The energy dependence of the absolute cross section exhibits a regular increase from $E_p \simeq 4$ Mev to about 12 Mev, then they begin to decrease as the incident energy increases.

This is well explained by our treatment. Our choice of 45 Mev distorting potential yields a resonance peak of the total cross section at about $E_p=15$ Mev. Experimental cross sections in an energy region $E_p=10\sim17$ Mev are well reproduced by taking $v\simeq7$ Mev, and $\zeta=0.03$, but for the low energy cross sections of $E_p<10$ Mev calculated cross sections are too small. A choice of $U\simeq47$ Mev, $\zeta\simeq0.05$ and $v\simeq10$ Mev will give more favorable agreement in a wider energy region.

Calculated angular distributions show 90° symmetry in a range of $E_p=10\sim15$ Mev with the some remarkable patterns. An addition of a suitable imaginary potential and a rounding-off of the potential edge will smear these patterns in some amount without appreciably destroying the 90° symmetry.



·*c*: Experimental angular distribution for inelastically scattered protons from C^{12} with excitation of 4.43 Mev level.

6.13 Mev, 6.51 Mev: taken from reference 21) 9.5 Mev: from 19), 12 Mev: from 24) 14.7 Mev, 19.4 Mev: from 23).

Fig. 4. Angular distribution for inelastically scattered protons from C^{12}.

In the experimental data for Mg^{24}, there is also an increase of the cross section which may be considered as a resonance at the incident energy about 8 to 9 Mev. This phenomenon is not explained by the single particle resonance of a single partial wave only, for its angular distribution is not 90° symmetric. Two or more waves must contribute appreciably to the cross section to reproduce asymmetries of data.

iv) *The l-dependent potentials*

So far, the distorting potential is treated as l-independent, but some l-dependent potential may be realized, for example, $\boldsymbol{l}\cdot\boldsymbol{s}$ force which is considered in the analysis of polarization experiments. In the case of l-dependent distorting potential, the distances of adjacent resonances and ordering of resonance levels differ from those in the case of l-independent distorting potential. Especially, in the former case, two single particle resonances with different parity may occur almost at the same energy. In this case, the angular distribution of the scattered particles is not 90° symmetric because of the interference between two resonating waves.

The experimental angular distributions for protons are not always 90° symmetric, even in the case that the absolute value is very large and it seems to be just the case of such resonance.

For the sake of simplicity, we examine the effect of two successive resonances which are realized almost at the same energy, neglecting the contributions of other waves.

When waves of $l=2$ and $l=3$ resonate simultaneously, for instance, the angular distribution is given by

$$\begin{aligned}\sigma(\theta)\propto{}&(0.04\,|\alpha|^2+0.02\,|\beta|^2)P_0+(-0.012\,|\alpha|^2+0.01\,|\beta|^2)P_2(\cos\theta)\\&+(0.03\,|\alpha|^2-0.02\,|\beta|^2)P_4(\cos\theta)+0.03\,|\beta|^2P_6(\cos\theta)\\&+2\mathcal{R}e(\alpha\beta^*\epsilon)[0.04P_1(\cos\theta)-0.02P_3(\cos\theta)+0.03P_5(\cos\theta)].\end{aligned}$$

Here, α is the radial integral $e^{-2Im\delta_2}\mathcal{R}_{22}$, β is $e^{-2Im\delta_3}\mathcal{R}_{33}$ and ϵ is the phase factor

$e^{i(\delta_2+\delta_{3'}-\delta_3{}^*-\delta_{2'}^*)}$ of $(7-a)$. For the case of $l=1$ and $l=2$,

$$\sigma(\theta) \propto (0.11\,|\alpha|^2+0.04\,|\beta|^2)P_0+(0.02\,|\alpha|^2-0.01\,|\beta|^2)P_2(\cos\theta)+0.03\,|\beta|^2P_4(\cos\theta)$$
$$+2\mathcal{R}e(\alpha\beta^*\epsilon)[0.06P_1(\cos\theta)+0.03P_3(\cos\theta)],$$

where $\alpha=e^{-2Im\delta_1}\mathcal{R}_{11}$, $\beta=e^{-2Im\delta_2}\mathcal{R}_{22}$ and $\epsilon=e^{i(\delta_1+\delta_{2'}-\delta_2{}^*-\delta_{1'}^*)}$. The term which causes the 90° asymmetry comes from the square bracket in this expression. Usually, $\alpha, \beta>0$ and if the phase factor ϵ are positive quantity, the angular distribution will be forward peaked. For other cases of two waves resonance one, can obtain the similar expressions. When two waves with same parity are concerned the resultant angular distribution is exactly 90° symmetric.

§ 5. Conclusion

In the preceding section, we have examined the effects of parameters involved in our calculation. In view of this examination we can conclude that the calculation based on the single particle model of inelastic scattering gives a suitable explanation for the experiment, in its qualitative feature. But it should be kept in mind that our results are considerably dependent on so many parameters as summarized below.

The parameters involved in our calculations are i) the depth (and radius) of the distorting potential, ii) the imaginary part which is added to the real potential, and iii) the range of interaction of two-body interaction V which is the only origin of the inelastic excitation.

As for the general tendencies of these parameters, the depth (and radius) of the potential determines the positions and ordering of the single particle resonances and the phase shifts of the incident (and outgoing) partial waves. The shape of the distorting potential may not alter the general feature of the result which is given by the real square well potential.

To describe the nuclear interaction of the incident nucleon in terms of the self-consistent field we should probably take the effective mass of a " single particle " in the nucleus to be different from the original mass. The change of the mass, however, may partly be accounted for by the change of the depth of the distorting potential, namely by taking the depth of the potential as a flexible parameter. The imaginary part of the distorting potential which is thought to be the averaged contribution of the higher order effects of the excitation affects appreciably the energy dependence of the total cross section and the angular distribution. It smooths out the sharp resonance peaks of the cross section and destroys the 90° symmetry of the angular distribution in the neighbourhood of the resonance energy. Variation of the range of two-body interactions gives an effect on the interference terms between different partial waves, hence also on the angular distribution in the non-resonance region. But its effect on the total cross section is usually small.

These parameters may be fixed by referring to experimental data. If the parameters are once fixed by certain experimental data, it is necessary to study other data by using the same parameters in order to examine the validity of the "single particle model" for nuclear reactions. But with the poor accumulation of experimental data and the present stage of the theoretical analysis, it is too early to expect a fair fit between the theory and the experiments, although some features of experimental data have been explained by our treatment.

The cross section is proportional to the square of the strength of the two-body interaction. It is adjusted to give the absolute value of the experimental cross sections and is smaller than (about some ten percents of) the strength of the two-body interaction which is necessary to explain the free nucleon-nucleon scattering experiments. Although the strength of the interaction V thus determined from experiments seems to be of reasonable order of magnitude, it is not yet justified that our perturbation treatment for V is permitted because of the difficulty to estimate the higher order term. It is a future subject to set forward an analysis in connection with the close examination of $(2-c)$. Present analyses are performed under the assumption that the diagonal part of higher order effect may be smeared by averaging and contained in the distorting complex potential, leaving the validity of the assumption unexamined. Relations between the distorting potentials for elastic scattering and for inelastic scattering must also be made clear by the study of $(2-c)$.

Though the remarkable resonance peaks are weakened by adding the small imaginary part to the potential, the ordering and the energy distance of the successive resonances remain unchanged, once the nature of the distorting potential is fixed. The ordering and positions of resonances are mainly dependent on the real part of the distorting potential. Therefore the $\boldsymbol{l}\cdot\boldsymbol{s}$ force, which is required in the analysis of the polarization experiments will also contribute to some extent. We examined the effects very qualitatively by adding l-dependent force to the distorting potential. The angular distribution with 90° asymmetry for protons in a certain resonance energy region may be explained by taking account of $\boldsymbol{l}\cdot\boldsymbol{s}$ force.

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Covariant Solutions of the Bethe-Salpeter Equation

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A fully covariant investigation is made of the Bethe-Salpeter equation for a pair of nucleons, with pseudo-scalar interaction. The "ladder" approximation is adopted, but pair creation and nucleonic recoil are accounted for exactly. Matrix solutions are obtained, with varying degrees of explicitness, for instantaneous and delayed interaction, vanishing and non-vanishing meson mass, and for vanishing and non-vanishing total energy. Important properties are disclosed which are either obscured or do not appear at all in non-relativistic approximation.

First the Bethe-Salpeter equation is reduced to a pair of coupled differential equations in which the Dirac matrices appear only in the coupling. In the instantaneous interaction approximation, these can be reduced to a single covariant equation in a single variable, showing the radical influence of nucleon recoil on pair effects. When the instantaneous interaction approximation is discarded, new features appear. There is a discrete infinity of stable states corresponding to each one of the non-relativistic theories, requiring a new quantum number for their enumeration. Jastrow's hypothesis of a repulsive "core" interaction is rigorously established, and the singularity is isolated.

It is shown how to obtain solutions corresponding to states of higher angular momentum from those with $J=0$, making use of the relativistic quantum enumeration. The conclusion is drawn that the relativistic quantum number is a property of the state of any pair of interacting particles, and its possible connection with the "strangeness" number is discussed.

§ 1. Introduction

It has been suspected for some time that the covariant wave equations, which have been derived on the basis of quantized field theory,[1] have certain features differing essentially from those of corresponding Schrödinger equations. It is true that by making non-covariant approximations it has been possible to obtain analogues of the Schrödinger equations, even to the extent of predicting effective potentials. Among others Salpeter and Bethe,[2] Lévy[3] and Klein[4] have treated the inter-nucleon interaction in this way. But, on removal of the non-covariant approximations difficulties arise. It was first noticed by Goldstein,[5] in a special (unrealistic) case, that the Bethe-Salpeter $(B-S)$ equation appeared not to possess solutions of the type expected for the prediction of bound states. A variety of diagnoses of this trouble have been made, blaming sometimes the extreme value of the binding energy assumed by Goldstein and sometimes the highly singular nature of the field-theoretical potential. The remedy suggested by Goldstein himself was to introduce a less singular potential which by a limiting process might be made to coincide with the exact theoretical potential. Though such a procedure does not work[6] in the extreme case studied by Goldstein, possibly it would do for less extreme binding energies. McCarthy and Green[7] have developed a method for solving the eigenvalue problem, based on this possibility.

Wick,[8] Scarf[9] and Cutkosky[10] have meanwhile studied similar equations for spinless particles, and have shown that the same difficulties do not arise. Wick, however, pointed out that there was an essential difference between the $B-S$ equations for fermions and spinless particles, inasmuch as the Fredholm method, which is particularly suited to the isolation of solutions corresponding to bound states, requires modification[7] before it can be applied to the equation for two fermions. It has not, in fact, yet been proved beyond doubt that the covariant $B-S$ equation could determine the binding energy of the deuteron.

The questions which remain can be finally answered only by producing exact solutions of the $B-S$ equation for non-vanishing total energy. Subject only to the " ladder " approximation, the effect of which is already known,[4] such solutions are obtained in this paper. They enable one to affirm the existence of solutions to the eigenvalue problem (for non-vanishing total energy), and show the presence of a repulsive " core " interaction of the type proposed by Jastrow[11] and Lévy.[3] The effective core potential is determined for states with $J=0$, and a method is prepared for extending the calculations to states of higher angular momentum.

The repulsive force at short internucleonic distances can be ascribed to the creation of virtual pairs and not to the delayed nature of the interaction ; it appears even in the instantaneous interaction approximation adopted by nearly all previous authors. But the neglect of nucleon recoil which is also commonly made also modifies the nuclear forces, particularly those associated with virtual pair creation and discussed by Arnowitt and Gasiorowicz.[12]

An interesting feature of the solutions obtained is that they require for their classification a new quantum number, which does not appear in non-relativistic approximations at all. It is an analogue of the quantum number proposed by Pais[13] for the classification of the " strange " particles, but also appeared in early discussions of relativistic wave equations by Born and his associates.[14],[15] It seems clear that such a quantum number will appear in the solution of any covariant wave equation, since it corresponds to a universal constant of the motion : the invariant square of the relativistic angular momentum tensor. In the final section of this paper, we discuss the possibility of classifying and describing the new particles, without introducing any concept beyond those already implicit in relativistic quantum mechanics.

§ 2. Reduction of the $B-S$ equation

The $B-S$ equation representing the interaction of a pair of nucleons can be written, in the " ladder " approximation,

$$(\boldsymbol{p}_1-M)\,\Psi\,(\boldsymbol{p}_2-M)+F(x)\,\gamma_5\Big(\sum_i \tau_i \Psi \tau_i\Big)\gamma_5=0. \tag{2·1}$$

Here p_1 and p_2 are four-vector momentum operators, conjugate to the space-time coordinate vectors x_1 and x_2 for the two particles; $\boldsymbol{p}_1=\gamma\cdot p_1$ and $\boldsymbol{p}_2=\gamma\cdot p_2$ both operate on Ψ; $x=x_2-x_1$; $F(x)$ is the relativistic interaction potential; $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ is anti-hermitean, since γ_4 is hermitean but γ_1, γ_2 and γ_3 are anti-hermitean; and τ_i are the isotopic spin matrices (for charge-independent pseudo-scalar meson theory). According as Ψ represents a charge singlet or charge triplet state, the matrix operator $\sum' \tau_i \cdots \tau_i$ is replaced by the eigenvalue $+3$ or -1 respectively, and $(2\cdot1)$ reduces to

$$(\boldsymbol{p}_1-M)\Psi(\boldsymbol{p}_2-M)=f(x)\gamma_5\Psi\gamma_5, \tag{2·2}$$

where $f(x)=-3F(x)$ or $+F(x)$. If one adopts the momentum representation for the centre-of-mass coordinates, but the coordinate representation for the relative coordinates of the two nucleons, one has

$$\boldsymbol{p}_1=\boldsymbol{p}+i\boldsymbol{\nabla}; \quad \boldsymbol{p}_2=\boldsymbol{p}-i\boldsymbol{\nabla}, \tag{2·3}$$

where p is half the resultant energy-momentum 4-vector, $\boldsymbol{p}=\gamma\cdot p$ and $\boldsymbol{\nabla}=\gamma\cdot\nabla$. So

$$p_1^2+p_2^2=2(p^2-\Box); \quad p_1^2-p_2^2=4ip\cdot\nabla. \tag{2·4}$$

Our object is to obtain solutions of $(2\cdot1)$, not restricted to the simple radially symmetric form which we have studied previously,[16)] but allowing Ψ to depend on angular co-ordinates, together with the appropriate Dirac matrices. An important feature of the method we shall adopt is to express Ψ in the form

$$\Psi=(\boldsymbol{p}_1+M)\omega+\omega(\boldsymbol{p}_2+M)+\theta, \tag{2·5}$$

where ω and θ are matrices satisfying

$$\{\omega,\ \gamma_5\}=0; \quad [\theta,\ \gamma_5]=0, \tag{2·6}$$

but otherwise unrestricted. It is easy to see that $(2\cdot5)$ is uniquely equivalent to the most general form (involving 16 independent Dirac matrices) which one could write down for Ψ.

On substituting from $(2\cdot5)$ into $(2\cdot2)$ one requires the identities

$$\begin{aligned}(\boldsymbol{p}_1-M)\theta(\boldsymbol{p}_2-M)&=1/2(\boldsymbol{p}_1-M)(\boldsymbol{p}_1\theta+\theta\boldsymbol{p}_2)+1/2\cdot(\boldsymbol{p}_1\theta+\theta\boldsymbol{p}_2)(\boldsymbol{p}_2-M)\\&\quad-1/2(p_1^2+p_2^2-2M^2)\theta,\\(p_1^2-M^2)\omega(\boldsymbol{p}_2-M)&+(\boldsymbol{p}_1-M)\omega(p_2^2-M^2)\\&=1/2\cdot(p_1^2+p_2^2-2M^2)\omega(\boldsymbol{p}_2-M)+1/2\cdot(\boldsymbol{p}_1-M)(p_1^2+p_2^2-2M^2)\omega\\&\quad-1/2(p_1^2-p_2^2)(\boldsymbol{p}_1\omega-\omega\boldsymbol{p}_2);\end{aligned} \tag{2·7}$$

the resulting equation is satisfied provided

$$\begin{aligned}&\{\Box+M^2-p^2-f(x)\}\omega=1/2(\boldsymbol{p}_1\theta+\theta\boldsymbol{p}_2);\\&\{\Box+M^2-p^2+f(x)\}\theta=2ip\cdot\nabla(\boldsymbol{p}_1\omega-\omega\boldsymbol{p}_2)+i[(\boldsymbol{\nabla}f),\ \omega].\end{aligned} \tag{2·8}$$

This pair of coupled differential equations can be made the starting point for a discussion of the general covariant solutions of the $B-S$ equation. For some purposes we shall find it convenient to introduce ω' satisfying

$$\{\Box+M^2-p^2+f(x)\}\omega'=2p\cdot\nabla\omega\,; \tag{2·9}$$

then, if

$$\varphi=\theta-i(\boldsymbol{p}_1\omega'-\omega'\boldsymbol{p}_2), \tag{2·10}$$

one has

$$\{\Box+M^2-p^2+f(x)\}\varphi=i\{(\nabla f)\,(\omega+i\omega')-(\omega-i\omega')\,(\nabla f)\} \tag{2·11}$$

and

$$\{\Box+M^2-p^2-f(x)\}\omega=2p\cdot\nabla\omega'+1/2\{\boldsymbol{p},\ \varphi\}+1/2\cdot i[\nabla,\ \varphi]. \tag{2·12}$$

§ 3. The instantaneous interaction approximation

We remark now that most previous work, including that of Lévy[3] and Klein,[4] has been based on three distinct approximations. The first replaces the real interaction by an instantaneous one; the second neglects the recoil of the nucleons in the emission of virtual mesons; and the third treats virtual pair creation by a perturbation procedure. Arnowitt and Gasiorowicz,[12] while retaining the first two approximations, have treated pair creation rigorously and concluded that the perturbation procedure may be poorly convergent. On physical grounds one might expect, however, that pair effects would be considerably influenced by nucleon recoil, and we propose, therefore, in this section to make a calculation in which only the instantaneous interaction approximation is retained. In the following sections even this approximation will be dispensed with; but for the present we set

$$\begin{gathered}f(x)=2iV(r)\,\delta(p\cdot x/E),\\ r^2=-x^2,\quad E^2=p^2,\end{gathered} \tag{3·1}$$

so that $2V(r)$ is an analogue of Yukawa's classical potential energy. This is not, in itself, a non-covariant approximation, though it has the appearance of one when one adopts the centre-of-mass frame with $p=0$, so that $p_4=E$; this we shall proceed to do.

The present discussion will, for the sake of simplicity, be restricted to states with $J=0$; the results for states of higher angular momentum appear to be closely analogous. Only those solutions will be sought which are unchanged when the signs of x_4, γ_4 and E are reversed; the others, which change sign under this transformation, correspond to nucleon-antinucleonic interactions. Therefore, if one writes

$$\begin{aligned}\omega_a&=1/2(\omega+\gamma_4\omega\gamma_4)=\omega_a\gamma_4,\\ \omega_b&=1/2(\omega-\gamma_4\omega\gamma_4)=\omega_\beta\gamma\cdot x,\\ \omega_a'&=1/2(\omega'+\gamma_4\omega'\gamma_4)=\omega_\alpha'\gamma_4,\\ \omega_b'&=1/2(\omega'-\gamma_4\omega'\gamma_4)=\omega_\beta'\gamma\cdot x,\\ \varphi_a&=1/2(\gamma_4\varphi+\varphi\gamma_4)=\varphi_\alpha\gamma_4,\\ \varphi_b&=1/2\cdot i(\gamma_4\varphi-\varphi\gamma_4)=\varphi_\beta\gamma\cdot x,\end{aligned} \tag{3·2}$$

then ω_a, ω_b', φ_a and φ_b should be even functions of x_4, and ω_b and ω_a' should be odd functions which vanish on the space-like surface $x_4=0$. In general ω_α, ω_β, ω_α', ω_β', φ_α and φ_β could involve the matrix γ_5, but when one considers isotropic mixtures of the spin states, they are just scalar functions of x_4 and r.

The equations satisfied by the new set of field variables can be written down immediately from (2·12), (2·9) and (2·11) ; they are

$$(\Box+M^2-E^2-f)\,\omega_a+2E\Gamma_4\omega_a'=E\varphi_a-\gamma_4\gamma\cdot\nabla\varphi_b\,,$$

$$(\Box+M^2-E^2\pm f)\,\omega_a'-2E\nabla_4\omega_a=0\,,$$

$$(\Box+M^2-E^2\mp f)\,\omega_b+2E\nabla_4\omega_b'=\nabla_4\varphi_b\,,$$

$$(\Box+M^2-E^2+f)\,\omega_b'-2E\nabla_4\omega_b=0\,,$$

$$(\Box+M^2-E^2+f)\,\varphi_a=2f\Gamma_4\omega_a'-2\gamma_4(\gamma\cdot\Gamma f)\,\omega_b'\,,$$

$$(\Box+M^2-E^2+f)\,\varphi_b=2f\nabla_4\omega_b-2\gamma_4(\gamma\cdot\nabla f)\,\omega_a. \qquad (3\cdot3)$$

The ambiguous signs in the 2nd and 3rd equations are arbitrary, because f vanishes when multiplied by an odd function of x_4. These equations are easily solved for $x_4\neq 0$, when f vanishes. The general formal solutions of the last two are

$$\varphi_a=\exp(iS|x_4|)\,\alpha,$$

$$\varphi_b=\exp(iS|x_4|)\,\beta, \qquad (3\cdot4)$$

where α and β are functions of r only, and S is the integral operator which may be represented symbolically by

$$S=(M^2-E^2-\Delta)^{1/2}. \qquad (3\cdot5)$$

It should be noticed that terms like $\exp(-iS|x_4|)\,\alpha'$ and $\exp(-iS|x_4|)\,\beta'$ are excluded from the solutions by the boundary conditions $\varphi_a\to0$ and $\varphi_b\to0$ as $x_4\to\pm\infty$, since M is supposed to have a small positive imaginary part. Using (3·4), the first four of eqn. (3·3) are readily solved for the auxiliary variables $\omega_a\pm i\omega_a'$ and $\omega_b\pm i\omega_b'$; one has

$$\omega_a=\exp(iT|x_4|)\,(\alpha_1\cos Ex_4+i\,\mathrm{sgn}\,x_4\,\alpha_2\sin Ex_4)\,,$$

$$\omega_a'=\exp(iT|x_4|)\,(\alpha_1\sin Ex_4-i\,\mathrm{sgn}\,x_4\,\alpha_2\cos Ex_4)\,.$$

$$+(2iES)^{-1}\,\mathrm{sgn}\,x_4\exp(iS|x_4|)\,(E\alpha-\gamma_4\gamma\cdot\nabla\beta)\,,$$

$$\omega_b=\exp(iT|x_4|)\,(\mathrm{sgn}\,x_4\,\beta_1\cos Ex_4+i\beta_2\sin Ex_4)\,,$$

$$\omega_b'=\exp(iT|x_4|)\,(\mathrm{sgn}\,x_4\,\beta_1\sin Ex_4-i\beta_2\cos Ex_4)$$

$$+(2E)^{-1}\exp(iS|x_4|)\,\beta\,, \qquad (3\cdot6)$$

where T is the integral operator

$$T=(M^2-\Delta)^{1/2}. \qquad (3\cdot7)$$

When (3·4) and (3·6) are substituted back into (3·3), the validity of the latter at $x_4=0$ requires

$$
\begin{aligned}
&(T-V)\alpha_1+E\alpha_2=0,\\
&2ES\alpha_2+E\alpha=\gamma_4\gamma\cdot\nabla\beta,\\
&\quad\beta_1=0,\\
&(S+V)\beta=2iE(T+V)\beta_2,\\
&(S+V)\alpha=E^{-1}V(E\alpha-\gamma_4\gamma\cdot\nabla\beta)\\
&\qquad+2V(E\alpha_1+T\alpha_2)-E^{-1}(\gamma_4\gamma\cdot\nabla V)(\beta-2iE\beta_2),\\
&(S+V)\beta=2iV(T\beta_1+E\beta_2)-2(\gamma_4\gamma\cdot\nabla V)\alpha_1.
\end{aligned}
\tag{3·8}
$$

The terms in α and β contribute nothing to the solution for θ, given by (2·10), and hence for Ψ when $x_4 \neq 0$, but are nevertheless required to neutralize the discontinuity in ω' at $x_4=0$ arising from the coefficient of α_2. When α and β are eliminated, one is left with

$$
\begin{aligned}
&E\alpha_2=-(T-V)\alpha_1,\\
&ET\beta_2=i(\gamma_4\gamma\cdot\nabla V)\alpha_1,\\
&(S^2+VT)\alpha_2+EV\alpha_1=i(T+V)\gamma_4\gamma\cdot\nabla\beta_2,
\end{aligned}
\tag{3·9}
$$

from whence α_2 and β_2 are also easily eliminated, giving

$$TV^{-1}(S^2T\alpha_1+\nabla V\cdot\nabla\alpha_1)=VT^2\alpha_1-\nabla V\cdot\nabla\alpha_1. \tag{3·10}$$

This result is exact, within the limits of the ladder approximation and the instantaneous interaction approximation. At non-relativistic energies, one can write

$$
\begin{aligned}
&T\approx M-(2M)^{-1}\Delta,\\
&E\approx M+1/2\cdot W.
\end{aligned}
\tag{3·11}
$$

Neglecting squares and products of Δ and W, and also potential contributions of order $(\mu/M)^2$, where μ is the meson mass, one then obtains

$$\{MW+(1-M^{-2}V^2)\Delta+V^2\}\alpha_1=M^{-2}(M+V)\nabla V\cdot\nabla\alpha_1. \tag{3·12}$$

This is the equivalent Schrödinger equation, showing a classical potential $-(1-V^2/M^2)^{-1}\cdot(1+W/M)(V^2/M)$ and also a velocity dependent term. It differs radically from equations derived by Lévy and Klein treating pair effects by a perturbation method. It also differs somewhat from the equation obtained by Arnowitt and Gasiorowicz,[12)] but agrees with it in predicting a singularity of the potential, for $V(r)=M$. At distances shorter than this the forces are repulsive, simulating the "core" postulated by Jastrow and Lévy.

It can be seen, *a posteriori*, that our result could not be expected to agree exactly with that of Arnowitt and Gasiorowicz. The equations which determine α and β are

$$
\begin{aligned}
&(S+V)\beta=-2M^{-1}(M+V)(\gamma_4\gamma\cdot\nabla V)\alpha_1,\\
&E\alpha=\gamma_4\gamma\cdot\nabla\beta+2S(M-V)\alpha_1,
\end{aligned}
\tag{3·13}
$$

in an approximation similar to that of (3·12). Now there is, unfortunately, no good non-relativistic approximation to the operator S which appears in these equations, since M^2 and E^2 are of the same order, and their difference is of the same order as the square of the relative momentum $(-\Delta)$. Approximating S by zero in (3·8) and (3·9), one obtains an equivalent Schrödinger equation more closely resembling those of previous authors. The present calculation shows that, whereas pair effects do cause a failure of the perturbation expansion of the effective potential, pair and recoil effects together introduce a still more deeply rooted difficulty: there are some equations to which there is no good non-relativistic approximation at all.

The instantaneous interaction approximation is one which considerably relieves the difficulty of solving $B-S$ equation. Yet neither its quantitative nor its qualitative effect is very obvious, even in interactions at low energy. We shall therefore discard the approximation in subsequent sections, to see what new features arise.

§ 4. A general method of solution

Again in this section we shall restrict attention to states with $J=0$. In a fully covariant theory this is sufficient, since by applying a Lorentz transformation to certain states with $J=0$, one attains states with $J\neq 0$. In § 7 we shall exploit this fact in order to infer solutions for states of higher angular momentum from those obtained in this section. For the sake of simplicity we consider isotropic mixtures of the spin states, and solutions ω and θ of (2·8) will then have the form

$$\omega=\omega_1\boldsymbol{x}+\omega_2\boldsymbol{p},$$
$$\theta=\theta_1+i\theta_2[\boldsymbol{x},\ \boldsymbol{p}], \tag{4·1}$$

where ω_1, ω_2, θ_1 and θ_2 are functions only of s and z given by

$$s=x^2,\quad z=(x^2p^2)^{-1/2}x\cdot p. \tag{4·2}$$

As in the previous section, nucleon-nucleonic interactions are distinguished from nucleon-anti-nucleonic interactions by the fact that θ_1, θ_2 and ω are even functions of z, so that ω_1 is the only odd function, vanishing for $x_4=0$. Accepting the exact field-theoretical prediction, $f(x)$ is a function $g(s)$ of s only.

Using (4·1), eqs. (2·8) reduce to

$$\{\Box+M^2-E^2-g(s)\}\omega=\theta_1\boldsymbol{p}+2\{p\cdot\nabla(\theta_2\boldsymbol{x})-\nabla\cdot(\theta_2 x)\boldsymbol{p}\}, \tag{4·3a}$$

$$\{\Box+M^2-E^2+g(s)\}\theta_1+4p\cdot\nabla\{\nabla\cdot(\omega_1 x)+p\cdot\nabla\omega_2\}=0, \tag{4·3b}$$

$$\{\Box+M^2-E^2+4\nabla_s+g(s)\}\theta_2+2p\cdot\nabla\omega_1-2g'(s)\omega_2=0, \tag{4·3c}$$

where, in (4·3c), ∇_s means differentiation with respect to s, holding $p\cdot x$ constant, and $E^2=p^2$. Changing the field variables to ξ, η, σ and τ, defined by

$$\omega_1=s^{-1/2}(\xi-z\cdot\partial\eta/\partial z);\quad \omega_2=1/E\cdot\partial\eta/\partial z;$$
$$\theta_1=\sigma;\quad \theta_2=1/Es\cdot\partial\tau/\partial z \tag{4·4a}$$

and introducing the operators

$$\mathcal{P} \equiv 4\,(s^2 \cdot \partial^2/\partial s^2 + 2s \cdot \partial/\partial s) + (M^2 - E^2)\,s,$$

$$\mathcal{Q} = (1-z^2)\,\partial^2/\partial z^2 - 3z \cdot \partial/\partial z, \tag{4·4b}$$

the above eqs. become

$$\{\mathcal{P} + \mathcal{Q} - 3 - sg(s)\}\,\xi - 2\mathcal{Q}\,(\eta + \tau) = EX, \tag{4·5a}$$

$$\{\mathcal{P} + \mathcal{Q} + 1 - sg(s)\}\,\eta + 2\xi + 2\,(2s \cdot \partial/\partial s + 1)\,\tau = EY, \tag{4·5b}$$

$$\{\mathcal{P} + \mathcal{Q} + sg(s)\}\,\sigma = ES, \tag{4·5c}$$

$$\{\mathcal{P} + \mathcal{Q} - 1 - 4s \cdot \partial/\partial s + sg(s)\}\,\tau - 2s^2 g'(s)\,\eta = ET, \tag{4·5d}$$

where

$$X = sz\sigma\,; \quad Y = \int_0^z s\sigma\,dz,$$

$$S = -4E^{-1}\,sp \cdot \nabla [s^{-1/2} \{\mathcal{Q}\eta + (\eta + (2s \cdot \partial/\partial s + 3)\,\xi\}],$$

$$T = -2s^2 \int p \cdot \nabla \omega_1\,dz \tag{4·6a}$$

and

$$p \cdot \nabla = (p^2 x^2)^{1/2} \{2z \cdot \partial/\partial s + (1-z^2)/s \cdot \partial/\partial z\}. \tag{4·6b}$$

(i) *Solution for* $p \to 0$, $\mu = 0$

To illustrate the general method of solution, we shall first obtain the exact solution for the limit $p \to 0$ and $g(s) = 4\lambda/s$, which has not yet been given in a complete form, though Goldstein solved (4·5c), which is then uncoupled from the others. Introducing $\zeta = \eta + \tau$, one has

$$\{\mathcal{P} + \mathcal{Q} - 3 - 4\lambda\}\,\xi - 2\mathcal{Q}\zeta = EX\,[=0].$$

$$\{\mathcal{P} + \mathcal{Q} + 1 + 4\lambda\}\,\zeta + 2\xi = E\,(Y+T)\,[=0]. \tag{4·7}$$

These coupled eqs. can be solved by replacing the differential operators $\mathcal{P}$ and $\mathcal{Q}$ by eigenvalues P and Q, which must be related by

$$D(P, Q) \equiv (P+Q-3-4\lambda)\,(P+Q+1+4\lambda) + 4Q = 0. \tag{4·8}$$

The eigenfunctions of $\mathcal{Q}$ are the Gegenbauer polynomials $C^1_{k-1}(z)$, but as ξ and ζ must be odd functions of z, only even integral values of $k\,(k=2n)$ are allowed. The odd Gegenbauer polynomials are orthogonal in the interval $0 < z < 1$, or on the contour $0 \to i\infty \to \infty \to 1$ which embraces the physically significant values. They are singular at the light cone, where z transits from $i\infty$ to ∞, but are normally multiplied by functions of s which vanish strongly at $s=0$, so the actual solutions of (4·7) have no singularity there. Moreover, any odd function of z can be expanded in terms of the odd Gegenbauer polynomials. The eigenvalue of $\mathcal{Q}$ corresponding to the eigenfunction $C^1_{2n-1}(z)$ is $Q_n = 1 - 4n^2$.

The eigenfunctions of $\mathcal{P}$ which vanish at $s=0$ are of the form $s^{-1/2}J_{\nu+1/2}(\epsilon s^{1/2})$, where J represents the Bessel function and $\epsilon=(M^2-E^2)^{1/2}$; the value of ν is connected to the eigenvalue of $\mathcal{P}$ by $P=4\nu(\nu+1)$. For bound states, $M^2>E^2$, and if $s^2=-r^2$, the eigensolution tending to zero exponentially for large values of r is $s^{-1/2}H^{(1)}_{\nu+1/2}(\epsilon s^{1/2})$; this, however, is singular at $s=0$ and there is in fact no solution of the $B-S$ equation representing a bound state when $E=0$. For more realistic values of E, it will appear that this difficulty does *not* arise.

To determine η, and hence $\tau=\zeta-\eta$, when ξ and ζ have been determined, one may either use (4·5b), or substitute

$$\tau=s\cdot\partial\tau_1/\partial s+\tau_2 \tag{4·9}$$

in (4·5d), which is then satisfied provided

$$\begin{aligned}&(\mathcal{P}+\mathcal{Q}+3-4\lambda)\tau_1-4\tau_2=0,\\&(\mathcal{P}+\mathcal{Q}-1-4\lambda)\tau_2-\mathcal{P}\tau_1+8\lambda\zeta=ET[=0].\end{aligned} \tag{4·10}$$

Here again $\mathcal{P}$ and $\mathcal{Q}$ may be replaced by their eigenvalues to determine τ_1 and τ_2 in terms of ζ. There is also an independent solution with $\xi=\zeta=0$ and $\eta=C^1_{2n-1}(z)\cdot J_\nu(\epsilon s^{1/2})$; $\mathcal{Q}$ has the same eigenvalue $Q=Q_n$ as before, but P and Q are connected by the relation

$$D'(P,\ Q)=P+Q+1-4\lambda-2(P+1)^{1/2}, \tag{4·11}$$

instead of (4·8).

In the limit $p\to 0$, (4·5c) reduces to a generalization of Goldstein's equation, already discussed by Green.[6] Again the eigenfunctions of $\mathcal{Q}$ are Gegenbauer polynomials $C^1_{k-1}(z)$, but as σ must be an even function of z, k must be an odd integer $2n+1$ and the corresponding eigenvalue of $\mathcal{Q}$ is $Q_n'=-4n(n+1)$.

(ii) *Solution for* $p\to 0$, $\mu\neq 0$

When $p\to 0$, but $g(s)=2\pi i\lambda\mu s^{-1/2}H^{(1)}_1(\mu s^{1/2})$, as required by relativistic meson theory,* ξ, η, σ and τ may again be eigenfunctions of $\mathcal{Q}$, but the exact radial dependence of these functions has to be determined by numerical methods. The required equations appear in § 7. However, the character of the solutions for $|s|\ll\mu^{-2}$ does not depend on the meson mass μ, and is the same as in (i) above.

(iii) *Solution for* $E\neq 0$, $\mu=0$

It has already appeared, in (i) and (ii) above, that there is a discrete infinity of solutions of the $B-S$ eqn., one for each eigenvalue Q_n or Q_n' of $\mathcal{Q}$. This is a phenomenon which did not appear in the instantaneous interaction approximation, and indicates at least one fundamental respect in which the true solutions of the $B-S$ eqn. are modified by this approximation. It is quickly seen that it in no way depends on the artificial value (zero) assumed for E above. In fact the solutions of (4·5) can be obtained for $E\neq 0$ in the form of power series in E, regarding the "binding energy" $(M^2-E^2)/M$

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* The $\delta(s)$ singularity disappears on multiplication by a wave function vanishing at the light cone.

however as an independent parameter. The method will be indicated, assuming for simplicity $g(s)=4\lambda/s$, as for mass-less quanta. First the solution for $E=0$ is obtained, as described in (i) above. This solution is now put into $(4\cdot 6)$, using

$$z\, C^1_{k-1}(z)=1/2\{C_k^1(z)+C^1_{k-2}(z)\},$$

$$k\int_0^z C^1_{k-1}(z)\,dz=1/2\{C_k^1(z)-C^1_{k-2}(z)\},$$

$$(1-z^2)\cdot\partial C^1_{k-1}(z)/\partial z=1/2\cdot\{(k+1)\,C^1_{k-2}(z)-(k-1)\,C_k^1(z)\} \qquad (4\cdot 12)$$

to express X, Y, S and T in terms of eigenfunctions of $\mathcal{Q}$; these are then substituted as inhomogeneous terms in the right-hand sides of $(4\cdot 5c)$, $(4\cdot 7)$ and $(4\cdot 10)$. It is now a simple matter to integrate these equations—the quadratures can be performed at this stage—to obtain terms linear in E. By iteration, the coefficient of any power of E can be obtained. We do not give the details of this calculation, though we have carried it through several stages, since it gives little indication of the general character of the solution, except near $E=0$.

(iv) *Solution for* $E\neq 0$, $\mu\neq 0$

For general values of E and μ we confine ourselves in this section to finding solutions in ascending powers of s. In the following section connection formulae will be developed joining these solutions to those for large values of s. In the neighbourhood of $s=0$, one has

$$g(s)=4\lambda[s^{-1}-1/4\cdot\mu^2\{(\ln(-1/4\cdot\mu^2 s)+2\gamma-1\}+0(s)],$$

where γ is Euler's constant 0.577; however, within the region $|s|\mu^2<1$ it is sufficient to approximate $g(s)$ by $4\lambda/s$.

Let ξ_m, η_m, σ_m and τ_m be the solutions of $(4\cdot 5)$ which, in the limit $E=0$, are eigenfunctions of $\mathcal{Q}$ corresponding to the eigenvalues $1-4(m+1)^2$ (for ξ_m, η_m and τ_m) or $-4m(m+1)$ (for σ_m). Let all these functions be developed in terms of the Gegenbauer polynomials:

$$\xi_m=\sum_{n=0}^{\infty}\xi_{m,n}\,C^1_{2n+1}(z),\ \text{etc.},$$

$$\sigma_m=\sum_{n=0}^{\infty}\sigma_{m,n}\,C^1_{2n}(z), \qquad (4\cdot 13)$$

so that $\xi_{m,n}$, $\cdots$, $\sigma_{m,n}$ vanish for $m\neq n$ in the limit $E=0$. Let X_m, Y_m, S_m and T_m be the values of X, Y, S and T corresponding to the values ξ_m, η_m, σ_m and τ_m of ξ, η, σ and τ; and let these be developed in a similar way:

$$X_m=\sum_{n=0}^{\infty}X_{m,n}\,C^1_{2n+1}(z),\ \text{etc.},$$

$$S_m=\sum_{n=0}^{\infty}S_{m,n}\,C^1_{2n}(z). \qquad (4\cdot 14)$$

Let all the functions so defined be developed in powers of s, e. g.,

$$\xi_{m,n}=\sum_{j=0}^{\infty}\xi^{(j)}_{m,n}s^{\nu+j},$$

$$S_{m,n}=\sum_{j=0}^{\infty}S^{(j)}_{m,n}s^{\nu+j},\tag{4·15}$$

where ν will depend on m and will be determined in the usual way from an indicial equation.

We shall now *describe* how to obtain the recurrence relation connecting $\xi^{(j+1)}_{m,n}$, $\eta^{(j+1)}_{m,n}$ and $\zeta^{(j+1)}_{m,n}$ (where $\zeta=\eta+\tau$) with $\xi^{(j)}_{m,n}$, $\eta^{(j)}_{m,n}$ and $\zeta^{(j)}_{m,n}$, listing the actual results in the appendix to this paper. First $S^{(j)}_{m,n}$ is computed from $\xi^{(j)}_{m,n}$ and $\eta^{(j)}_{m,n}$, using (4·6), and $\sigma^{(j)}_{m,n}$ is found by integrating (4·5c). Next $T^{(j)}_{m,n}$ is derived using eq. (4·4a) for ω_1 and (4·6) for T. From (4·6) $X^{(j+1)}_{m,n}$ and $Y^{(j+1)}_{m,n}$ are also determined. Then (4·7) can be used to find $\xi^{(j+1)}_{m,n}$ and $\zeta^{(j+1)}_{m,n}$, and $\eta^{(j+1)}_{m,n}$ is found most easily by substituting $\tau=\zeta-\eta$ in (4·5b).

From dimensional considerations alone it is clear that $\xi^{(j)}_{m,n}$, $\eta^{(j)}_{m,n}$ and $\zeta^{(j)}_{m,n}$ will be homogeneous polynomials of degree j in E^2 and $\epsilon^2=M^2-E^2$; the initial values $\xi^{(0)}_{m,n}$, $\eta^{(0)}_{m,n}$ and $\zeta^{(0)}_{m,n}$ needed to begin the iteration procedure can therefore be determined by setting $E=0$ and $\epsilon=0$ in (4·7) and (4·5b). It has been seen that $\xi_{m,n}$, $\eta_{m,n}$ and $\zeta_{m,n}$ vanish if $E=0$ and $m\neq n$, so there are only $\xi^{(0)}_{m,m}$, $\eta^{(0)}_{m,m}$ and $\zeta^{(0)}_{m,m}$ to be considered. One solution can be obtained by taking $\xi^{(0)}_{m,m}=\zeta^{(0)}_{m,m}=0$, but $\eta^{(0)}_{m,m}\neq 0$, and ν as a root of

$$D'\{4\nu(\nu+1),\ 1-4(m+1)^2\}=0,\tag{4·16a}$$

where $D'(P,\ Q)$ is defined by (4·11). Another solution is started by taking $\xi^{(0)}_{m,m}$ and $\zeta^{(0)}_{m,m}$ in a ratio determined from (4·7), and ν as a root of

$$D\{4\nu(\nu+1),\ 1-4(m+1)^2\}=0\tag{4·16b}$$

with $D(P,\ Q)$ defined in (4·8).

Still another solution can be begun by taking $\xi^{(0)}_{m,m}=\eta^{(0)}_{m,m}=\zeta^{(0)}_{m,m}=0$ and $\sigma^{(0)}_{m,m}\neq 0$, but this will not be required at present.

§ 5. The eigenvalue problem for bound states

We wish to show now that the eigenvalue problem associated with the $B-S$ equation in the form (4·3) possesses a solution, even though, as we have already observed, there is none in the limiting case with $E=0$. We shall look for solutions which are not singular for $s=0$, and decrease exponentially for large values of r (where $(r^2=-s)$ when $M^2>E^2$. Obviously only such solutions could satisfactorily represent a bound state. It is clear on dimensional grounds that the solutions must be functions of E^2s and $(M^2-E^2)s$, and their behaviour near $s=0$ can therefore be determined correctly by restricting consideration to small values of E, as was done in the previous section. It was seen that there is a discrete infinity of solutions, corresponding to the eigenvalues of the operator $\mathcal{Q}$ in the limit $E=0$. Some of these vary as imaginary powers, some as negative and some as positive powers of s, near $s=0$, and of these only the latter are acceptable. For large values of r, it is easily seen that there are two classes of solutions, varying as $e^{\epsilon r}$ and the other as $e^{-\epsilon r}$, where $\epsilon=(M^2-E^2)^{1/2}$; the condition for the existence of a bound state is that a solution acceptable near $s=0$ should join smoothly to a solution varying as $e^{-\epsilon r}$ for large r.

A positive power of s near $s=0$ corresponds to a repulsive interaction at short distances, similar to the repulsive 'core' interaction whose existence was postulated, but not rigorously demonstrated by Lévy and determined phenomenologically by Jastrow.$^{11)}$ From the precise behaviour of the solution near $s=0$, the equivalent 'core' potential can be determined and this will be one of the objects of the present calculations. We shall first, however, in our relativistic formalism, obtain a precise condition for an eigenvalue of the energy.

Let S_m be the special value of the function S, defined in the previous section and appearing in (4·13). Then S_m will be an even function of x_4 and it will be assumed that S_m be acceptable, in the same sense described above, in respect of its behaviour near $s=0$; this restricts us to the real positive roots of the indicial equations (4·15a) and (4·15b).

Let $\sigma_n{}^A$ and $\sigma_n{}^B$ be two independent solutions of the homogeneous equation

$$\{\mathcal{P}-4n(n+1)+sg(t)\}\sigma=0. \tag{5·1}$$

It will be convenient to choose $\sigma_n{}^A$ so that it varies as $e^{-\epsilon r}$ for larger r, and to normalize $\sigma_n{}^A$ and $\sigma_n{}^B$ in such a way that their Wronskian is unity for $r=0$. Then the solution of

$$\{\mathcal{P}+\mathcal{Q}+sg(s)\}\sigma_m=ES_m \tag{5·2}$$

which vanished at $s=0$ is

$$\sigma_m=\sum_n \sigma_{m,n} C_{2n}^1(z),$$

$$\sigma_{m.n}=\Big\{\sigma_n{}^A(r)\int_0^r S_{m,n}(r)\,\sigma_n{}^B(r)\,r\,dr$$

$$-\sigma_n{}^B(r)\int_0^r S_{m,n}(r)\,\sigma_n{}^A(r)\,r\,dr\Big\}. \tag{5·3}$$

A solution corresponding to a bound state will be a linear combination of such solutions:

$$\sigma=\sum_{m=0} B_m\sigma_m \tag{5·4}$$

with the coefficient B_m chosen so that σ vanishes at $r=\infty$, i. e. so that

$$\sum_m B_m I_{m,n}=0, \tag{5·5a}$$

where

$$I_{m,n}=\int_0^\infty S_{m,n}(r)\,\sigma_n^A(r)\,r\,dr. \tag{5·5b}$$

The condition for a bound state is therefore that the determinant

$$\det I=|I_{m,n}|=0. \tag{5·6}$$

The formulae required for $S_{m,n}(r)$ and the computation of $I_{m,n}$ are listed in the appendix to this paper.

As a first approximation, the determinantal equation might be solved by setting the diagonal terms equal to zero. For practical purposes, however it is unnecessary to calculate the $I_{m,n}$ and solve this determinantal equation; it is sufficient to determine an effective potential in the neighbourhood of $r=0$.

§ 6. Scattering and effective "Core" potential

In scattering problems there is no restriction upon the solutions of the covariant wave equation like that implied by (5·5) and (5·6): there is an independent solution for each value of the relativistic quantum number m. If $E^2>M^2$, each of the solutions σ_m satisfies the boundary conditions; although the polynomials $C_{2n}^1(z)$, with $z=x_4/s^{1/2}$, are singular on the light cone, they are multiplied by sufficiently high powers of s to ensure that there is no singularity of the solutions at $s=0$. For each value of m we therefore define a relativistic 'core' potential $h_m(s,\ z)$ by the equation

$$\begin{gathered}\{\Box+M^2-E^2+g(s)+h_m(s,\ z)\}\sigma_m=0,\\ sh_m(s,\ z)\sigma_m=-ES_m.\end{gathered} \tag{6·1}$$

It is, however, natural to expand σ_m in Gegenbauer polynomials as in (4·12), and one then obtains for $\sigma_{m,n}$ the similar equations

$$\begin{gathered}[\not{D}+s\{g(s)+h_{m,n}(s)\}]\sigma_{m,n}=0,\\ sh_{m,n}(s)\sigma_{m,n}=-\{4n(n+1)\sigma_{m,n}+ES_{m.n}\}.\end{gathered} \tag{6·2}$$

The relativistic wave equation, with $s=-r^2$, becomes

$$\left\{-\left(\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}\right)+g(-r^2)-h_{m,n}(-r^2)\right\}\sigma_{m,n}=(E^2-m^2)\sigma_{m,n}$$

Near $s=0$, $$h_{m,n}(s)\sim-4\{\lambda+\nu(\nu+1)\}s^{-1};$$

a more exact value for somewhat larger values of s can be obtained from the formulae (A1) and (A2) of the appendix.

It is clear that $h_{m,n}(-r^2)$ has a repulsive term which more than cancels the attraction term $-4\lambda/r^2$ in $g(-r^2)$ near $r=0$. Here there is a rigorous demonstration of the phenomenon suggested by Lévy that relativistic effects combine to make the effective potential repulsive at short distances. It seems likely, though we have not succeeded in proving rigorously, that $h_{m,n}(-r^2)$ becomes small compared with $g(-r^2)$ for larger values of r, so that the attractive potential persists for values of r comparable with μ^{-1}. If one believes the results obtained with the instantaneous interaction approximation, the reversal of the forces occurs for the value of r given by $V(r)=M$.

§ 7. States of higher angular momentum

The discussion of §§ 4–6 has been limited to states with $J=0$. We shall now show how to solve equations (2·8) for states of higher angular momentum, restricting ourselves for the present to the case $p=0$, when exact solutions can be obtained. It will be obvious that such solutions furnish a starting point for development in powers of E^{2}, similar to those obtained in § 5 for $J=0$.

The angular momentum, including the spin angular momentum, is given by the tensor

$$\mathcal{H}_{\lambda\mu}=i(x_\lambda \nabla_\mu - x_\mu \nabla_\lambda)+\mathcal{S}_{\lambda\mu}, \tag{7·1}$$

where $\mathcal{S}_{\lambda\mu}$ is defined by

$$\mathcal{S}_{\lambda\mu}\Psi=1/4\cdot i\{(\gamma_\lambda\gamma_\mu-\gamma_\mu\gamma_\lambda)\Psi-\Psi(\gamma_\lambda\gamma_\mu-\gamma_\mu\gamma_\lambda)\}, \tag{7·2}$$

if Ψ is any bispinor (Dirac matrix). Let equation (2·2) be written symbolically in the form

$$\mathcal{B}\Psi=0\,;$$

then although $\mathcal{H}_{\lambda\mu}$ is constant of motion, all its components do not commute with the Bethe-Salpeter operator $\mathcal{B}$, in general. When $p=0$, however, solutions of $\mathcal{B}\Psi=0$ can be found which are simultaneous eigenfunctions of the operators $\mathcal{H}_z$, J and $\mathcal{H}$ defined by

$$\begin{aligned}&\mathcal{H}_z=\mathcal{H}_{12},\\ &J(J+1)=\mathcal{H}_{23}^2+\mathcal{H}_{31}^2+\mathcal{H}_{12}^2,\\ &\mathcal{H}(\mathcal{H}+2)=-1/2\cdot\mathcal{H}^{\lambda\mu}\mathcal{H}_{\lambda\mu}=\mathcal{H}_{14}^2+\mathcal{H}_{24}^2+\mathcal{H}_{34}^2-J(J+1),\end{aligned} \tag{7·3}$$

when $p\neq0$, J commutes with $\mathcal{B}$ in the barycentric frame, but $\mathcal{H}$ does not. However, it is always possible to define an operator $\mathcal{K}$ which reduces to $\mathcal{H}$ when $p=0$ and commutes with $\mathcal{B}$; it has the property that

$$\mathcal{K}\sigma_m=k\sigma_m=2m\sigma_m \tag{7·4}$$

where σ_m is one of the solutions obtained in § 4. Thus $\mathcal{K}$ is the operator whose eigenvalue is the relativistic quantum number.

When $p=0$, if Ψ is one solution of $\mathcal{B}\Psi=0$, $\mathcal{H}_{34}\Psi$ and $(\mathcal{H}_{14}\pm i\mathcal{H}_{24})\Psi$ are also solutions. It is easy to see that if Ψ is a solution with $J=0$ but $\mathcal{H}\neq0$, $\mathcal{H}_{34}\Psi$ and $(\mathcal{H}_{14}\pm i\mathcal{H}_{24})\Psi$ are solutions with $J=1$ and $\mathcal{H}_z=0$ and ±1 respectively. For, by hypothesis $\mathcal{H}_{12}\Psi=\mathcal{H}_{23}\Psi=\mathcal{H}_{31}\Psi=0$, so using

$$[\mathcal{H}_{\lambda\mu},\ \mathcal{H}_{\nu\rho}]=-i(g_{\lambda\nu}\mathcal{H}_{\mu\rho}-g_{\mu\rho}\mathcal{H}_{\lambda\rho}-g_{\lambda\rho}\mathcal{H}_{\mu\nu}+g_{\mu\nu}\mathcal{H}_{\lambda\nu})$$

one finds

$$J(J+1)\mathcal{N}_{34}\Psi=2\mathcal{N}_{34}\Psi$$

and

$$\mathcal{N}_z\mathcal{N}_{34}\Psi=0,$$

but

$$\mathcal{N}_z(\mathcal{N}_{14}\pm i\mathcal{N}_{24})\Psi=\pm(\mathcal{N}_{14}\pm i\mathcal{N}_{24})\Psi.$$

Also if Ψ is a solution with $J=0$ and $\mathcal{N}>J'$, then solutions with $J=J'$ are obtained on multiplying Ψ by certain homogeneous polynomials of degree J' in $\mathcal{N}_{14}$, $\mathcal{N}_{24}$ and $\mathcal{N}_{34}$. For example when $J'=2$, these polynomials are $(\mathcal{N}_{14}\pm i\mathcal{N}_{24})^2$ for $\mathcal{N}_z=\pm 2$, $(\mathcal{N}_{14}\pm i\mathcal{N}_{24})\mathcal{N}_{34}$ for $N_z=\pm 1$, and $2\mathcal{N}_{34}^2-\mathcal{N}_{14}^2-\mathcal{N}_{24}^2$ for $N_z=0$.

These results enable us to infer solutions for $J\neq 0$ from those already obtained for $J=0$. The solutions for $J=0$ are explicitly, in barycentric coordinates

$$\begin{aligned}
\Psi_m&=(i\Gamma+M)\omega_m+\omega_m(-i\Gamma+M)+\theta_m,\\
\omega_m&=s^{-1/2}\xi_m\boldsymbol{x}\,C_{2m+1}^1(z)+2\eta_m(\gamma_4-x_4x/s)\,C_{2m}^2(z),\\
\theta_m&=\sigma_m C_{2m}^1(z)+2is^{-1}\tau_m[\boldsymbol{x},\gamma_4]\,C_{2m}^2(z),
\end{aligned}\tag{7·5}$$

where $z=x_4s^{-1/2}$ and according to (4·5) σ_m is a solution of

$$\{\not{\mathcal{D}}-4m(m+1)+sg(s)\}\sigma_m=0\tag{7·6}$$

and ξ_m, η_m, and $\zeta_m=\tau_m+\eta_m$ are coupled by

$$\begin{aligned}
&\{\not{\mathcal{D}}-2-4(m+1)^2-sg(s)\}\xi_m+2\{4(m+1)^2-1\}\zeta_m=0,\\
&\{\not{\mathcal{D}}+2-4(m+1)^2+sg(s)\}\zeta_m+2\xi_m=2s\{g(s)+sg^1(s)\}\eta_m,\\
&\not{\mathcal{D}}-4s\cdot\partial/\partial s-4(m+1)^2-sg(s)\}\eta_m+2\xi_m+2(2s\cdot\partial/\partial s+1)\zeta_m=0.
\end{aligned}\tag{7·7}$$

$$(\not{\mathcal{D}}=4s^2\cdot\partial^2/\partial s^2+8s\cdot\partial/\partial s+\epsilon^2 s)$$

When $p\neq 0$ the $\mathcal{N}_{\lambda\mu}$ no longer commutes with $\mathcal{B}$, so that new solutions of $B-S$ equation are no longer be derived by applying operator involving $\mathcal{N}_{\lambda 4}$ to the solution for $J=0$. However it seems certain that operator $\mathcal{K}_{\lambda\mu}$ commuting with $\mathcal{B}$ and reducing to $\mathcal{N}_{\lambda\mu}$ when $p=0$ must exist; such operators would bear the same relation

$$-1/2\cdot\mathcal{K}^{\lambda\mu}\mathcal{K}_{\lambda\mu}=\mathcal{K}(\mathcal{K}+2)\ \text{to}\ \mathcal{K}\ \text{as}\ \mathcal{N}_{\lambda\mu}$$

bear to $\mathcal{N}$. In fact, the operators are easily determined for a pair of non-interacting particles and are given by

$$\mathcal{K}_{\lambda\mu}=\mathcal{N}_{\lambda\mu}+(2ip^2)^{-1}(x_\lambda p_\mu-x_\mu p_\lambda)\int_{-\infty}^{(p\cdot x)}d(p\cdot x)\,\{\Box+M^2-p^2\}.\tag{7·8}$$

§ 8. The relativistic quantum number

An important feature of these solutions of $B-S$ equation obtained without the use of the instantaneous interaction approximation is the emergence of a new quantum number k, which we have called the relativistic quantum number since it is invariant under

Lorentz transformation. The meaning of the new quantum number is that one has a discrete infinity of quantum mechanical states corresponding to each of the states predicted by non-relativistic quantum mechanics. It is easily seen that a pair of similar non-interacting particles are in a state characterized by $k=0$. For the $B-S$ wave function for such a pair of particles depends on the time coordinate as $\exp\{-iE(x_4^{(1)}+x_4^{(2)})\}$ and does not involve the relative time. Since the relativistic quantum number is a constant of motion, the value zero will be preserved in all interactions which do not involve the real creation of new particles.

The relativistic quantum number is not simply a characteristic of $B-S$ equation but of any relativistic wave equation for two elementary particles. It is a transparent consequence of the existence of an additional variable—the relative time—in the wave equation. It was first discussed to the authors' knowledge, in papers by Born and Fuchs[14]; its application to the classification of the new particles was suggested by Born and Green.[15] When the properties of these particles became better known, it was clear that the stability of the Λ- and θ-particles required selection rules additional to those predicted by ordinary quantum mechanics, and Pais[13] has already proposed that these selection rules should be associated with the quantization of the components of an angular momentum tensor in an abstract euclidean 4-dimensional space. Pais' theory is a generalization of an earlier theory of Pais himself[17] and Gell-Mann,[18] which differs from it in the substitution of an isotopic angular momentum for the general angular momentum tensor. The properties of our tensor $\mathcal{H}_{\lambda\mu}$ are substantially the same as those of Pais' tensor, and if one could regard the strange particles as bound or excited states of two or more particles, there might indeed be a close relation between them. But, for the $B-S$ equation at least, the relevant constant of the motion is not connected exactly with the tensor $\mathcal{H}_{\lambda\mu}$, except when $p=0$, but with the tensor $\mathcal{K}_{\lambda\mu}$ which commutes with the $B-S$ operator.

It may be added, by way of confirmation, that a fourth constant of the motion appears also in the solutions of Wick's modification of the Bethe-Salpeter equation.[19] There, again, non-relativistic approximation excludes the solutions corresponding to all but one value of this constant; but there seems to be no reason to prefer the surviving solution to the others.

Appendix

The various coefficients appearing in the § 4 will now be given here.

To determine $S_{m,n}^{(j)}$ which appeared in eq. (4·14) we utilise eq. (4·6b) to get the result

$$S_{m,n}^{(j)}=(\nu_j+n+1)[\{1-4(n+1)^2\}\eta_{m,n}^{(j)}+(2\nu_j+3)\xi_{m,n}^{(j)}]$$
$$+(\nu_j-n)[\{1-4n^2\}\eta_{m,n-1}^{(j)}+(2\nu_j+3)\xi_{m,n-1}^{(j)}] \tag{A·1}$$

where $\nu_j=\nu+j$.

The coefficient $\sigma_{m,n}^{(j)}$ occurring in eq. (4·12) is given by

$$\sigma_{m,n}^{(j)}=\frac{ES_{m,n}^{(j)}}{4\{\nu_j(\nu_j+1)+\lambda-n(n+1)\}}-\epsilon^2\sigma_{m,n}^{(j-1)}\,. \tag{A·2}$$

In deducing (A·2) we have made use of (4·5c) ; $S_{m,n}^{(j)}$ is given in (A·1).

From (4·4) and (4·6b) we have for $T_{m,n}^{(j)}$ the following expression :

$$\begin{aligned}T_{m,n}^{(j+1)}=&E\Big[\frac{(\nu_j+n+2)}{2n+3}\xi_{m,n+1}^{(j)}-\Big\{\frac{2\nu_j+1}{2n+3}+(n-\nu_j+2)\Big\}\eta_{m,n+1}^{(j)}\Big]\\&-E\Big[\frac{(2\nu_j+4n+3)}{(2n+1)(2n+3)}\xi_{m,n}^{(j)}-\Big\{\frac{2(2\nu_j+1)}{(2n+1)(2n+3)}+\nu_j\Big\}\eta_{m,n}^{(j)}\Big]\\&-E\Big[\frac{(\nu_j-n)}{2n+1}\xi_{m,n-1}^{(j)}-\Big\{\frac{2\nu_j+1}{2n+1}+n+\nu_j\Big\}\eta_{m,n-1}^{(j)}\Big]\,.\end{aligned} \tag{A·3}$$

From (4·6b) we deduce

$$\begin{aligned}X_{m,n}^{(j+1)}&=1/2\,(\sigma_{m,n}^{(j)}+\sigma_{m,n+1}^{(j)}),\\Y_{m,n}^{(j+1)}&=\frac{1}{2}\Big(\frac{\sigma_{m,n}^{(j)}}{2n+1}-\frac{\sigma_{m,n+1}^{(j)}}{2n+3}\Big).\end{aligned} \tag{A·4}$$

Again with the help (A·4) and (4·7) we find the recurrence relation between $\xi_{m,n}^{(j+1)}$ and $\xi_{m,n}^{(j)}$:

$$\xi_{m,n}^{(j+1)}=$$

$$\frac{\{4(\nu_j+1)(\nu_j+2)-4(n+1)^2+2+4\lambda\}(EX_{m,n}^{(j+1)}-\epsilon^2\xi^{(j)})+2(1-4(n+1)^2)EY_{m,n}^{(j+1)}+ET_{m,n}^{(j+1)}-\varepsilon^2\zeta_{m,n}^{(j)})}{D\{4(\nu_j+1)(\nu_j+2),\ 1-4(n+1)^2\}}\,, \tag{A·5}$$

the recurrence relation between $\zeta_{m,n}^{(j+1)}$ and $\zeta_{m,n}^{(j)}$ is given by

$$\zeta_{m,n}^{(j+1)}=\frac{\{4(\nu_j+1)(\nu_j+2)-4(n+1)^2-2-4\lambda\}(EY_{m,n}^{(j+1)}+ET_{m,n}^{(j+1)}-\varepsilon^2\xi_{m,n}^{(j)})-2(EX_{m,n}^{(j+1)}-\varepsilon^2\xi_{m,n}^{(j)})}{D\{4(\nu_j+1)(\nu_j+2),\ 1-4(n+1)^2\}} \tag{A·6}$$

where $D(P, Q)$ is given by eq. (4·8).

Finally for the recurrence relation of $\eta_{m,n}^{(j+1)}$ and $\eta_{m,n}^{(j)}$ we have from (4·5b)

$$\eta_{m,n}^{(j+1)}=\frac{EY_{m,n}^{(j+1)}-\epsilon^2\eta_{m,n}^{(j)}-2\xi_{m,n}^{(j+1)}-(2\nu_j+3)\zeta_{m,n}^{(j+1)}}{D'\{4(\nu_j+1)(\nu_j+2),\ 1-4(n+1)^2\}} \tag{A·7}$$

where $D'(P, Q)$ is given by (4·11).

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The Repulsive Potential between Two Normal Helium Atoms

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The repulsive potential between two normal helium atoms is calculated using the LCAO MO method. The main idea is that we use two different effective charges for σ_g and σ_u orbitals, dropping the restriction of a common charge for the both orbitals. It is found that remarkable improvements are achieved for the theoretical calculation but there still remains a large discrepancy between theory and experiment. A critical discussion of the present status of theoretical calculations is attempted in some detail.

§ 1. Introduction and general remarks

The liquid helium and the atomic nucleus present interesting common features in several respects as the quantum mechanical many-body problems. They are both "condensed" systems and the behaviors of the constituent particles are believed to correlate each other in some very complicated way. Yet, strange to say, it seems that some gas-like models or independent particle models can claim their rights as meaningful first approximations of real situation for both cases, that is, the Bose-gas model for the liquid helium and the Fermi-gas model and the so-called shell model for the nucleus. To find the reason for this is one of the important problems awaiting a clear-cut answer from theoretical physicists.

In the problem of the nuclear matter one of the most serious difficulties is that we lack the exact knowledge of the interaction between nucleons. The present meson theory provides only a partial and incomplete answer to it and the difficulty is closely related to the fundamental problem of the quantum field theory in general. The two-body force might be determined through the analysis of scattering experiments but there is no positive evidence that the many-body effect could be negligible in the nuclear matter.

As for the interaction between helium atoms, there is no obstacle, in principle, that prevents us to reach the final answer by theoretical calculations. The fact is, however, that we do not know the interaction between helium atoms with sufficient accuracy, just like in the problem of the nucleon interaction. One of the reasons for this situation is probably that the two helium atoms do not form a stable molecule, hence they are devoid of apparent chemical interest. However, the He-He interaction is the simplest case of the interaction between two closed-shell electron clouds, which must be quite important also for chemistry.

About thirty years ago, Slater[1] calculated the repulsive potential between two normal helium atoms and obtained the following approximate expression,

$$V(R)=7.7\times10^{-10}\exp(-4.60R)\ \text{erg}\quad (R\ \text{in}\ \mathring{A}) \tag{1·1}$$

for the potential as a function of the internuclear distance R. Later, the inverse six-power dispersion energy was calculated separately[2)] and added to the above potential:

$$[770\exp(-4.60R)-(1.49/R^6)]\times10^{-12}\ \text{erg}\quad (R\ \text{in}\ \mathring{A}). \tag{1·2}$$

This is so-called Slater-Kirkwood potential and has been widely used.

In Slater's calculation of the repulsive valence force a very accurate atomic wave function was used, but certain multiple exchange integrals were neglected and several integrals were only approximately calculated. Rosen[3)] criticized this point and performed his calculation with a simpler but less accurate atomic wave function so that all integrals could be evaluated. He used, for the spatially unsymmetrized wave function for the unperturbed helium atom,

$$\Psi\sim\exp[-(2.15r_1+1.19r_2)/a_0]\quad (a_0:\ \text{Bohr radius}) \tag{1·3}$$

and obtained results which can be summarized in the form,

$$V(R)=9.25\times10^{-10}\exp(-4.40R)\ \text{erg}\quad (R\ \text{in}\ \mathring{A}). \tag{1·4}$$

This formula gives considerably higher values of interaction energy than Slater's formula (1·1). Both Slater and Rosen performed their calculation using the valence bond method.

As for the experimental data, rather an accurate potential at small internuclear distances is provided by the recent work of Amdur and Harkness[4] (A–H). The result is

$$\left.\begin{array}{lll} 2.884/R^{1.79}\ \text{eV} & \text{for} & 0.52<R<1.02 \\ 4.713/R^{5.94}\ \text{eV} & \text{for} & 1.27<R<1.59 \end{array}\right\} (R\ \text{in}\ \mathring{A}). \tag{1·5}$$

When we compare the experimental values calculated with the above formulas with Rosen's theoretical result, we find serious disagreement between them, especially at small separations. At $R=1.0$ au (atomic units) (1·4) gives 56 eV while (1·5) yields only 9 eV and so the discrepancy amounts to 47 eV.

Griffing and Wehner[5)] (G-W) were the first who adopted the antisymmetrized molecular orbital (ASMO) method for the present problem. They used simple Slater-type 1s-functions, χ_a and χ_b, as AO's and constructed two LCAO MO's:

$$\begin{array}{l} \varphi_1\sim\chi_a+\chi_b;\ \sigma_g \\ \varphi_2\sim\chi_a-\chi_b;\ \sigma_u \end{array} \tag{1·6}$$

where

$$\chi_a=(Z^3/\pi)^{1/2}\exp(-Zr_a) \tag{1·7}$$

and similarly for χ_b with $Z=27/16$. Using these molecular orbitals they calculated the He-He interaction energy by ASMO method and could obtain a better result than previous calculations. It is well known that, so long as we use an atomic orbital of the form (1·7), the ASMO method is just equivalent to the simple Heitler-London theory in the present problem. Comparing the result of G-W with that of Rosen, we notice rather a curious fact: Rosen started with a wave function for an isolated He atom which is better than the one of G-W, and the result is that his energy values are considerably higher than the values obtained by the simpler calculation of G-W.

Recently Sakamoto and Ishiguro[6] (S-I) performed an interesting calculation on the present subject. They attempted to take into account the deformation of atomic orbitals due to the approach of another atom by making use of the modified atomic orbital of Inui[7] and Nordsieck[8], which was successful in the treatment of the ground state of H_2 molecule. The repulsive interaction energy of the He-He system obtained by S-I is lower than that of G-W but it is to be noted that the difference between these two results is rather small, namely within 1 eV over nearly the whole range of the internuclear distance in discussion and if we compare the so far best theoretical result of S-I with the experiment of A-H, we still find a large discrepancy which amounts to about 30 eV at the internuclear distance of 0.5 Å. If the experimental result of Amdur and Harkness is reliable, this discrepancy must be attributed to some defects in the theoretical estimations.

In this connection the present author has attempted a simple united-atom treatment of the problem[9] at small internuclear distances where the discrepancy between theory and experiment remains quite large. In place of φ_1 and φ_2 given in $(1\cdot6)$ and $(1\cdot7)$, we here use the following united-atom wave functions,

$$\begin{aligned}\varphi_1&=(1s\sigma\,;\,\zeta)=N(1,\,\zeta)\exp(-\zeta r)\,Y_{00};\ \sigma_g\\ \varphi_2&=(2p\sigma\,;\,\eta)=N(2,\,\eta)\,r\exp(-\eta r)\,Y_{10}\,;\ \sigma_u\end{aligned}\tag{1·8}$$

where $N(n,\,\zeta)=(2\zeta)^{n+(1/2)}/[(2n)!]^{1/2}$ and the coordinates of electrons are measured from the middle point between two nuclei. ζ and η are variable parameters. It is clear that this approximation is a very crude one, especially in φ_1, because it is unable to describe the concentration of the electron cloud at the positions of two nuclei. In spite of this apparent crudeness, this simple treatment gives a better result at the internuclear distance $R=0.8$ au than the elaborate valence bond theory of S-I does. Even at $R=1.0$ au our result is definitely better than that of Rosen. A slight improvement on φ_1 yields 34.7 eV as the repulsive interaction energy at $R=1.0$ au, which is 3.7 eV lower than the corresponding value of S-I, 38.4 eV. This is rather an unexpected situation and it may not be unreasonable to infer that, at the internuclear distance of about 1 au, the total wave functions used by various authors in their more or less elaborate calculations would bear some definite defects.

Griffing and Wehner used the molecular orbitals given in $(1\cdot6)$ and $(1\cdot7)$ and took the same fixed value of the effective charge for $\chi_a+\chi_b$ $(1s\sigma_g)$ and $\chi_a-\chi_b$ $(1s\sigma_u)$ orbitals. This procedure seems to be too restrictive. It is instructive for us to remind that in the H_2^+ ion problem there exists a remarkable difference between the manners of variations of effective charges for $(1s\sigma_g)$ and $(1s\sigma_u)$ orbitals of LCAO type: at smaller internuclear distances the value for $(1s\sigma_g)$ rises up steeply and the one for $(1s\sigma_u)$ falls down rapidly. This is displayed, for example, in Fig. 4.6 on P. 85 of Coulson's text book[10]. In addition, for the present He-He system we must take account of the exclusion effect of the Pauli principle and this should reinforce the requirement of the separate variation of effective charges for σ_g and σ_u orbitals. In short, σ_g and σ_u orbitals must reduce to $1s$ and $2p$ atomic orbitals, respectively, of Be in the united-atom limit and to cope with this physical requirement we should, at least, drop the severe restriction of equal effective charges.

Thus we are naturally led to the adoption of two different values of effective charges for $(1s\sigma_g)$ and $(1s\sigma_u)$ orbitals. This simple modification of the calculation of Griffing and Wehner brings us a significant improvement of the theoretical values of the He-He repulsive interaction, which will be described in the following.

§ 2. Calculation and result

The system consists of doubly charged helium nuclei a and b separated by a distance R with four electrons. Our problem is to construct a suitable wave function which makes the energy of the system as low as possible within limited labor. In the present calculation we are going to use, as the total wave function, an antisymmetrized product of orthonormal molecular spin orbitals, which are products of MO's with the usual spin eigenfunctions. As stated in the last part of the preceding section, the MO's assume the following forms:

$$\begin{aligned}\varphi_g&=N_g(\chi_a+\chi_b)\ ;\ \sigma_g\\ \varphi_u&=N_u(\chi_a'-\chi_b')\ ;\ \sigma_u\end{aligned}\tag{2·1}$$

where

$$N_g=1/[2+2(\chi_a|\chi_b)]^{1/2},\quad N_u=1/[2-2(\chi_a'|\chi_b')]^{1/2},$$

$$(\chi_a|\chi_b)=\int\chi_a\chi_b dV,\qquad (\chi_a'|\chi_b')=\int\chi_a'\chi_b' dV.$$

χ and χ' are simple $1s$-functions;

$$\begin{aligned}\chi&=(\zeta^3/\pi)^{1/2}\exp(-\zeta r),\\ \chi'&=(\zeta'^3/\pi)^{1/2}\exp(-\zeta' r).\end{aligned}\tag{2·2}$$

The point is that we assign different values of the effective charges ζ and ζ' for σ_g and σ_u orbitals separately. In the calculation of G-W. $\zeta=\zeta'=27/16$.

The total wave function is thus:

$$\psi=(1/24)^{1/2}\sum_\lambda(-1)^\lambda P_\lambda[\varphi_g(1)\alpha(1)\varphi_g(2)\beta(2)\varphi_u(3)\alpha(3)\varphi_u(4)\beta(4)]\tag{2·3}$$

where P_λ is the operation permuting the electron coordinates, and α and β are the usual orthonormal spin eigenfunctions. The energy operator for the atomic units*

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* We use the following atomic units:

Length: $a_0=0.529171\times10^{-8}$cm (Bohr radius),

Energy: $e^2/a_0=27.2100$ eV (twice the ionization energy of hydrogen).

$$H=\sum_{i=1}^{4} H(i)+\sum_{i>j=1}^{4}(1/r_{ij})+(4/R), \qquad (2\cdot4)$$

where

$$H(i)=-(1/2)\Delta_i-(2/r_{ai})-(2/r_{bi}).$$

Then the total energy of the system is

$$E=\int \psi H\psi dV=2H_g+2H_u+J_{gg}+J_{uu}+4J_{gu}-2K_{gu}+(4/R), \qquad (2\cdot5)$$

where

$$H_g=\int \varphi_g(1)H(1)\varphi_g(1)dV_1,$$

$$H_u=\int \varphi_u(1)H(1)\varphi_u(1)dV_1,$$

$$J_{gg}=\iint \varphi_g(1)\varphi_g(2)(1/r_{12})\varphi_g(1)\varphi_g(2)dV_1dV_2,$$

$$J_{uu}=\iint \varphi_u(1)\varphi_u(2)(1/r_{12})\varphi_u(1)\varphi_u(2)dV_1dV_2,$$

$$J_{gu}=\iint \varphi_g(1)\varphi_u(2)(1/r_{12})\varphi_g(1)\varphi_u(2)dV_1dV_2,$$

$$K_{gu}=\iint \varphi_g(1)\varphi_u(2)(1/r_{12})\varphi_u(1)\varphi_g(2)dV_1dV_2.$$

Next we decompose these integrals into more basic integrals expressed in χ and χ' and then start the routine work of evaluating a number of integrals.

It seems convenient to introduce the following notations and definitions according to the work of Roothaan and Rudenberg[11],

$$\chi_a\to 1s_a\to s_a, \quad \chi_b\to 1s_b\to s_b,$$

$$\chi_a'\to 1s_a'\to s_a', \quad \chi_b'\to 1s_b'\to s_b',$$

$$\int \chi_a\chi_b dV=(\chi_a|\chi_b)=(s_a|s_b), \qquad (2\cdot6)$$

$$\int \chi_a(1/r_a)\chi_b dV=(\chi_a|1/r_a|\chi_b)=(s_a|1/r_a|s_b), \qquad (2\cdot7)$$

$$\int \chi_b(1/r_a)\chi_b dV=(\chi_b|1/r_a|\chi_b)=(s_b|1/r_a|s_b), \qquad (2\cdot8)$$

and the similar formulas for the case of χ'. The above are the one-electron integrals. As for the two-electron integrals, Coulomb, exchange and hybrid types of integrals appear in the present calculation. These integrals can be expressed generally in the form,

$$[\Omega(1)|\bar{\Omega}(2)]=\iint \Omega(1)(1/r_{12})\bar{\Omega}(2)dV_1dV_2, \qquad (2\cdot9)$$

where Ω and $\bar{\Omega}$ are charge distributions consisting of the product of two atomic orbitals. For example,

$$[s_a(1)s_a(1)|s_b(2)s_b(2)]=\iint \chi_a(1)\chi_a(1)\,(1/r_{12})\,\chi_b(2)\chi_b(2)\,dV_1dV_2, \quad \text{(Coulomb)} \tag{2·10}$$

$$[s_a(1)s_b(1)|s_a(2)s_b(2)]=\iint \chi_a(1)\chi_b(1)\,(1/r_{12})\,\chi_a(2)\chi_b(2)\,dV_1dV_2, \quad \text{(exchange)} \tag{2·11}$$

$$[s_a(1)s_a(1)|s_a(2)s_b(2)]=\iint \chi_a(1)\chi_a(1)\,(1/r_{12})\,\chi_a(2)\chi_b(2)\,dV_1dV_2, \quad \text{(hybrid)} \tag{2·12}$$

and others may be inferred according to the above examples. Now we can write down the terms in (2·5) by using the integrals defined above:

$$\begin{aligned}
H_g&=2N_g^2\{(1/2)(\alpha/R)^2-2(\alpha/R)-2(s_b|1/r_a|s_b)+[(\alpha/R)-4](s_a|1/r_a|s_b)\\
&\qquad -(1/2)(\alpha/R)^2(s_a|s_b)\},\\
H_u&=2N_u^2\{(1/2)(\beta/R)^2-2(\beta/R)-2(s_b'|1/r_a|s_b')+[(\beta/R)-4](s_a'|1/r_a|s_b')\\
&\qquad -(1/2)(\beta/R)^2(s_a'|s_b')\},\\
J_{gg}&=2N_g^4\{[s_as_a|s_as_a]+4[s_as_a|s_as_b]+[s_as_a|s_bs_b]+2[s_as_b|s_as_b]\},\\
J_{uu}&=2N_u^4\{[s_a's_a'|s_a's_a']-4[s_a's_a'|s_a's_b']+[s_a's_a'|s_b's_b']+2[s_a's_b'|s_a's_b']\},\\
J_{gu}&=2N_g^2N_u^2\{[s_as_a|s_a's_a']+[s_as_a|s_b's_b']-2([s_as_a|s_a's_b']-[s_a's_a'|s_as_b])-2[s_as_b|s'_as_b']\},\\
K_{gu}&=2N_g^2N_u^2\{[s_as_a'|s_as_a']-[s_as_a'|s_bs_b']-2([s_as_a'|s_as_b']-[s_as_a'|s_a's_b])\\
&\qquad +[s_as_b'|s_as_b']-[s_as_b'|s_a's_b]\},\\
N_g^2&=1/[2+2(s_a|s_b)],\quad N_u^2=1/[2-2(s_a'|s_b')],\quad \alpha=\zeta R,\quad \beta=\zeta' R.
\end{aligned}$$

The explicit analytical expressions for the numerical evaluation of these integrals are found in the Appendix. Because all of them are the integrals between $1s$ atomic orbitals, the integrations can be performed without difficulty. Some of them can be checked both in the analytical forms and in the numerical values with the results of Hirschfelder and Linnett[12]. A little troublesome are $[s_as_b'|s_as_b']$ and $[s_as_b'|s_a's_b]$ in K_{gu} because they are expressed in the form of some infinite series, while all of others are expressed in closed analytical forms in terms of several auxiliary functions. Fortunately the convergence of the infinite series is sufficiently rapid, especially in the combined form of $[s_as_b'|s_as_b']-[s_as_b'|s_a's_b]$, and only the first three or four terms are necessary for the present purpose.

It is desirable to determine the best values of α and β through the continuous variation of these variable parameters, but we did not do that. The first reason is that we entirely rely upon the numerical table of Kotani, Amemiya, Ishiguro and Kimura[13] (KAIK), where the auxiliary functions are tabulated with the step of 0.25 of ZR. Besides, while the present theoretical calculation yields a result much better than the previous calculations as will be shown presently, there still remains a large discrepancy between theory and experiment, and we want to reserve our labor for a more improved treatment we are planning now. This is the second reason or excuse why we did not try to look for the strictly best values of α and β. The merit of the present calculation is that it reveals to us what was wrong in the previous theoretical treatments rather than its numerical result itself.

Actual calculations were performed at three values of the internuclear distance R, 1.0, 1.5 and 2.0 au, and the results are summarized in Table 1. E is the total energy of the He-He system and $V(R)$ is the repulsive potential energy defined by $E(R)-E(\infty)$. We took $E(\infty)=-5{,}69531$ au in accordance with Sakamoto-Ishiguro[6] and Griffing-Wehner[5]. The result of the present calculation may be expressed in the following approximate formula :

Table 1. The numerical results of the present calculation.

R(au)	1.0	1.5	2.0
$\alpha=\zeta R$	2.25	3.0	3.5
$\beta=\zeta' R$	1.25	2.0	3.0
E(au)	−4.70584	−5.32781	−5.58131
$V(R)$	0.98947 au (26.92 eV)	0.36750 au (10.0 eV)	0.11400 au (3.10 eV)

$$V(R)=8.58817\exp(-2.16097R)\ \text{au},\ 1<R<2\ \ (R\text{ in au}), \qquad (2\cdot13)$$

This is compared with the corresponding formula of S-I :

$$V(R)=15.02222\exp(-2.36478R)\ \text{au},\ \ (R\text{ in au}).$$

The potential curves due to these two formulas are found in Fig. 1.

§ 3. Discussion

Table 2 and Fig. 1 are prepared for the purpose of comparison of various theoretical estimations between them and with the experimental result reported by Amdur and Harkness[4].

Table 2. He-He repulsive potential (in eV)

R(au)	(1)	(2)	(3)	(4)	(5)	(6)
0.8889	55.58	72.96	51.84	49.94	10.96	34.23
1.0	42.45	56.24	—	38.41	9.011	26.92
1.1858	27.08	36.53	24.86	24.78	6.647	18.02
1.2	21.12	35.30	—	23.94	6.502	17.48
1.4	16.08	22.16	—	14.92	4.934	11.34
1.6	9.892	13.91	—	9.294	3.885	7.363
1.77788	6.423	9.261	6.212	6.115	3.216	5.013
1.8	6.088	8.730	—	5.792	3.147	4.778
2.0	3.747	5.481	—	3.609	2.606	3.102

(1) Slater
(2) Rosen
(3) Griffing and Wehner
(4) Sakamoto and Ishiguro
(5) Experimental result of Amdur and Harkness

(according to Table V of S-I)

(6) Present calculation according to the formula (2·13) of this paper:

$$V(R)=8.58817\exp(-2.16097\,R)\ \text{au},\quad (R\ \text{in au})$$

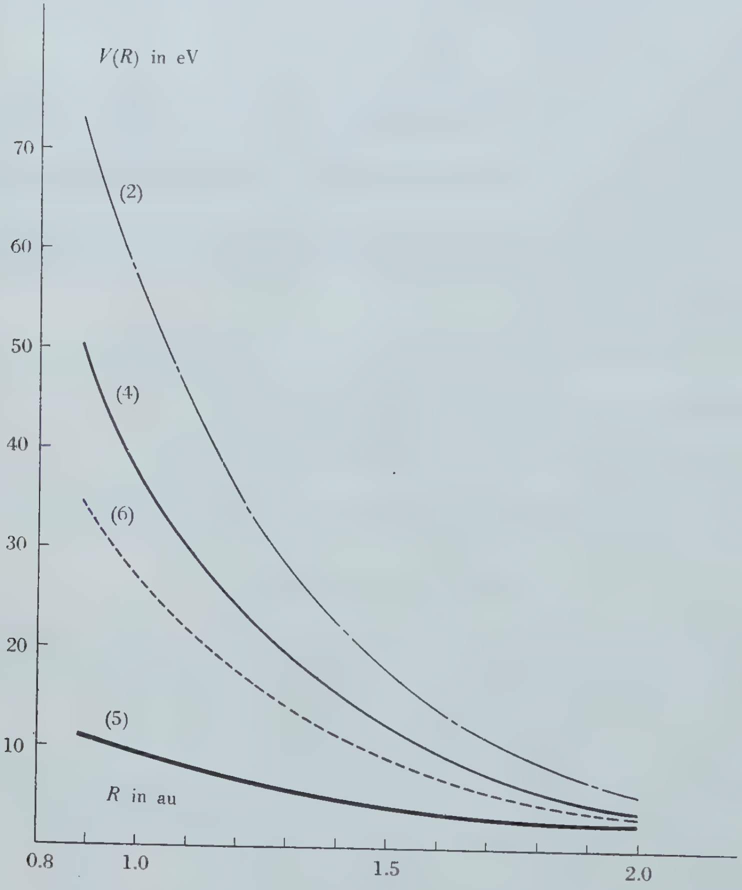

Fig. 1.

It is observed that at $R=2.0$ au there is no serious discrepancy between various calculations but at $R=1.0$ au there arise remarkable diversities and differences. At $R=1.0$ au the value 27 eV of the present calculation is about 11 eV better than the value 38 eV of S-I and is a half of the very large value 56 eV of Rosen.

In this connection, Fig. 2 may be helpful to realize the reason for this significant improvement of the theoretical value at small separation. As stressed before, the essential difference between G-W's treatment and the present one is that we drop the severe restriction of equal effective charge both for σ_g and σ_u orbitals in G-W's treatment. In their calculation $\zeta=\zeta'=27/16=1.6875$ for all values of internuclear distance R, while in our present treatment $\zeta=2.25$ and $\zeta'=1.25$ at $R=1.0$ au. (See Table 1). The situation is shown schematically in the middle part of Fig. 2. The necessity of the separate variations of ζ and ζ' may be recognized most directly when we imagine the united-atom limit, a Be atom, of the present He-He system (right-hand side of Fig. 2).

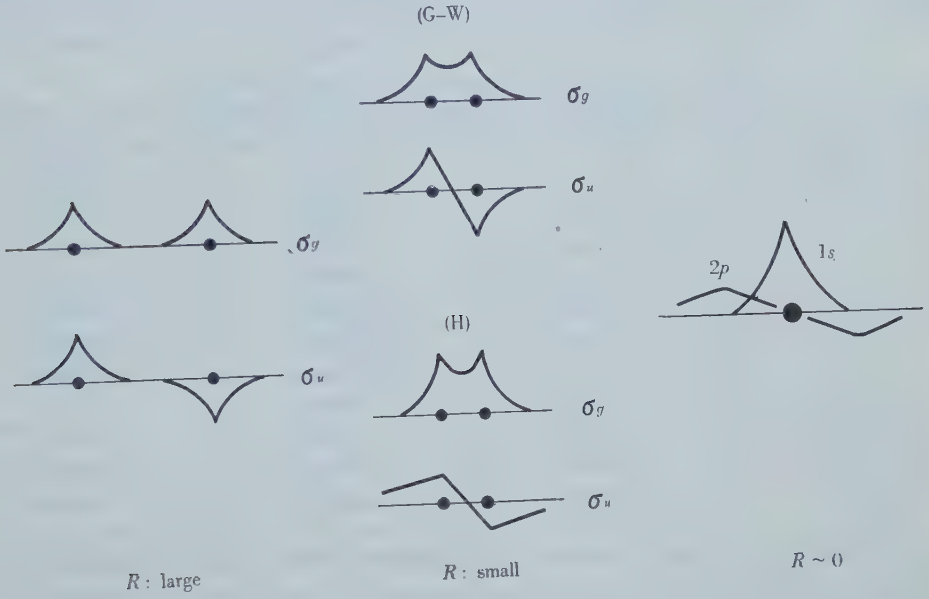

Fig. 2.

In a Be atom, it is usually conceived that two of four electrons occupy the innermost 1*s* orbital and the electron cloud is tightly bound to the nucleus under the influence of bare nuclear charge. The other two electrons are forced to occupy other states because of Pauli exclusion principle. Thus, the ground state configuration of Be is $(1s)^2(2s)^2$ and the level of $(1s)^2(2p)^2$ lies several eV above the ground level. For 2*s* or 2*p* electrons the nuclear charge is screened considerably by the inner two 1*s* electrons. Now let us return to the He-He system. Imagine that we build up the whole system by dividing the procedure into two steps. First, we allot two electrons to the bare field of two nuclei to form, say, a $(\text{He-He})^{++}$ system. This is analogous to the well-examined H_2 problem. In the case of H_2, it is well-known that, as the nuclei approach each other, the effective charge of the 1*s* atomic orbitals increases quite appreciably from the original value 1 at smaller values of the internuclear distance R. It would be natural to expect that the same would happen for the present $(\text{He-He})^{++}$ system. If it is true, the assumption of $\zeta=27/16$ for all values of R would not be the most reasonable value but a little larger values should be taken for small separations. Next, suppose we add the remaining two electrons to the $(\text{He-He})^{++}$ system. Except for too small values of R, these two electrons are expected to occupy $2p\sigma$ state. Then, even if a $2p\sigma$ electron moves in full attraction of the bare nuclei, the effective charge ζ' would decrease considerably from the original value $\zeta'=2$ at small separations, as exemplified in the case of H_2^+ problem[10]. In the present problem this tendency will be reinforced because of the screening of the bare nuclear field by the inner electrons and of the Pauli exclusion principle. Thus, the use of a fixed value $\zeta'=27/16$ is inadequate also for the $2p\sigma$ orbital and, contrary to the case of the $1s\sigma$ orbital, we should take smaller values of ζ' for small separations. Table 3 may help us to understand the situation explained above a little more quantitatively.

Table 3. The values of various energy terms at $R=1.0$ au.

$E_{total}=E_g+E_u+4J_{gu}-2K_{gu}+4,\ E_g=2H_g+J_{gg},\ E_u=2H_u+J_{uu}.$

$\alpha=\zeta R=\zeta$	2.25	2.5	1.75
$\beta=\zeta' R=\zeta'$	1.25	1.0	1.75
H_g	-4.316344	-4.345730	-4.040239
H_u	-2.489686	-2.275115	-2.661907
J_{gg}	$+1.149554$	$+1.239626$	$+0.950354$
J_{uu}	$+0.745940$	$+0.614447$	$+0.985012$
J_{gu}	$+0.862256$	$+0.767899$	$+0.935962$
K_{gu}	$+0.219149$	$+0.166879$	$+0.256689$
$E_g=2H_g+J_{gg}$	-7.483134	-7.451834	-7.130124
$E_u=2H_u+J_{uu}$	-4.233432	-3.935783	-4.338802
$4J_{gu}-2K_{gu}$	$+3.010726$	$+2.737838$	$+3.230470$
E_{total}	-4.705840	-4.649779	-4.238456
$V(R=1)$	26.92 eV	28.45 eV	39.64 eV

Here $E_g=2H_g+J_{gg}$ and $E_u=2H_u+J_{uu}$ are the energies of two electrons occupying $1s\sigma\,(\sigma_g)$ and $2p\sigma\,(\sigma_u)$ orbitals respectively. The case of $\zeta=\zeta'=1.75$ may serve as a substitute for the case of $\zeta=\zeta'=27/16=1.6875$. In fact the value $V=39.6$ eV in this case is very close to those of G-W and S-I. Griffing and Wehner attempted to improve the calculation by minimizing the energy with respect to the effective charge for each distance. But their conclusion was that " the variation of effective nuclear charge is unimportant in the He_2 problem from the view-point of decreasing the energy." This probably due to the cancellation of the effects of opposite direction on σ_g and σ_u orbitals as may be seen in Table 3. Of course, it is quite another problem why the value $\zeta=\zeta'=27/16$, determined by a simple variational calculation on an isolated He atom, happened to be nearly the best one under the restriction of equal effective charges.

In the treatment of Sakamoto and Ishiguro[6], the total wave function is

$$\psi\sim\begin{vmatrix} a(1)\alpha(1) & a(1)\beta(1) & b(1)\alpha(1) & b(1)\beta(1) \\ a(2)\alpha(2) & a(2)\beta(2) & b(2)\alpha(2) & b(2)\beta(2) \\ a(3)\alpha(3) & a(3)\beta(3) & b(3)\alpha(3) & b(3)\beta(3) \\ a(4)\alpha(4) & a(4)\beta(4) & b(4)\alpha(4) & b(4)\beta(4) \end{vmatrix}. \qquad (3\cdot1)$$

Here α and β are the usual spin functions and a and b are the modified AO's introduced by Inui[7] and Nordsieck[8], namely

$$a=e^{-\alpha\xi-\beta\eta},\qquad b=e^{-\alpha\xi+\beta\eta}, \tag{3·2}$$

where ξ and η are the prolate spherical coordinates defined by

$$\xi=(r_a+r_b)/R, \quad \eta=(r_a-r_b)/R,$$

and conversely

$$r_a=(R/2)(\xi+\eta), \quad r_b=(R/2)(\xi-\eta).$$

After simple manipulations of addition and subtraction between four columns we obtain, as described in Seitz's text book[14],

$$\phi\sim\begin{vmatrix} \varphi_g(1)\alpha(1) & \varphi_g(1)\beta(1) & \varphi_u(1)\alpha(1) & \varphi_u(1)\beta(1) \\ \varphi_g(2)\alpha(2) & \varphi_g(2)\beta(2) & \varphi_u(2)\alpha(2) & \varphi_u(2)\beta(2) \\ \varphi_g(3)\alpha(3) & \varphi_g(3)\beta(3) & \varphi_u(3)\alpha(3) & \varphi_u(3)\beta(3) \\ \varphi_g(4)\alpha(4) & \varphi_g(4)\beta(4) & \varphi_u(4)\alpha(4) & \varphi_u(4)\beta(4) \end{vmatrix} \tag{3·3}$$

where

$$\begin{aligned} \varphi_g&=a+b=e^{-\alpha\xi-\beta\eta}+e^{-\alpha\xi+\beta\eta}=2e^{-\alpha\xi}\cosh\beta\eta, \\ \varphi_u&=a-b=e^{-\alpha\xi-\beta\eta}-e^{-\alpha\xi+\beta\eta}=-2e^{-\alpha\xi}\sinh\beta\eta. \end{aligned} \tag{3·4}$$

This is to be compared with the corresponding functions, φ_1 and φ_2, in G-W:

$$\begin{aligned} \varphi_1&\sim e^{-Zr_a}+e^{-Zr_b}=e^{-(ZR/2)(\xi+\eta)}+e^{-(ZR/2)(\xi-\eta)}=2e^{-(ZR/2)\xi}\cosh(ZR/2)\eta, \\ \varphi_2&\sim e^{-Zr_a}-e^{-Zr_b}=e^{-(ZR/2)(\xi+\eta)}-e^{-(ZR/2)(\xi-\eta)}=-2e^{-(ZR/2)\xi}\sinh(ZR/2)\eta, \end{aligned} \tag{3·5}$$

and, further, with φ_g and φ_u in the present treatment:

$$\begin{aligned} \varphi_g&\sim e^{-\zeta r_a}+e^{-\zeta r_b}=e^{-(\zeta R/2)(\xi+\eta)}+e^{-(\zeta R/2)(\xi-\eta)}=2e^{-(\zeta R/2)\xi}\cosh(\zeta R/2)\eta, \\ \varphi_u&\sim e^{-\zeta' r_a}-e^{-\zeta' r_b}=e^{-(\zeta' R/2)(\xi+\eta)}-e^{-(\zeta' R/2)(\xi-\eta)}=-2e^{-(\zeta' R/2)\xi}\sinh(\zeta' R/2)\eta. \end{aligned} \tag{3·6}$$

At $R=1.0$ au, α and β in (3·4) are determined to be 0.81747 and 0.92350 respectively according to S-I, while $(ZR/2)=27/32=0.84375$ in (3·5) and $(\zeta R/2)=1.125$ and $(\zeta' R/2)=0.625$ in (3·6). Now it is clear that there is no significant difference between the total wave function of G-W and of S-I. Then it follows that both of these calculations yield almost the same result in regard to the energy values. The above comparison is also helpful to understand why Sakamoto and Ishiguro were not so successful in spite of their inclusion of two variable parameters, while the present calculation is more successful with the same number of variable parameters. In S-I's treatment the deformation of the electron cloud happened to be small, not because the deformation is really small but because their parameters, α and β, could not afford to produce the necessary deformations explained above for σ_g and σ_u orbitals. One of the simplest ways to improve S-I's calculation would be to take two different values of β in the manner that $\beta > \alpha > \beta'$, instead of one. This modification creates no troubles in the calculation of molecular integrals because β and β' are the coefficients of the variable η. We cannot, however, expect much for this modification because it is α, the coefficient of ξ, which determines the degrees of spread of the σ_g and σ_u orbitals. We cannot expect to obtain a drastically improved result as long as we assign the common α to both σ_g and σ_u orbitals.

We have seen that the present calculation yields a significantly improved result and the reason for this achievement has been discussed. However, if we compare our present result directly with the experiment of Amdur and Harkness4) we find still a large discrepancy remaining between them, especially at small internuclear distances. Then our next problem will be to find how to diminish this large gap. We shall try to count out some of possible causes responsible for this discrepancy, assuming that such discrepancy really exists. The first is that our efforts in looking for the best values of the variational parameters were not exhaustive as mentioned in § 2, but this could be a source of error of 2 or 3 eV at best. The second is that the treatment of the correlation between two inner electrons remains unsatisfactory because we used simple LCAO MO's. The possibility of improvement on this point remains open. The third is the neglect of contributions from other configurations, especially $(1s\sigma)^2(2s\sigma)^2$ and $(1s\sigma)^2(2p\pi_\pm)^2$. At small separations it is probable that the $(1s\sigma)^2(2s\sigma)^2$ configuration plays an important role, because it is the ground state configuration of a Be atom, the united-atom limit of the He-He system. In the present paper we are exclusively discussing the exchange force region of the He-He interaction because we find there a large disagreement between theory and experiment. However, our final aim is a consistent quantum mechanical treatment of the whole range of the interaction force, from the repulsive exchange force region to the attractive van der Waals force region. This has been accomplished for H_2 by Hirschfelder and Linnett12). For this purpose the inclusion of the $(1s\sigma)^2(2p\pi_\pm)^2$ configurations will be necessary. We are attempting to examine these three possibilities.

Finally we should like to spend a few words about the problem of the non-additivity of the force between helium atoms in the exchange force region. Recently Shostak15) examined the problem in the case of the linear configuration of three normal helium atoms. His conclusion is that the non-additivity is considerably large at small internuclear distances. However, it is to be noted that his argument is based on the two-body force due to the calculation of Griffing and Wehner5). It has been shown in the present paper that their calculation is not of sufficient accuracy, especially at small internuclear distances. The problem of the non-additivity must be reexamined carefully by using wave functions of sufficient accuracy.

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Appendix

The expressions are given for the evaluation of numerical values of various integrals in § 2. In the following $\alpha=\zeta R$ and $\beta=\zeta' R$.

(a) H_g, H_u, J_{gg}, J_{uu}.

$$(s_a|s_b)=\frac{\alpha^3}{2}\left[A_2(\alpha)-\frac{1}{3}A_0(\alpha)\right]=\left(1+\alpha+\frac{1}{3}\alpha^2\right)e^{-\alpha},$$

$$(s_b|1/r_a|s_b)=\frac{1}{R}\frac{\alpha^3}{2}[A_1(\alpha)B_0(\alpha)+A_0(\alpha)B_1(\alpha)]=\frac{1}{R}[1-(1+\alpha)e^{-2\alpha}],$$

$$(s_a|1/r_a|s_b)=\frac{1}{R}\alpha^3A_1(\alpha)=\frac{1}{R}\alpha(1+\alpha)e^{-\alpha},$$

$$[s_as_a|s_as_a]=\frac{5}{8}\zeta=\frac{1}{R}\frac{5}{8}\alpha,$$

$$[s_as_a|s_as_b]=\frac{1}{R}\frac{\alpha^3}{4}[4A_1(\alpha)+2\{A_0(2\alpha)B_1(\alpha)-A_1(2\alpha)B_0(\alpha)\}+$$
$$+\alpha\{A_0(2\alpha)B_2(\alpha)-A_2(2\alpha)B_0(\alpha)\}],$$

$$[s_as_a|s_bs_b]=\frac{1}{R}\frac{\alpha^3}{4}\left[A_1(\alpha)B_0(\alpha)+A_0(\alpha)B_1(\alpha)-2A_1(2\alpha)-\alpha\left\{A_2(2\alpha)-\frac{1}{3}A_0(2\alpha)\right\}\right]$$
$$=\frac{1}{R}\left[1-\left(1+\frac{11}{8}\alpha+\frac{3}{4}\alpha^2+\frac{1}{6}\alpha^3\right)e^{-2\alpha}\right],$$

$$[s_as_b|s_as_b]=\frac{1}{R}\frac{\alpha^6}{90}[45W_0{}^0(2,2;\alpha)-30W_0{}^0(2,0;\alpha)+5W_0{}^0(0,0;\alpha)$$
$$+4W_2{}^0(0,0;\alpha)].$$

These are the formulas for H_g and J_{gg}. The corresponding formulas for H_u and J_{uu} can be obtained similarly by replacing $s\rightarrow s'$ and $\alpha\rightarrow\beta$ in the above formulas.

(b) J_{gu}.

$$[s_as_a|s_a's_a']=\frac{\zeta\zeta'}{(\zeta+\zeta')^3}(\zeta^2+3\zeta\zeta'+\zeta'^2)=\frac{1}{R}\frac{\alpha\beta}{(\alpha+\beta)^3}(\alpha^2+3\alpha\beta+\beta^2),$$

$$[s_as_a|s_b's_b']=\frac{1}{R}\frac{\beta^3}{4}[2\{A_1(\beta)B_0(\beta)+A_0(\beta)B_1(\beta)\}$$
$$+2\{A_0(\alpha+\beta)B_1(\alpha-\beta)-A_1(\alpha+\beta)B_0(\alpha-\beta)\}$$
$$+\alpha\{A_0(\alpha+\beta)B_2(\alpha-\beta)-A_2(\alpha+\beta)B_0(\alpha-\beta)\}],$$

$$[s_as_a|s_a's_b']=\frac{1}{R}\frac{\beta^3}{4}[4A_1(\beta)+2\{A_0(\alpha+\beta)B_1(\alpha)-A_1(\alpha+\beta)B_0(\alpha)\}$$
$$+\alpha\{A_0(\alpha+\beta)B_2(\alpha)-A_2(\alpha+\beta)B_0(\alpha)\}],$$

$$[s_a's_a'|s_as_b]=\frac{1}{R}\frac{\alpha^3}{4}[4A_1(\alpha)+2\{A_0(\alpha+\beta)B_1(\beta)-A_1(\alpha+\beta)B_0(\beta)\}$$
$$+\beta\{A_0(\alpha+\beta)B_2(\beta)-A_2(\alpha+\beta)B_0(\beta)\}],$$

$$[s_as_b|s_a's_b']=\frac{1}{R}\frac{(\alpha\beta)^3}{90}[45W_0^0(2,2;\alpha,\beta)-15W_0^0(2,0;\alpha,\beta)$$
$$-15W_0^0(0,2;\alpha,\beta)+5W_0^0(0,0;\alpha,\beta)+4W_2^0(0,0;\alpha,\beta)].$$

(c) K_{gu}

$\gamma=(1/2)(\alpha+\beta),\quad \delta=(1/2)(\alpha-\beta),$

$$[s_as_a'|s_as_a']=\frac{20\zeta^3\zeta'^3}{(\zeta+\zeta')^5}=\frac{1}{R}\frac{20(\alpha\beta)^3}{(\alpha+\beta)^5},$$

$$[s_as_a'|s_bs_b']=\frac{1}{R}\frac{4(\alpha\beta)^3}{(\alpha+\beta)^3}[A_1(\gamma)B_0(\gamma)+A_0(\gamma)B_1(\gamma)-2A_1(2\gamma)$$
$$+\gamma\{(1/3)A_0(2\gamma)-A_2(2\gamma)\}],$$

$$[s_as_a'|s_as_b']=\frac{1}{R}\frac{2(\alpha\beta)^3}{(\alpha+\beta)^3}[2\{A_1(\gamma)B_0(\delta)-A_0(\gamma)B_1(\delta)\}$$
$$+2\{A_0(2\gamma)B_1(\alpha)-A_1(2\gamma)B_0(\alpha)\}+\gamma\{A_0(2\gamma)B_2(\alpha)-A_2(2\gamma)B_0(\alpha)\}],$$

$$[s_as_a'|s_a's_b]=\frac{1}{R}\frac{2(\alpha\beta)^3}{(\alpha+\beta)^3}[2\{A_1(\gamma)B_0(\delta)+A_0(\gamma)B_1(\delta)\}$$
$$+2\{A_0(2\gamma)B_1(\beta)-A_1(2\gamma)B_0(\beta)\}+\gamma\{A_0(2\gamma)B_2(\beta)-A_2(2\gamma)B_0(\beta)\}],$$

$$[s_as_b'|s_as_b']=\frac{1}{R}\frac{(\alpha\beta)^3}{8}\sum_{\tau=0}^{\infty}(2\tau+1)[G_\tau^0(0,\delta)^2W_\tau^0(2,2;\gamma)$$
$$-2G_\tau^0(0,\delta)G_\tau^0(2,\delta)W_\tau^0(2,0;\gamma)+G_\tau^0(2,\delta)^2W_\tau^0(0,0;\gamma)],$$

$$[s_as_b'|s_a's_b]=\frac{1}{R}\frac{(\alpha\beta)^3}{8}\sum_{\tau=0}^{\infty}(-1)^\tau(2\tau+1)[G_\tau^0(0,\delta)^2W_\tau^0(2,2;\gamma)$$
$$-2G_\tau^0(0,\delta)G_\tau^0(2,\delta)W_\tau^0(2,0;\gamma)+G_\tau^0(2,\delta)^2W_\tau^0(0,0;\gamma)].$$

Here $A_n(\alpha)$, $B_n(\alpha)$, $G_\tau^\nu(m,\alpha)$, $W_\tau^\nu(m,n;\alpha)$ and $W_\tau^\nu(m,n;\alpha,\beta)$ are the functions defined in KAIK's table[13]), namely

$$A_n(\alpha)=\int_1^\infty e^{-\alpha\xi}\xi^n d\xi,$$

$$B_n(\alpha)=\int_{-1}^{+1}e^{-\alpha\eta}\eta^n d\eta,$$

$$G_\tau^\nu(m,\alpha)=\int_{-1}^{+1}e^{-\alpha\eta}P_\tau^\nu(\eta)\eta^m(1-\eta^2)^{\nu/2}d\eta,$$

$$W_\tau^\nu(m,n;\alpha)\equiv W_\tau^\nu(m,n;\alpha,\alpha),$$

$$W_\tau^\nu(m,n;\alpha,\beta)=\int_1^\infty\int_1^\infty Q_\tau^\nu(\xi_>)P_\tau^\nu(\xi_<)e^{-\alpha\xi_1-\beta\xi_2}\xi_1^m\xi_2^n(\xi_1^2-1)^{\nu/2}(\xi_2^2-1)^{\nu/2}d\xi_1d\xi_2,$$

where $\xi_>$ is the larger of ξ_1 and ξ_2 and $\xi_<$ the smaller. $A_n(\alpha)$, $B_n(\alpha)$, $G_\tau^\nu(m, \alpha)$ and $W_\tau^\nu(m, n; \alpha)$ are tabulated in KAIK's table. The evaluations of $W_\tau^\nu(m, n; \alpha, \beta)$ are made by using a series of recurrence formulas found also in the above mentioned table. $[s_a s_b' | s_a s_b']$ and $[s_a s_b', s_a' s_b]$ in K_{gu} are expressed in infinite series. Fortunately the convergence of these series is of sufficient rapidity for the present purpose, especially in the combined form;

$$
\begin{aligned}
&[s_a s_b' | s_a s_b'] - [s_a s_b' | s_a' s_b] \\
&= \frac{1}{R} \frac{(\alpha\beta)^3}{4} \sum_{s=0}^{\infty} (4s+3) \Big[G_{2s+1}^{0}(0, \delta)^2 W_{2s+1}^{0}(2, 2; \gamma) \\
&\qquad - 2G_{2s+1}^{0}(0, \delta) G_{2s+1}^{0}(2, \delta) W_{2s+1}^{0}(2, 0; \gamma) + G_{2s+1}^{0}(2, \delta)^2 W_{2s+1}^{0}(0, 0; \gamma) \Big].
\end{aligned}
$$

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On Einstein's Relativistic Theory of the Non-Symmetric Field

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We propose a new formulation of Einstein's relativistic theory of the non-symmetric field. Here the field quantities are different in transformation character from the corresponding quantities in his formalism. The transposition invariance is no more a postulate but a consequence of the theory. The field equations derived from the variational principle coincide formally with those obtained by Einstein and Straus.

§ 1. Introduction

In general relativity, when deriving the gravitational equations according to the variational principle, one takes as the independent field quantities a symmetric tensor density $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ of weight 1 and a symmetric connexion $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$. About ten years ago, Einstein propounded a unified field theory in which $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ and $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$ were replaced by the corresponding non-symmetric quantities $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$.[1] There he followed the concepts and methods of the symmetric theory, e. g., the covariance of the field equations under arbitrary transformations of coordinates, the variational principle for the derivation of the field equations, etc. On the other hand, he introduced a new basic requirement which was not seen in the symmetric theory. It is the so-called "transposition invariance". We shall illustrate this principle in the following.

In the symmetric theory, it follows from the variation relative to $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$ that

$$\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}{}_{,\lambda}+\overset{\circ}{\mathfrak{g}}{}^{\rho\nu}\overset{\circ}{\Gamma}{}^{\mu}_{\rho\lambda}+\overset{\circ}{\mathfrak{g}}{}^{\mu\rho}\overset{\circ}{\Gamma}{}^{\nu}_{\lambda\rho}-\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}\overset{\circ}{\Gamma}{}^{\rho}_{\lambda\rho}=0. \tag{1·1}$$

In this equation the order of indices does not matter, because $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ and $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$ are symmetric with respect to μ and ν. But there arises some arbitrariness in the choice of the order, when one tries to generalize (1·1) to the non-symmetric case. In this connection, Einstein claimed that the equations for the non-symmetric field should remain valid under the simultaneous changes $\mathfrak{g}^{\mu\nu}\rightarrow\mathfrak{g}^{\nu\mu}$ and $\Gamma^{\lambda}_{\mu\nu}\rightarrow\Gamma^{\lambda}_{\nu\mu}$ (transposition with respect to $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$). This is the "transposition invariance" for the equations. Under this postulate the equation (1·1) must be extended so that the order of indices is as in (1·1).

The transposition invariance of the field equations is assured if they are derivable from a Lagrangian density invariant for transposition. So Einstein first formulated the non-symmetric theory as follows: In the symmetric theory, the gravitational equations for empty space are derived from the Lagrangian density

$$\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}, \tag{1·2}$$

where $\overset{\circ}{R}_{\mu\nu}$ is the Ricci tensor formed from $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$. One may be inclined to adopt as the Lagrangian density in the non-symmetric case the quantity that results from (1·2) by removing the condition of symmetry for $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ and $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$. Unfortunately, however, such a quantity is not invariant under the transposition with respect to $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$. Hence Einstein added an artificial term to this quantity to get a Lagrangian density invariant under transposition.

In his recent formulation of the theory, he introduced a quantity $U^{\lambda}_{\mu\nu}$ depending on $\Gamma^{\lambda}_{\mu\nu}$ alone.2) The above generalization of (1·2), when expressed in terms of $\mathfrak{g}^{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$, is left invariant by the transposition with respect to *these* two quantities, i. e., $\mathfrak{g}^{\mu\nu}\to\mathfrak{g}^{\nu\mu}$ and $U^{\lambda}_{\mu\nu}\to U^{\lambda}_{\nu\mu}$ (not $\Gamma^{\lambda}_{\mu\nu}\to\Gamma^{\lambda}_{\nu\mu}$!). The theory developed on this basis arouses our interest because of its logical simplicity, as Einstein expected. Nevertheless, the character of $U^{\lambda}_{\mu\nu}$ is not familiar to mathematicians, much less to physicists. It is this point that prevents us from a full understanding of the theory.

We thus see some artificialness in the above two formulations of the non-symmetric theory. This feature seems to result from, among others, taking the transposition invariance as a guiding principle. Accordingly, it is desirable to formulate the theory in such a way that the transposition invariance is no more a postulate but a result of the theory. It is true that the formalism of Einstein in 1948 is satisfactory from this standpoint, but there the field quantities are assumed to be complex.3) So one may possibly expect a new formulation in which not only the above requirement is satisfied but the field quantities are real. Such an expectation will be realized in the present paper.

Now, as is well known, $\mathfrak{g}^{\mu\nu}$ is not one irreducible entity but splits into two independent quantities—the symmetric and the skew-symmetric parts. Therefore, the removal of the symmetry condition for $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ is equivalent to the introduction of a skew-symmetric tensor density $\mathfrak{f}^{\mu\nu}$ in addition to a symmetric one $\mathfrak{h}^{\mu\nu}$ (denoted by $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ in the symmetric case). Our theory begins with giving up such a generalization of $\overset{\circ}{\mathfrak{g}}{}^{\mu\nu}$ and introducing a new quantity in the following manner.

In Maxwell's electromagnetic theory, the field is usually represented by a skew-symmetric tensor $F_{\mu\nu}$. It is possible as well to describe the field by the dual of $F_{\mu\nu}$, i. e.,

$$\tilde{\mathfrak{F}}^{\mu\nu}=\tilde{\mathfrak{E}}^{\mu\nu\rho\sigma}F_{\rho\sigma}/2\,!,\quad (F_{\mu\nu}=\tilde{\mathfrak{e}}_{\mu\nu\rho\sigma}\tilde{\mathfrak{F}}^{\rho\sigma}/2\,!), \tag{1·3}$$

where $\tilde{\mathfrak{E}}^{\mu\nu\rho\sigma}$ and $\tilde{\mathfrak{e}}_{\mu\nu\rho\sigma}$ are skew-symmetric quantities defined by $\tilde{\mathfrak{E}}^{1234}=\tilde{\mathfrak{e}}_{1234}=1$. One can easily see that this $\tilde{\mathfrak{F}}^{\mu\nu}$ is a skew-symmetric pseudo-tensor density of weight 1. Corresponding to this alternative representation, we shall replace the above $\mathfrak{f}^{\mu\nu}$ by a skew-symmetric pseudo-tensor density $\tilde{\mathfrak{f}}^{\mu\nu}$ of weight 1. Then one of the field quantities becomes the sum of $\mathfrak{h}^{\mu\nu}$ and $\tilde{\mathfrak{f}}^{\mu\nu}$, which is no more a tensor density but transforms according to the law introduced in the next section. And the concept of transposition can be implied in the transformation character of the field quantities. Thus any theory depending on these quantities alone will satisfy the requirement of transposition invariance, if it is covariant under arbitrary transformations of coordinates.

On this line of thought we are going to establish a relativistic theory of the non-symmetric field. In § 2 the transformation law will be defined for various field quantities in such a way that the transposition invariance follows automatically. On this basis we shall then study into the parallelism and curvature in § 3. Generalizing (1·2) in a reasonable way, we shall next obtain a Lagrangian density which is invariant under the transposition. Then in § 4 the variational principle yields the field equations, which formally coincide with those of Einstein and Straus.4) Lastly the following remark will be made: When the transformation character of the field quantities is such as defined above, the electromagnetic field tensor formerly considered by us5) and the electric current density defined by Einstein3) are both subject to the same transformation law as in Maxwell's theory. In other words, they are a skew-symmetric tensor of order 2 and a vector density respectively.

§ 2. Transformation laws for field quantities

In this section we shall define the transformation character of field quantities in such a way that the transposition invariance follows from the requirement of general covariance.

We start with a symmetric tensor density $\mathfrak{h}^{\mu\nu}$ and a skew-symmetric pseudo-tensor density $\tilde{\mathfrak{f}}^{\mu\nu}$, both of weight 1. They transform, on change of coordinates, according to the equations

$$\begin{aligned} '\mathfrak{h}^{\mu\nu} &= |J|\,\mathfrak{h}^{\rho\sigma}(\partial\,'x^{\mu}/\partial x^{\rho})\,(\partial\,'x^{\nu}/\partial x^{\sigma}),\\ '\tilde{\mathfrak{f}}^{\mu\nu} &= J\,\tilde{\mathfrak{f}}^{\rho\sigma}(\partial\,'x^{\mu}/\partial x^{\rho})\,(\partial\,'x^{\nu}/\partial x^{\sigma}), \end{aligned} \tag{2·1}$$

where $J=\det(\partial x^{\mu}/\partial\,'x^{\nu})$. Let $\mathfrak{g}^{\mu\nu}$ be the sum of these two quantities, thus

$$\mathfrak{h}^{\mu\nu}+\tilde{\mathfrak{f}}^{\mu\nu}=\mathfrak{g}^{\mu\nu}; \quad \mathfrak{g}^{\underline{\mu\nu}}=\mathfrak{h}^{\mu\nu}, \quad \mathfrak{g}^{\underset{\smile}{\mu\nu}}=\tilde{\mathfrak{f}}^{\mu\nu}, \tag{2·2}$$

where and throughout the remainder of the present study we indicate the symmetric or skew-symmetric part of a quantity by putting — or $\smile$ under a couple of indices in question. $\mathfrak{g}^{\mu\nu}$ behaves like a tensor density for coordinate transformations with $J>0$. When $J<0$, however, it is subject to the interchange of indices in addition to the transformation as a tensor density. This behaviour is contrary to Einstein's formulation, and is significant for the subsequent development of our problem.

The determinant of $\mathfrak{g}^{\mu\nu}$, denoted by

$$\mathfrak{g}=\det(\mathfrak{g}^{\mu\nu}), \tag{2·3}$$

is a scalar density of weight 2. It is evident for $J>0$. For $J<0$ it results from the well-known fact that the values of a determinant is unaltered by the interchange of rows and columns. Hence the condition

$$\mathfrak{g}<0 \tag{2·4}$$

is independent of the choice of coordinates.

Under this condition we introduce the quantity $g^{\mu\nu}$ through

$$g^{\mu\nu} = \mathfrak{g}^{\mu\nu}/\sqrt{-\mathfrak{g}}. \tag{2·5}$$

It is obvious that $g^{\underline{\mu\nu}}$ and $g^{\underset{\vee}{\mu\nu}}$ are a tensor and a pseudo-tensor respectively. We next define the adjoint of $g^{\mu\nu}$ by

$$g^{\mu\alpha} g_{\nu\alpha} = \delta_\nu^{\mu}, \quad (g^{\alpha\mu} g_{\alpha\nu} = \delta_\nu^{\mu}). \tag{2·6}$$

If we assume that this is left invariant by coordinate transformations with $J<0$ as well as by those with $J>0$, $g_{\mu\nu}$ cannot be a tensor. In fact we have by taking (2·4) into account

$$\begin{aligned} 'g_{\underline{\mu\nu}} &= g_{\underline{\rho\sigma}} (\partial x^\rho/\partial' x^\mu)(\partial x^\sigma/\partial' x^\nu), \\ 'g_{\underset{\vee}{\mu\nu}} &= (J/|J|)\, g_{\underset{\vee}{\rho\sigma}} (\partial x^\rho/\partial' x^\mu)(\partial x^\sigma/\partial' x^\nu). \end{aligned} \tag{2·7}$$

This shows that the symmetric and the skew-symmetric parts of $g_{\mu\nu}$ are a tensor and a pseudo-tensor respectively. Thus the transformation character of $g_{\mu\nu}$ is similar, in some degree, to that of $\mathfrak{g}^{\mu\nu}$. In other words, for $J<0$, $g_{\mu\nu}$ is subject not only to the tensor transformation but also to the interchange of indices.

Next we take up the equation

$$g_{\mu\nu,\lambda} - g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda} - g_{\mu\rho}\Gamma^{\rho}_{\lambda\nu} = 0, \tag{2·8}$$

which was adopted by Einstein as a generalization of the well-known relation in the symmetric case. In order that this equation may admit a unique $\Gamma^{\lambda}_{\mu\nu}$ for a given $g_{\mu\nu}$, it is necessary and sufficient that[6]

$$\mathfrak{g} \neq 0 \quad \text{if } \mathfrak{k} \neq 0; \quad \mathfrak{g}(\mathfrak{g} - 2\mathfrak{h}) \neq 0 \quad \text{if } \mathfrak{k} = 0, \tag{2·9}$$

where

$$\mathfrak{h} = \det(g_{\underline{\mu\nu}}) < 0, \quad \mathfrak{k} = \det(g_{\underset{\vee}{\mu\nu}}). \tag{2·10}$$

One can easily see that $\mathfrak{h}$ and $\mathfrak{k}$ are scalar densities of weight 2, and that the condition (2·9) and the inequality in (2·10) are independent of the coordinate system used.

Now, we require that (2·8) is covariant for arbitrary transformations of coordinates, where it is taken for granted that $g_{\mu\nu}$ has such a transformation character as stated above. Then it follows under the condition (2·9) that

$$\begin{aligned} '\Gamma^{\lambda}_{\underline{\mu\nu}} &= \Gamma^{\rho}_{\underline{\sigma\tau}} (\partial' x^\lambda/\partial x^\rho)(\partial x^\sigma/\partial' x^\mu)(\partial x^\tau/\partial' x^\nu) + (\partial' x^\lambda/\partial x^\rho)(\partial^2 x^\rho/\partial' x^\mu \partial' x^\nu), \\ '\Gamma^{\lambda}_{\underset{\vee}{\mu\nu}} &= (J/|J|)\, \Gamma^{\rho}_{\underset{\vee}{\sigma\tau}} (\partial' x^\lambda/\partial x^\rho)(\partial x^\sigma/\partial' x^\mu)(\partial x^\tau/\partial' x^\nu). \end{aligned} \tag{2·11}$$

Thus $\Gamma^{\lambda}_{\underline{\mu\nu}}$ and $\Gamma^{\lambda}_{\underset{\vee}{\mu\nu}}$ transform like an ordinary linear connexion and a pseudo-tensor of order 3 respectively. This may be expressed also as follows: $\Gamma^{\lambda}_{\mu\nu}$ is subject to the interchange of the lower indices besides the transformation like a linear connexion, when the coordinates undergo a transformation with $J<0$.

It will be seen from the above that the concept of transposition is implied in the transformation character of field quantities. For, when carrying out a coordinate transformation with $J<0$, we have merely to transform $\mathfrak{g}^{\mu\nu}$ ($\Gamma_{\mu\nu}^{\lambda}$, etc.) like an ordinary tensor density (a connexion, etc.), and then to interchange the indices μ and ν. Accordingly, as long as a Lagrangian density is made only of these quantities, the transposition invariance of the Lagrangian will be assured by its property of being a scalar density. The variational principle then yields the field equations, which satisfy the requirement of transposition invariance as well as of general covariance. This plan will be carried out in § 4.

§ 3. Pseudo-covariant differentiation

In the ordinary theory of linear connexions, the covariant differential of a quantity Φ (indices suppressed) transforms in the same way as Φ. But it is not the case when we use the quantity $\Gamma_{\mu\nu}^{\lambda}$ with the transformation law (2·11) in place of the ordinary connexion. This feature is contrary to Einstein's formulation, so we shall explain it in some detail.

To begin with we tentatively consider for any vector v^{λ}

$$dv^{\lambda}+\Gamma_{\mu\nu}^{\lambda}v^{\mu}dx^{\nu}, \tag{3·1}$$

which is formally analogous to the covariant differential with respect to the ordinary connexion. Since $\Gamma_{\underset{\smile}{\mu\nu}}^{\lambda}$ is a pseudo-tensor, (3·1) is a sum of a vector and a pseudo-vector, and is changed into

$$(\partial' x^{\lambda}/\partial x^{\rho})\left[dv^{\rho}+\Gamma_{\nu\mu}^{\rho}v^{\mu}dx^{\nu}\right] \tag{3·2}$$

by a transformation with $J<0$. (Attention should be paid to the interchange of the lower indices of $\Gamma_{\mu\nu}^{\lambda}$!) Thus the process defined by (3·1) is not invariant for a transformation with $J<0$. One should rather consider that (3·1) forms an invariant operation together with the expression in the bracket of (3·2).

In this way we have been led to introduce (3·1) and a similar expression in (3·2) at the same time. Correspondingly, we define two kinds of differential operations such as

$$v^{\lambda}_{+;\,\mu}=v^{\lambda}{}_{,\mu}+\Gamma_{\nu\mu}^{\lambda}v^{\nu}\,; \quad v^{\lambda}_{-;\,\mu}=v^{\lambda}{}_{,\mu}+\Gamma_{\mu\nu}^{\lambda}v^{\nu}. \tag{3·3}$$

Each of these operations is invariant under transformations with $J>0$, but transforms into each other by those with $J<0$. In this respect they differ from the ordinary covariant derivative, so we shall call them *pseudo-covariant derivatives*. In contrast with this the process defined by

$$v^{\lambda}{}_{|\mu}=v^{\lambda}{}_{,\mu}+\Gamma_{\underline{\mu\nu}}^{\lambda}v^{\nu} \tag{3·4}$$

has a meaning independent of the coordinate system chosen.

The two operations in (3·3) are of the same form as the two kinds of covariant derivatives introduced by Einstein. In his formulation, each of them has individually an invariant meaning, and the two kinds are considered for convenience' sake. In our formulation, on the other hand, it is impossible to preserve the covariant character without either of them, and the use of the one involves to introduce the other.

It is easy to generalize (3·3) and (3·4) to a tensor, a tensor density, and so on. For later use we give two examples:

$$g_{\underset{+}{\mu}\,\underset{-}{\nu};\lambda}=g_{\mu\nu,\lambda}-g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda}-g_{\mu\rho}\Gamma^{\rho}_{\lambda\nu},$$
$$\mathfrak{g}^{\mu\;\nu}_{+\,-;\lambda}=\mathfrak{g}^{\mu\nu}{}_{,\lambda}+\mathfrak{g}^{\rho\nu}\Gamma^{\mu}_{\rho\lambda}+\mathfrak{g}^{\mu\rho}\Gamma^{\nu}_{\lambda\rho}-\mathfrak{g}^{\mu\nu}\Gamma^{\rho}_{\underline{\lambda\rho}}.$$

As to parallel displacements one can also consider three types corresponding to (3·3) and (3·4). Of course the (+)- or the (−)-parallelism has no invariant meaning, if considered separately. That is, the (+)-parallelism in one coordinate system does not coincide with the (+)- but with the (−)-parallelism in a system with the reverse screwsense (a system derived from the former by a transformation with $J<0$).

The first quantity of (3·3) is a sum of a tensor and a pseudo-tensor, and we can obtain its pseudo-covariant derivative: e. g.,

$$v^{\lambda}_{\underset{+}{+};\,\underset{+}{\mu};\,\nu}=v^{\lambda}_{+;\,\mu,\nu}+v^{\rho}_{+;\,\mu}\Gamma^{\lambda}_{\rho\nu}-v^{\lambda}_{+;\,\rho}\Gamma^{\rho}_{\mu\nu}.$$

Interchanging μ and ν and subtracting, we get

$$v^{\lambda}_{\underset{+}{+};\,\underset{+}{\mu};\,\nu}-v^{\lambda}_{\underset{+}{+};\,\underset{+}{\nu};\,\mu}=R^{\lambda}{}_{\rho\mu\nu}v^{\rho}-2\Gamma^{\rho}_{\underset{\smile}{\mu\nu}}v^{\lambda}_{+;\,\rho},\tag{3·5}$$

where

$$R^{\lambda}{}_{\mu\nu\sigma}=\Gamma^{\lambda}_{\mu\nu,\sigma}-\Gamma^{\lambda}_{\mu\sigma,\nu}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma}-\Gamma^{\rho}_{\mu\sigma}\Gamma^{\lambda}_{\rho\nu}.\tag{3·6}$$

(3·5) is a generalized identity of Ricci.

The quantity (3·6) corresponds to the curvature tensor in the ordinary theory of linear connexions. In the present case, however, it is no more a tensor but a sum of a tensor $K^{\lambda}{}_{\mu\nu\sigma}$ and a pseudo-tensor $\widetilde{K}^{\lambda}{}_{\mu\nu\sigma}$, thus

$$R^{\lambda}{}_{\mu\nu\sigma}=K^{\lambda}{}_{\mu\nu\sigma}+\widetilde{K}^{\lambda}{}_{\mu\nu\sigma},\tag{3·7}$$

where

$$K^{\lambda}{}_{\mu\nu\sigma}=\mathring{R}^{\lambda}{}_{\mu\nu\sigma}+\Gamma^{\rho}_{\underset{\smile}{\mu\nu}}\Gamma^{\lambda}_{\underset{\smile}{\rho\sigma}}-\Gamma^{\rho}_{\underset{\smile}{\mu\sigma}}\Gamma^{\lambda}_{\underset{\smile}{\rho\nu}},\quad \mathring{R}^{\lambda}{}_{\mu\nu\sigma}=\Gamma^{\lambda}_{\underline{\mu\nu},\sigma}-\Gamma^{\lambda}_{\underline{\mu\sigma},\nu}+\Gamma^{\rho}_{\underline{\mu\nu}}\Gamma^{\lambda}_{\underline{\rho\sigma}}-\Gamma^{\rho}_{\underline{\mu\sigma}}\Gamma^{\lambda}_{\underline{\rho\nu}},\tag{3·8}$$

$$\widetilde{K}^{\lambda}{}_{\mu\nu\sigma}=\Gamma^{\lambda}_{\underset{\smile}{\mu\nu}|\sigma}-\Gamma^{\lambda}_{\underset{\smile}{\mu\sigma}|\nu}.\tag{3·9}$$

The quantity $R^{\lambda}{}_{\mu\nu\sigma}$ appears also when a vector is transported (+)-parallel to itself along a small closed circuit.

Contracting $R^{\lambda}{}_{\mu\nu\sigma}$ for λ and σ, we have a quantity $R_{\mu\nu}$, which is a generalization of the Ricci tensor. It is also a sum of a tensor and a pseudo-tensor, i. e.,

$$R_{\mu\nu}=K_{\mu\nu}+\widetilde{K}_{\mu\nu};\quad K_{\mu\nu}=K^{\rho}{}_{\mu\nu\rho},\quad \widetilde{K}_{\mu\nu}=\widetilde{K}^{\rho}{}_{\mu\nu\rho}.\tag{3·10}$$

A similar formula to (3·5) follows, if one carries out a transformation with $J<0$, thus (primes omitted)

$$v^{\lambda}_{\underset{-}{-};\,\underset{-}{\mu};\,\nu}-v^{\lambda}_{\underset{-}{-};\,\underset{-}{\nu};\,\mu}=(K^{\lambda}{}_{\rho\mu\nu}-\widetilde{K}^{\lambda}{}_{\rho\mu\nu})v^{\rho}+2\Gamma^{\rho}_{\underset{\smile}{\mu\nu}}v^{\lambda}_{-;\,\rho}.$$

§ 4. Derivation of the field equations

In this section we derive the field equations on the basis of the variational principle. Here the Lagrangian density is introduced by generalizing (1·2) in a very reasonable manner. The field quantities $\mathfrak{g}^{\mu\nu}$ and $\Gamma_{\mu\nu}^{\lambda}$ are supposed, in the course of variation, to be independent of each other. And the foregoing relation (2·8) between them results from the variation.

Now, in the symmetric theory, the Lagrangian density (1·2) is used for the derivation of the gravitational equations for empty space. Therefore, one might try to generalize it by removing the symmetry condition for $\overset{\circ}{\mathfrak{g}}^{\mu\nu}$ and $\overset{\circ}{\Gamma}{}_{\mu\nu}^{\lambda}$ to obtain the Lagrangian density for the non-symmetric field. But the resulting expression is evidently a sum of a scalar density $\mathfrak{L}$ and a pseudo-scalar density $\widetilde{\mathfrak{L}}$. So we must choose either of them as a Lagrangian density in order to make the field equations covariant. For its determination we consider the transition from the non-symmetric to the symmetric case. In this limit, $\mathfrak{L}$ goes over to the Lagrangian density in the symmetric theory, while $\widetilde{\mathfrak{L}}$ is reduced to zero. This seems to show that it is reasonable to adopt $\mathfrak{L}$ as the Lagrangian density in our case. Its concrete expression can easily be calculated as follows:

$$\mathfrak{L}=\mathfrak{h}^{\mu\nu}K_{\underline{\mu\nu}}+\widetilde{\mathfrak{f}}^{\mu\nu}\widetilde{K}_{\underset{\vee}{\mu\nu}}, \tag{4·1}$$

where

$$K_{\underline{\mu\nu}}=\overset{\circ}{R}{}^{\rho}{}_{\underline{\mu\nu}\rho}-\Gamma_{\underset{\vee}{\mu\sigma}}^{\rho}\Gamma_{\underset{\vee}{\rho\nu}}^{\sigma},\quad \widetilde{K}_{\underset{\vee}{\mu\nu}}=\Gamma_{\underset{\vee}{\mu\nu}|\rho}^{\rho}-(\Gamma_{\underset{\vee}{\mu\rho}|\nu}^{\rho}-\Gamma_{\underset{\vee}{\nu\rho}|\mu}^{\rho})/2. \tag{4·2}$$

As to the variation with respect to $\widetilde{\mathfrak{f}}^{\mu\nu}$ we shall follow the case of Maxwell's equations in special relativity.[3] There we have two methods of deriving the equations from the variational principle: $\delta\int\sum_{\mu,\nu}F_{\mu\nu}F_{\mu\nu}d^4x=0$. The one method consists in the variation with respect to the 4-vector potential ϕ_μ introduced by

$$F_{\mu\nu}=\phi_{\mu,\nu}-\phi_{\nu,\mu}. \tag{4·3}$$

Then the variation produces one half of the field equations (i.e., $\sum_{\nu}F_{\mu\nu,\nu}=0$). The other half,

$$F_{\mu\nu,\lambda}+F_{\nu\lambda,\mu}+F_{\lambda\mu,\nu}=0, \tag{4·4}$$

is obtained from (4·3) after elimination of ϕ_μ. The alternative method is to carry out the variation relative to $F_{\mu\nu}$ by taking (4·4) as an auxiliary condition. In this way one can avoid the introduction of ϕ_μ. It goes without saying that the two methods lead to the same system of equations.

Similarly, we may also have two methods in the non-symmetric theory. To see this we introduce the dual of ϕ_μ by

$$\widetilde{\phi}^{\mu\nu\sigma}=\widetilde{\mathfrak{E}}^{\mu\nu\rho\sigma}\phi_\rho, \tag{4·5}$$

which is a pseudo-tensor density of weight 1, skew-symmetric in all indices. Then (4·3) and (4·4) have the forms $\widetilde{\mathfrak{F}}^{\mu\nu}=\widetilde{\phi}^{\mu\nu\rho}{}_{,\rho}$ and $\widetilde{\mathfrak{F}}^{\mu\rho}{}_{,\rho}=0$ respectively, where $\widetilde{\mathfrak{F}}^{\mu\nu}$ is given by (1·3). If we observe that the quantity $\tilde{\mathfrak{f}}^{\mu\nu}$ has the same transformation character as $\widetilde{\mathfrak{F}}^{\mu\nu}$, the following two methods may be suggested :[3] (1) Make the variation with respect to $\tilde{\mathfrak{f}}^{\mu\nu}$ under the auxiliary condition

$$\tilde{\mathfrak{f}}^{\mu\nu}{}_{,\nu}=0. \tag{4·6}$$

(2) Introduce a pseudo-tensor density $\tilde{\mathfrak{f}}^{\mu\nu\lambda}$ of weight 1 through

$$\tilde{\mathfrak{f}}^{\mu\nu}=\tilde{\mathfrak{f}}^{\mu\nu\rho}{}_{,\rho}, \tag{4·7}$$

and then carry out the variation with respect to $\tilde{\mathfrak{f}}^{\mu\nu\lambda}$ in place of $\tilde{\mathfrak{f}}^{\mu\nu}$. Both methods will produce the same result.

In what follows we shall adopt the latter method. By a simple calculation it follows that

$$\begin{aligned}\delta\mathfrak{L}=&K_{\underline{\mu\nu}}\delta\mathfrak{h}^{\mu\nu}+(\widetilde{K}_{\underset{\vee}{\mu\nu}}\delta\tilde{\mathfrak{f}}^{\mu\nu\lambda})_{,\lambda}-\widetilde{K}_{\underset{\vee}{\mu\nu},\lambda}\delta\tilde{\mathfrak{f}}^{\mu\nu\lambda}+(\mathfrak{h}^{\mu\nu}\delta\Gamma^{\ \lambda}_{\underline{\mu\nu}}-\mathfrak{h}^{\mu\lambda}\delta\Gamma^{\ \nu}_{\underline{\mu\nu}})_{,\lambda}\\&-(\mathfrak{h}^{\mu\nu}{}_{|\lambda}-\delta_{\lambda}^{\nu}\mathfrak{h}^{\mu\rho}{}_{|\rho}-\delta_{\lambda}^{\nu}\tilde{\mathfrak{f}}^{\rho\sigma}\Gamma^{\ \mu}_{\underset{\vee}{\rho\sigma}}+2\tilde{\mathfrak{f}}^{\mu\rho}\Gamma^{\ \nu}_{\lambda\rho})\delta\Gamma^{\ \lambda}_{\underline{\mu\nu}}\\&+(\tilde{\mathfrak{f}}^{\mu\nu}\delta\Gamma^{\ \lambda}_{\underset{\vee}{\mu\nu}}-\tilde{\mathfrak{f}}^{\mu\lambda}\delta\Gamma^{\ \nu}_{\underset{\vee}{\mu\nu}})_{,\lambda}-(\tilde{\mathfrak{f}}^{\mu\nu}{}_{|\lambda}-\delta_{\lambda}^{\nu}\tilde{\mathfrak{f}}^{\mu\rho}{}_{|\rho}+2\mathfrak{h}^{\rho\nu}\Gamma^{\ \mu}_{\underset{\vee}{\rho\lambda}})\delta\Gamma^{\ \lambda}_{\underset{\vee}{\mu\nu}}.\end{aligned} \tag{4·8}$$

Here apply the variational principle : $\delta\int\mathfrak{L}d^4x=0$, where $\delta\mathfrak{h}^{\mu\nu}$, $\delta\tilde{\mathfrak{f}}^{\mu\nu\lambda}$ and $\delta\Gamma^{\ \lambda}_{\mu\nu}$ vanish at the boundary of domain of integration. Then the results are

$$K_{\underline{\mu\nu}}=0,\quad \widetilde{K}_{\underset{\vee}{\mu\nu},\lambda}+\widetilde{K}_{\underset{\vee}{\nu\lambda},\mu}+\widetilde{K}_{\underset{\vee}{\lambda\mu},\nu}=0, \tag{4·9}$$

$$\begin{aligned}&\mathfrak{h}^{\mu\nu}{}_{|\lambda}-\delta_{\lambda}^{\mu}(\mathfrak{h}^{\nu\rho}{}_{|\rho}+\tilde{\mathfrak{f}}^{\rho\sigma}\Gamma^{\ \nu}_{\underset{\vee}{\rho\sigma}})/2-\delta_{\lambda}^{\nu}(\mathfrak{h}^{\mu\rho}{}_{|\rho}+\tilde{\mathfrak{f}}^{\rho\sigma}\Gamma^{\ \mu}_{\underset{\vee}{\rho\sigma}})/2\\&\qquad+\tilde{\mathfrak{f}}^{\mu\rho}\Gamma^{\ \nu}_{\underset{\vee}{\lambda\rho}}+\tilde{\mathfrak{f}}^{\nu\rho}\Gamma^{\ \mu}_{\underset{\vee}{\lambda\rho}}=0,\end{aligned} \tag{4·10a}$$

$$\tilde{\mathfrak{f}}^{\mu\nu}{}_{|\lambda}+(\delta_{\lambda}^{\mu}\tilde{\mathfrak{f}}^{\nu\rho}{}_{|\rho}-\delta_{\lambda}^{\nu}\tilde{\mathfrak{f}}^{\mu\rho}{}_{|\rho})/2-\mathfrak{h}^{\mu\rho}\Gamma^{\ \nu}_{\underset{\vee}{\rho\lambda}}+\mathfrak{h}^{\nu\rho}\Gamma^{\ \mu}_{\underset{\vee}{\rho\lambda}}=0. \tag{4·10b}$$

If we contract (4·10b) for λ and ν, we get by virtue of (4·7) or (4·6)

$$2\mathfrak{h}^{\mu\nu}\Gamma^{\ \lambda}_{\underset{\vee}{\nu\lambda}}=-\tilde{\mathfrak{f}}^{\mu\nu}{}_{,\nu}=0.$$

Since $\det(\mathfrak{h}^{\mu\nu})\doteq\mathfrak{h}\doteqdot 0$, this equation reduces to

$$\Gamma^{\ \nu}_{\underset{\vee}{\mu\nu}}=0. \tag{4·11}$$

After contraction of (4·10a) for λ and ν, we find on using (4·11)

$$\mathfrak{h}^{\mu\nu}{}_{|\nu}+\tilde{\mathfrak{f}}^{\nu\lambda}\Gamma^{\ \mu}_{\underset{\vee}{\nu\lambda}}=0.$$

In view of these results the equations (4·10) can be written as

$$\mathfrak{g}^{\mu\nu}{}_{|\lambda}+\mathfrak{g}^{\rho\nu}\Gamma^{\ \mu}_{\underset{\vee}{\rho\lambda}}-\mathfrak{g}^{\mu\rho}\Gamma^{\ \nu}_{\underset{\vee}{\rho\lambda}}=0,\quad \text{i. e.,}\ \mathfrak{g}^{\underset{+}{\mu}\underset{-}{\nu}}{}_{;\lambda}=0. \tag{4·12}$$

This equation is equivalent to (2·8).

When (4·12) (or (2·8)) and (4·11) hold, both $K_{\underset{\vee}{\mu\nu}}$ and $\widetilde{K}_{\underline{\mu\nu}}$ vanish, so $R_{\underline{\mu\nu}}$ and $R_{\underset{\vee}{\mu\nu}}$ coincide with $K_{\underline{\mu\nu}}$ and $\widetilde{K}_{\underset{\vee}{\mu\nu}}$ respectively. Thus (4·9) takes the form

$$R_{\underline{\mu\nu}}=0, \quad R_{\underset{\vee}{\mu\nu},\lambda}+R_{\underset{\vee}{\nu\lambda},\mu}+R_{\underset{\vee}{\lambda\mu},\nu}=0. \tag{4·13}$$

Summarizing the results, we may conclude that our field equations are (2·8), (4·11) (or (4·6)), and (4·13). They are of the same form as the equations obtained by Einstein and Straus.[4] It is easy to see that they are equivalent to the Euler equations (4·9) and (4·10) with the auxiliary condition (4·6).

§ 5. Concluding remarks

In this paper, Einstein's relativistic theory of the non-symmetric field has been reformulated so that the transposition invariance need not be postulated. The field quantities are different in transformation character from the quantities in Einstein's formulation. In what follows we shall state salient features of our formalism.

The present author formerly clarified that it was preferable to identify the quantity $\tilde{\mathfrak{e}}_{\mu\nu\rho\sigma}\mathfrak{g}^{\rho\sigma}/2$ with the electromagnetic tensor $F_{\mu\nu}$ in Maxwell's theory.[5] The same view has been held by Hlavatý, who obtained an interesting result favourable to this identification.[7] Einstein also considered that the electric current density should be represented by $\widetilde{\mathfrak{E}}^{\mu\nu\rho\sigma}g_{\nu\rho,\sigma}/2$.[3] When $g_{\mu\nu}$ is a tensor as usual, these quantities are respectively a pseudo-tensor and a pseudo-vector density, in contrast with the case of Maxwell's theory. This difficulty does not arise in our formalism. In fact, the corresponding quantities are a skew-symmetric tensor and a vector density respectively.

Einstein based his new formulation on a certain change of connexion called the λ-transformation.[2] If we tentatively admit this change in our formalism, the new $\Gamma^{\lambda}_{\mu\nu}$ arising from the original one by the λ-transformation does not transform according to (2·11) under a change of coordinates. Thus the λ-transformation loses its meaning in our case. This fact seems to be all the more desirable, because no interpretation has been put so far on the λ-transformation from the physical point of view.

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Inelastic Scattering of Neutrons and Surface Direct Processes

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In the case of inelastic scattering of neutrons in which the residual nucleus is left in a definite low-lying excited state, the surface direct processes through single particle excitation and through same sort of many particle excitations are formulated in a similar manner as in deuteron-stripping reaction. The distorted wave method is employed and its validity for the mentioned processes is discussed in some detail. The cross sections obtained are compared with that of another surface direct process accompanied by collective excitation and with that of the process through compound nucleus formation. It is shown that for intermediate and heavy nuclei (i) the collective excitation is much larger than the single particle excitation and (ii) the compound nucleus process is more important than the surface direct process in low energy region in which there are few open channels, while the latter is much larger than the former in higher energy region where many channels are open.

§ 1. Introduction

In recent experiments[1] many evidences have been accumulated which clearly indicate the inadequacy of the concept of compound nucleus in explaining some of the nuclear reactions at moderate energies. One of the typical examples is the forward peaks of angular distribution appearing in the case of inelastic scattering of particles in which the residual nucleus is left in a definite low-lying excited state. Such phenomena are evidently contradictory to the result of the statistical compound nucleus hypothesis.

It, therefore, seems necessary to modify the concept of the compound nucleus, for instance, by removing some of the underlying assumptions, i. e., statistical assumption and continuum assumption, etc. Another possible and more realistic way of modification is to take into account such processes that take place without formation of the compound nucleus. These are called direct processes and may be divided into two classes: volume direct and surface direct processes. The former takes place inside of the nucleus, while the latter occurs at the diffuse layer at the nuclear boundary. The surface direct process may further be classified into various processes according to the mechanisms by which the target nucleus is excited. In the case of inelastic scattering of neutrons, Austern, Butler and McManus[2] proposed the process by single particle excitation, whereas the direct process by collective excitation has been worked out by Hayakawa and Yoshida[3]. As these mechanisms are of the extreme natures, there may be intermediate mechanisms of nuclear excitation, which may be called " group excitation ". α-particle excitation may be considered as one of the examples, in which a quasi α-particle in a nucleus is excited to its higher level to form residual nucleus. The mechanisms which work in actual nuclear reactions will be governed by the natures of target and residual nuclei.

The purpose of the present paper is to formulate the surface direct processes by single particle and group excitations and, further, to investigate in what conditions, i. e., energy region, mass number, etc., the processes become of more importance in comparion with the compound nucleus process and with another surface direct process by collective excitation.

We shall deal only with the case of inelastic scattering of neutrons in which the residual nucleus is left in a definite low-lying excited state. Inelastic scattering of protons can be treated in a quite similar way.

By the use of the technique developed in the previous paper,[4] the exact formula of T-matrix of the surface direct process is presented in the next section. Section 3 is devoted to its approximate reductions and their validities are discussed in some detail. In section 4 the total and differential cross sections are presented and they are compared with those of the collective excitation and of the compound nucleus process in sections 5 and 6. Finally in section 7 some discussions are added.

The main results obtained in this paper may be summarized as follows.

(a) The shapes of angular distributions and excitation functions of surface direct processes between two given states are almost independent of the mechanisms by which the target nucleus is excited.

(b) The cross section of collective excitation is mush larger than that of single particle excitation.

(c) In low energy region the compound nucleus process is larger than the surface direct process.

(d) When the energy of incident neutron is sufficiently high so that there are many open channels, the surface direct process is much larger than the compound nucleus process. For heavy nuclei the latter may be negligibly small compared with the former in the energy region above 10 Mev.

§ 2. Formulation

In this section the surface direct process is formulated using the technique described in the previous paper[4)] which will be hereafter referred to as I.*

In the formal theory of nuclear reaction developed by Wigner and his collaborators[5)] the configuration space of a system is divided into two regions, internal and external ones, by a hypersurface S which is usually taken to be located as near actual nuclear surface as possible. It has been assumed that all nuclear interactions are confined to the internal region. However, an incident particle does interact with the tails of the wave function of a target nucleus which extend to infinity though in exponentially decaying manners. As was shown by Thomas[6)], it is those interactions in the external region which cause the stripping and pick-up processes in (d, p) and (p, d) reactions, respectively. Therefore, in the case of inelastic scattering of neutrons, it seems natural to call surface direct process the contributions arising from interactions in the external region.

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* More general cases of rearrangement collisions have been presented in the previous paper.

In treating inelastic scattering of neutrons, the Hamiltonian H of our total system is separated into two terms,

$$H=H_0+V,$$

where H_0 of the free Hamiltonian, representing a nucleus and a free neutron and V is the interaction between them. Schrödinger equation of the system,

$$H\Psi=E\Psi, \tag{1}$$

is represented, with suitable boundary conditions, in the integral form as follows[7] :

$$\begin{aligned}\Psi_a^{(+)}&=\Phi_a+(E+i\epsilon-H_0)^{-1}\cdot V\Psi_a^{(+)}\\&=[1+(E+i\epsilon-H)^{-1}\cdot V]\Phi_a,\end{aligned} \tag{2}$$

where Φ_a is the wave function of the initial state a, $g_a\cdot\exp(i\boldsymbol{k}_a\boldsymbol{r}_1)$, with the wave function g_a of the ground state of the nucleus.

The transition matrix element between the initial state a and the final state b with an excited state g_b of the nucleus can be written as follows :

$$T_{ba}=\langle\Phi_b|V|\Psi_a^{(+)}\rangle=\langle\Phi_b|V|[1+(E+i\epsilon-H)^{-1}\cdot V]\Phi_a\rangle, \tag{3}$$

where

$$\Phi_b=e^{i\boldsymbol{k}_b\cdot\boldsymbol{r}_1}\cdot g_b.$$

Further, the interaction V is divided into two parts in accordance with the situations* in which the neutron interacts with the nucleus either in the internal or in the external regions,

$$V=V_I+V_E, \tag{4}$$

where V_I and V_E are the internal- and external-region-interactions, respectively. Then, the equations (2) and (3) can be represented as follows[8] :

$$\begin{aligned}\Psi_a^{(+)}&=[1+(E+i\epsilon-H)^{-1}\cdot(V_I+V_E)]\Phi_a\\&=[1+(E+i\epsilon-H)^{-1}\cdot V_E][1+(E+i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_a,\end{aligned} \tag{5}$$

$$T_{ba}'\equiv T_{ba}^{(c)}+T_{ba}^{(d)}, \tag{6}$$

$$T_{ba}^{(c)}=\langle\Phi_b|V_I|[1+(E+i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_a\rangle, \tag{6a}$$

$$\begin{aligned}T_{ba}^{(d)}=\langle[1+(E-i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_b|V_E|&[1+(E+i\epsilon-H)^{-1}\cdot V_E]\\\times&[1+(E+i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_a\rangle.\end{aligned} \tag{6b}$$

The first term of (6)**, $T_{ba}^{(c)}$, contains only V_I and can be considered to involve a

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* $$V_I=\begin{cases}V, & (r_1<R),\\0, & (r_1\geqq R).\end{cases}\qquad V_E=\begin{cases}0, & (r_1<R),\\V, & (r_1\geqq R).\end{cases}$$

where r_1 is the radial coordinate of the neutron from centre-of-mass of the target nucleus and R is the radius of S.

** This term can be represented in the form of the many level formula of Kapur and Peierls[9], which was shown in I.

contribution of the compound nucleus together with that of the volume direct processes, while the second term, $T_{ba}^{(d)}$, represents the correction term due to the interaction in the external region and, therefore, may be reasonably interpreted as the surface direct process.

Since V_I vanishes in the external region and V_E is effective only in the external region, $[1+(E+i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_a$ and $[1+(E-i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_b$ in $T_{ba}^{(d)}$ are in their wave zones. Therefore, they can be represented as

$$[1+(E+i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_a=\Phi_a-\sum_{\beta}T_{\beta a}^{(c)}\cdot E_{\beta}^{(+)}, \tag{7a}$$

$$[1+(E-i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_b=-\{\Phi_b-\sum_{\gamma}T_{\gamma b}^{(c)}E_{\gamma}^{(+)}\}^*, \tag{7b}$$

where $E_{\beta}^{(+)}$ is the outgoing wave in channel β and $T_{\beta a}^{(c)}$ is the transition matrix element between states α and β arising from the internal-region-interaction V_I.

In treating the single particle excitation, the wave function of nucleus is expanded in a series of wave functions which are the products of the single particle and remaining core states in the external region;

$$g_a(\boldsymbol{r}_2\,\xi,\,I_a\,K_a)=\sum_{c}G_c(\xi)\,|\boldsymbol{r}_2\rangle_c, \tag{8}$$

$$|\boldsymbol{r}_2\rangle_c=\sum_{m_a}(l_a\,j\,m_a\,\nu|I_aK_a)\,(2M/\hbar^2R)^{1/2}$$
$$\cdot\gamma_{ca}\{h_{l_a}^{(1)}(i\kappa_ar)/h_{l_a}^{(1)}(i\kappa_aR)\}\mathcal{Y}_{j\nu c}Y_{l_am_a}(\Omega_2), \tag{9}$$

where $G_c(\xi)$ is the wave function of the core c, I_a the spin of the nucleus with its z-component K_a, and l_a and m_a are orbital angular momentum of the extra-particle and its z-component. $\mathcal{Y}_{j\nu c}$ is the wave function for the channel spin j and its z-component ν, which is a resultant of the spins of the core and the extra-nucleon. γ_{ca} is reduced width amplitude which has the dimension of (energy)$^{1/2}$ as was defined by Thomas[10].

The interaction V_E is taken to be a sum of two-body nuclear forces,

$$V_E=(\sum_{i=1}^{A}V_{i0})_E, \tag{10}$$

V_{i0} being determined by the nucleon-nucleon scattering data and the binding energy of the deuteron. In nuclear matter the two-body nuclear force might not be so strong as is inferred from free nucleon-nucleon scattering data. However, this effect may be negligible, since the density of nuclear matter is very small in the external region where the surface direct process takes place.

In the case of " group excitation ", the wave function of nucleus should be expanded, instead of (8), in a series of wave functions which are the products of the group and the remaining core states. Further, it is convenient to write (10) as a sum of effective potentials between the group, say, α-particle and a nucleon which is determined by experimental data of α-nucleon scattering.

The cross section of inelastic scattering is given by

$$d\sigma/d\Omega=(M/2\pi\hbar^2)^2\cdot(E_b/E_a)^{1/2}\cdot|T_{ba}|^2, \tag{11}$$

$$|T_{ba}|^2=|T_{ba}^{(c)}+T_{ba}^{(d)}|^2, \tag{12a}$$

where M is the nucleon mass, and E_a and E_b are the kinetic energies of incident and emitted nucleons, respectively. If the statistical assumption hold for $T_{ba}^{(c)}$, namely, if wave functions of the emitted particle from various levels of compound nucleus have random phases, the interference term between $T_{ba}^{(c)}$ and $T_{ba}^{(d)}$ will vanish and (12a) may be written as

$$|T_{ba}|^2=|T_{ba}^{(c)}|^2+|T_{ba}^{(d)}|^2. \tag{12b}$$

In this case the angular distribution will consist of the contribution due to the compound nucleus process, which is symmetric about 90° direction, superimposed by that of the surface direct process.

§ 3. Methods of approximation and their validities

This section is divided into two parts for the sake of convenience. In the first part we employ the Born approximation to treat the formula (6b), $T_{ba}^{(d)}$. Though its validity is not generally guaranteed except for some special cases, it is presented first for convenience for the discussion on the validity of the distorted wave method which will be given in the second part.

Throughout this paper, the anti-symmetrization of wave function and the tensor components in the external-region-interactions are entirely neglected. Further, the channel spin j and the spin state of the incident neutron is assumed to be unchanged during the process.

i) *The Born approximation.*

First, we shall evaluate the matrix element $T_{ba}^{(d)}$ using the nuclear force with the form of δ-function:

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2)=C\delta(\boldsymbol{r}_1-\boldsymbol{r}_2). \tag{13}$$

Then,

$$\begin{aligned}T_{ba}^{(d)}&=\int e^{i(\boldsymbol{k}_a-\boldsymbol{k}_b)\cdot\boldsymbol{r}_1}g_b{}^*(\boldsymbol{r}_2\xi,\ I_bK_b)\,g_a(\boldsymbol{r}_2\xi,\ I_aK_a)\,V(\boldsymbol{r}_1-\boldsymbol{r}_2)\,d\boldsymbol{r}_1d\boldsymbol{r}_2d\xi\\&=C(2M/\hbar^2R)\gamma_{ca}\cdot\gamma_{cb}\sum_{m_am_b}(l_ajm_a\nu|I_aK_a)\,(l_bjm_b\nu|I_bK_b)\\&\quad\times\int_R^\infty e^{i(k_a-k_b)r}\{h_{l_a}^{(1)}(i\kappa_ar)/h_{l_a}^{(1)}(i\kappa_aR)\}\{h_{l_b}^{(1)}(i\kappa_br)/h_{l_b}^{(1)}(i\kappa_bR)\}\\&\qquad\times Y_{l_bm_b}^*(\Omega)\,Y_{l_am_b}(\Omega)\,d\boldsymbol{r}.\end{aligned} \tag{15}$$

Using Racah's technique[11],the component sum of angular momenta, combined with the integration with respect to angular coordinate, can be easily performed with the result,

$$\begin{aligned}T_{ba}^{(d)}&=C(2M/\hbar^2R)\gamma_{ca}\cdot\gamma_{cb}(-1)^{j+I_a-I_b-K_a}\cdot(-i)^{l_a+l_b}\\&\quad\times\sum_L\mathcal{R}(l_al_bL,\ R)\,(I_bI_aK_b-K_a|L0)\,Z(l_aI_al_bI_b,\ jL).\end{aligned} \tag{16}$$

Here $Z(l_aI_al_bI_b,\ jL)$ is Z-coefficient of Biedenharn, Blatt and Rose,[12], and

$$\mathcal{R}(l_a l_b L, R) \equiv \int_R^\infty j_L(|\boldsymbol{k}_a-\boldsymbol{k}_b|\cdot r)\,\{h_{l_a}^{(1)}(i\kappa_a r)/h_{l_a}^{(1)}(i\kappa_a R)\}$$
$$\times\{h_{l_b}^{(1)}(i\kappa_b r)/h_{l_b}^{(1)}(i\kappa_b R)\}\,r^2\,dr, \tag{17a}$$

which can be approximated as

$$\mathcal{R}(l_a l_b L, R) \simeq j_L(|\boldsymbol{k}_a-\boldsymbol{k}_b|\cdot R)$$
$$\times\int^\infty \{h_{l_a}^{(1)}(i\kappa_a r)/h_{l_a}^{(1)}(i\kappa_a R)\}\,\{h_{l_b}^{(1)}(i\kappa_b r)/h_{l_b}^{(1)}(i\kappa_b R)\}\,r^2\,dr, \tag{17b}$$

since the integrand of (17b) decreases very rapidly, whereas $j_L(|\boldsymbol{k}_a-\boldsymbol{k}_b|r)$ is an oscillating function whose wave length is much larger than the decreasing rate of the former. The approximation (17b) may be expected to be very good when the single particle excitation is concerned. For group excitation the error introduced by (17b) may be considerably larger, since the binding energy of the group to the remaining core is small and, consequently, κ is small due to its large internal energy.

It should be noted that the angular distribution obtained by (17b) is, apart from the proportional factor, just the same as that given by Austern, Butler and McManus[2].

Next, the nuclear force with finite range is to be considered. For this purpose, we shall perform the Fourier transformation of $V(|\boldsymbol{r}_1-\boldsymbol{r}_2|)$,

$$V(r) = (2\pi)^{-3/2}\cdot\int G(P)\,e^{-i\boldsymbol{P}\cdot\boldsymbol{r}}\cdot d\boldsymbol{P}, \tag{18}$$

and its substitution in (14) leads to

$$T_{ba}^{(d)} = (2\pi)^{3/2}\cdot G(|\boldsymbol{k}_a-\boldsymbol{k}_b|)\times[\text{Equation}\,(16)/C]. \tag{19}$$

For Yukawa potential, $V(r) = V_0 e^{-\alpha r}/\alpha r$,

$$(2\pi)^{3/2}\cdot G(|\boldsymbol{k}_a-\boldsymbol{k}_b|) = (4\pi/\alpha)\cdot V_0[\alpha^2+|\boldsymbol{k}_a-\boldsymbol{k}_b|^2]^{-1}. \tag{20}$$

Due to the form factor (20), the forward peaks of angular distribution will slightly increase in comparison with the case of zero range nuclear force. The effect, however, may be negligible when the incident energy of neutron is below 5 Mev.

The strength of the potential with δ-function form cannot be determined by the two body problems at low energy, because it cannot form bound states. Therefore, we shall estimate the strength C by the use of the following relation which is obtained by equating (19) with (15),

$$C = (4\pi/\alpha^3)\cdot V_0, \tag{21}$$

with the restriction

$$\alpha^2 \gg |\boldsymbol{k}_a-\boldsymbol{k}_b|^2. \tag{22}$$

The equation (21) is also obtainable by taking average of the potential with respect to uniform wave function. It may be reasonable because of the following consideration. When the energies of incident and emitted neutron are low, their wave functions are nearly constant within the range of nuclear force. Therefore, in the equation (14), we may put $V(|\boldsymbol{r}_1-\boldsymbol{r}_2|)$ out of the integral. The same relation may also hold for the distorted wave method with the restriction (22).

In order to reduce the exact formula (6b) into the equation (14), i. e., T-matrix element in the Born approximation, it has been necessary to use the following two approximations :

(A) The interaction V_E is taken into account only as first order perturbation.

(B) The effects of the internal-region-interaction are completely neglected, namely, (7a) and (7b) are approximated by Φ_a and Φ_b, respectively.

The approximation (B) may be applicable only to the case in which the energy of incident neutron is sufficiently high and the traget is light nucleus.

As is well known, the nuclear force cannot be treated by the Born approximation in the case of nucleon-nucleon scattering. In our case, however, the first order Born approximation (A) may be expected to be rather good, because the wave function of the nucleus in the external region is very small compared with that in the internal one.

In order to explain the situation more concretely, it may be convenient to represent reduced width $\gamma^2_{\lambda s}$ in the unit of the sum rule limit of Teichmann and Wigner[13],

$$\gamma^2_{\lambda s}=(3\hbar^2/2MR^2)\,\theta^2_{\lambda s}. \tag{23}$$

It should be noted that the sum rule limit $3\hbar^2/2MR^2$ is derived by using uniform wave function. However, a wave function of bound state is not uniform, but is rapidly decreasing near the nuclear surface. Furthermore, the reduced width amplitude is proportional to the amplitude of the wave function at $r=R$. Therefore, for bound states, γ^2 does not approach to Teichmann-Wigner's sum rule limit, but considerably smaller than it even in the ideal single particle model. The upper limit of θ^2 may be expected to be of the order of 10^{-1}.

The radial part of the wave function in the tail region can be written, using θ_{ca} instead of γ_{ca}, as follows,

$$\sqrt{3}\cdot R^{-3/2}\cdot\theta_{ca}\{h^{(1)}_{l_a}(i\kappa_a r)/h^{(1)}_{l_a}(i\kappa_a R)\}. \tag{24}$$

Then, the transition matrix element between the states a and λ, $\langle\lambda|V_E|a\rangle$, which appears in the second order perturbation $\sum_{\lambda}\langle b|V_E|\lambda\rangle\langle\lambda|V_E|a\rangle(E-E_\lambda)^{-1}$, can be written, apart from the geometrical factor, as follows :

$$R^{-3}\theta_{ca}\cdot\theta_{c\lambda}\int d\boldsymbol{r}_1 e^{i(\boldsymbol{k}_a-\boldsymbol{k}_\lambda)\boldsymbol{r}_1}\int d\boldsymbol{r}_2 V_E(\boldsymbol{r}_1-\boldsymbol{r}_2)\,\{h^{(1)}_{l_a}(i\kappa_a r_2)/h^{(1)}_{l_a}(i\kappa_a R)\}$$
$$\times\{h^{(1)}_{l_\lambda}(i\kappa_\lambda r_2)/h^{(1)}_{l_\lambda}(i\kappa_\lambda R)\}\,Y_{l_a m_a}(\Omega_2)\,Y^*_{l_\lambda m_\lambda}(\Omega_2).$$

Here the quantities with the suffix λ correspond to the ones with the suffix a in the state a.

Adopting the nuclear force with the form of δ function, we may safely think as if the strength of the interaction V_0, defined in (21), would be reduced to $3(\alpha R)^{-3}\Theta_{ca}\Theta_{c\lambda}V_0$. For light nuclei with the ideal single particle model, which is apparently the worst case for the convergence of perturbation, $3(\alpha R)^{-3}\Theta_{ca}\Theta_{c\lambda}$ is, on an average, of the order of 10^{-1} or probably much smaller. Moreover, the restriction of the integral region $r \geq R$ may be favourable to the convergence. Thus, the approximate procedure of (A) may be expected to be rather good, provided the approximation (B) is applicable.

It may be worth noting that the success of the Butler theory[14] in the deuteron-stripping reaction may be due to similar circumstances.

ii) *Distorted wave method.*

Reserving the approximation (A), the approximation (B) will be improved in the following.

We shall approximate (7a) and (7b),

$$[1+(E+i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_a=\Phi_a-\sum_{\beta}T_{\beta a}^{(c)}\cdot E_{\beta}^{(+)}, \tag{7a}$$

$$[1+(E-i\epsilon-H_0-V_I)^{-1}\cdot V_I]\Phi_b=-\{\Phi_b-\sum_{\gamma}T_{\gamma b}^{(c)}\cdot E_{\gamma}^{(+)}\},^* \tag{7b}$$

by taking into account the wave function in the entrance channel only, namely, by setting $\beta=a$ in (7a) and $\gamma=b$ in (7b). The contributions from other channels will be entirely neglected. As was pointed out in section 2, the matrix elements $T_{aa}^{(c)}$ and $T_{bb}^{(c)}$ represent the processes due to the internal region interaction V_I only. Therefore, they differ from the diagonal matrix element, T_{aa} and T_{bb}, of the system which contain contribution of the external-region-interaction V_E. The differences, however, may be negligible, since the interaction of the external region is very small compared with that of the internal one.* Then, the distorted wave may be safely determined by the experimental data of nucleon scattering by nuclei. $\{\Phi_a-T_{aa}E_a^{(+)}\}$ may be represented as follows:

$$\begin{aligned}&2\pi\sum_{L_aM_a}i^{L_a+1}\cdot v_{L_a}(r_1)Y_{L_aM_a}(\Omega_1)\cdot Y_{L_aM_a}^*(\Omega_a)g(\mathbf{r}_2\xi\,;\,I_aK_a),\\&v_{L_a}(r_1)\equiv h_{L_a}^{(2)}(k_ar_1)-\eta_{L_a}h_{L_a}^{(1)}(k_ar_1),\end{aligned} \tag{25}$$

where Ω_a is the direction of $\boldsymbol{k}_a$. Eq. (25) is chosen so as to approach to Φ_a in the limit of $\eta_{L_a}\to 1$ for all L_a.

Then, using the δ-function interaction, the transition matrix element is given by

$$\begin{aligned}T_{ba}^{(d)}=4\pi^2C(2M/\hbar^2R)\gamma_{ca}\cdot\gamma_{cb}&\sum_{\substack{L_aL_bM_b\\m_am_b\nu}}i^{L_a-L_b}\sqrt{(2L_a+1)/4\pi}\\&\cdot(l_ajm_a\nu|I_aK_a)(l_bjm_b\nu|I_bK_b)\\&\times\mathcal{C}(L_aL_bl_al_b,\,0M_bm_am_b)\ \mathcal{R}(L_aL_bl_al_b,\,R)Y_{L_bM_b}(\Omega_b),\end{aligned} \tag{26}$$

where

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* The argument may not hold for non-diagonal matrix elements, because, though the absorption of wave takes place mainly in the internal region, its fraction in a channel may be small due to the large number of open channels. See § 6.

$$\mathcal{R}(L_aL_bl_al_b,\ R)\equiv\int_R^\infty v_{La}(r)\,v_{Lb}(r)\,\{h_{l_a}^{(1)}(i\kappa_a r)/h_{l_a}^{(1)}(i\kappa_a R)\}$$
$$\times\{h_{l_b}^{(1)}(i\kappa_b r)/h_{l_b}^{(1)}(i\kappa_b R)\}\,r^2dr, \tag{27}$$

and

$$\mathcal{A}(L_aL_bl_al_b;\ 0\,M_bm_am_a)\equiv\int Y_{La0}\,Y^*_{L_bM_b}Y_{l_am_a}Y^*_{l_bm_b}\,d\Omega. \tag{28}$$

Here the quantization axis of angular momenta is taken to the direction of $\boldsymbol{k}_a$ and, consequently, $M_a=0$.

In a similar way to that in i), (26) may be represented as follows:

$$T_{ba}^{(d)}=\pi\,C(2M/\hbar^2R)\gamma_{bc}\gamma_{ac}\sum i^{L_a-L_b+l_a-l_b-\lambda}(-1)^{j+K_b+M_b}$$
$$\times(L_aL_b0\,0|\lambda 0)\,(L_aL_b0-M_b|\lambda\mu)\,(I_aI_bK_a-K_b|\lambda\mu)$$
$$\times\sqrt{(2L_a+1)(2L_b+1)}\cdot(2\lambda+1)^{-1}\cdot\sqrt{(2L_a+1)/4\pi}\cdot Z(l_aI_al_bI_b,\ j\lambda)$$
$$\times\mathcal{R}(L_aL_bl_al_b,\ R)\,Y_{L_bM_b}(\Omega). \tag{29}$$

As is apparent in i), the convergence of the Born approximation is assured by the small amplitude of the wave function of the target nucleus in the external region. Therefore, if the amplitudes of the wave functions of incident and emitted neutrons do not differ so seriously from that of plane wave, the distorted wave method will converge rapidly, while the large increase of these amplitudes near nuclear surface, which is the case of resonances, may destroy the convergence. Then, it may be sufficient to discuss in what cases these amplitudes increase considerably to destroy the convergence.

As the distorted wave function in the external region can be represented as $j_L(kr)+T_Lh_L^{(1)}(kr)$, its amplitude near $r=R$ may be almost the same as that of plane wave when $|T_L|\ll 1$. If we adopt the distorted wave by a real potential, T_L can be written as $\sin\delta_L\cdot\exp(i\delta_L)$ and resonances occur at $\delta_L=\pi/2$, where the absolute value of T_L takes the maximum value unity and, consequently, the amplitude is in maximum. As is well known, the associated Hankel function of the first kind $h_L^{(1)}(kr)$ can be approximated by its asymptotic form $(kr)^{-1}\exp i(kr+\pi L/2)$ for $kr\geq L(L+1)/2$. Therefore, when k_aR and k_bR are much larger than unity, the convergence may be expected to be rather good even if kR corresponds just to resonance. Of course, the distorted waves with large angular momenta which take part in the reaction may sometimes be accompanied by large amplitudes near $r=R$. Such a case, however, does not play an important role in the final result of the process, since there are many other partial waves with smaller angular momenta, for which correct results may be obtained by the distorted wave method. Then, we may conclude that the distorted wave method may be a good approximation for intermediate and heavy nuclei except for the energy region near threshold, and for light nuclei with higher energy region.

As kR becomes much smaller than unity, $h_L^{(1)}(kr)$ can be approximated by $-i\cdot 1\cdot 3\cdot 5\cdots(2L-1)/(kr)^{L+1}$, whereas $j_L(kr)$ approachs to zero as $(kr)^L/\cdot 1\cdot 3\cdot 5\cdots(2L+1)$. Therefore, the amplitude of distorted wave may become very large if $kR\ll 1$, except in the case of $T_L \simeq 0$. Moreover, only few number of partial waves with small angular momenta can take part in the reaction. In view of the above consideration, the distorted wave method may not be valid for $k_a R \ll 1$ and $k_b R \ll 1$, especially in the case of resonance.

§ 4. Differential and total cross sections

In this section the differential and total cross sections are calculated in the case of the approximation ii) in the preceding section.

The differential cross section of the surface direct process by single particle excitation can be written as follows:

$$d\sigma(a,b)/d\Omega = \sum_{K_a K_b} (2I_a+1)^{-1} \cdot (M/2\pi\hbar^2)^2 (E_b/E_a)^{1/2} \cdot |T_{ba}^{(d)}|^2, \tag{30}$$

where $T_{ba}^{(d)}$ was given by (29).

It may be convenient to express (30) as a series of Legendre polynomials $P_L(\cos\theta)$, which may be performed in an analogous way to Blatt and Biedenharn's[15]. The result is

$$\begin{aligned} \frac{d}{d\Omega}\sigma(a,b) = {} & \frac{1}{16}\left(\frac{2M}{\hbar^2}V_0\right)^2 \left(\frac{2M}{\hbar^2}\gamma_{ca}^2\right)\left(\frac{2M}{\hbar^2}\gamma_{cb}^2\right)\frac{1}{R^2}\cdot\frac{1}{\alpha^6}\cdot\frac{1}{2I_a+1} \\ & \times \sum (-1)^{L_a'-L_a-\lambda} \frac{\sqrt{(2L_a+1)\,(2L_b+1)}}{2\lambda+1} (L_a L_b 0 0|\lambda 0)\,(L_a' L_b' 0 0|\lambda 0) \\ & \times Z(L_b L_b L_b' L_b', 0L)\, Z(L_a L_b L_a' L_b', \lambda L)\, Z(l_a I_a l_b I_b, j\lambda)^2 \\ & \times R_e[\mathcal{R}^*(L_a' L_b' l_a l_b, R)\, \mathcal{R}(L_a L_b l_a l_b, R)]\, P_L(\cos\theta), \end{aligned} \tag{31}$$

where the relation (21), i. e., $C = (4\pi/\alpha^3) V_0$, is utilized.

The total cross section of the process is readily obtained by taking the coefficient of $P_0(\cos\theta)$ multiplied by 4π as

$$\begin{aligned} \sigma(a,b) = {} & \frac{\pi}{4}\left(\frac{2M}{\hbar^2}V_0\right)^2 \cdot \left(\frac{2M}{\hbar^2}\gamma_{ca}^2\right)\left(\frac{2M}{\hbar^2}\gamma_{cb}^2\right)\frac{1}{R^2\alpha^6}\sqrt{\frac{E_b}{E_a}}\cdot\frac{1}{2I_a+1} \\ & \times \sum_{\lambda} \frac{Z(l_a I_a l_b I_b, j\lambda)^2}{2\lambda+1} \sum_{L_a L_b} (2L_a+1)\,(2L_b+1)\,(L_a L_b 0 0|\lambda 0)^2 |\mathcal{R}(L_a L_b l_a l_b, R)|^2. \end{aligned} \tag{32}$$

In the same manner as in (17), the radial integral $\mathcal{R}(L_a L_b l_a l_b, R)$,

$$\int_R^\infty v_{L_a}(r) \cdot v_{L_b}(r) \{h_{l_a}^{(1)}(i\kappa_a r)/h_{l_a}^{(1)}(i\kappa_a R)\}\{h_{l_b}^{(1)}(i\kappa_b r)/h_{l_b}^{(1)}(i\kappa_b R)\} r^2 dr,$$

may be approximated as follows:

$$\mathcal{R}(L_a L_b l_a l_b, R) \simeq v_{L_a}(R)\, v_{L_b}(R)\, \mathcal{R}(l_a l_b, R), \tag{33}$$

where

$$\mathcal{R}(l_a l_b, R) \equiv [h_{l_a}^{(1)}(i\kappa_a R) \cdot h_{l_b}^{(1)}(i\kappa_b R)]^{-1} \cdot \int_R^\infty h_{l_a}^{(1)}(i\kappa_a r)\, h_{l_b}^{(1)}(i\kappa_b r)\, r^2 dr, \tag{34}$$

which can be integrated analytically only if $l_a = l_b \equiv l$:

$$\mathcal{R}(ll, R) = [R^2/(\kappa_a^2 - \kappa_b^2)] \cdot [\kappa_b \cdot h_{l_b-1}^{(1)}(i\kappa_b R) \cdot h_{l_a}^{(1)}(i\kappa_a R)$$

$$- \kappa_a \cdot h_{l_a-1}^{(1)}(i\kappa_a R) \cdot h_{l_b}^{(1)}(i\kappa_b R)], \tag{34a}$$

$$\mathcal{R}(0\,0, R) = R^2 \cdot / (\kappa_a + \kappa_b). \tag{34b}$$

Using this approximation and representing the reduced widths γ_{ca}^2 and γ_{cb}^2 in terms of the unit of sum rule limit of Teichmann and Wigner, the total cross section (32) becomes

$$\sigma(a, b) = \sum_{\lambda} F_s(l_a I_a l_b I_b j \lambda, R) \sum_{L_a L_b} (2L_a + 1)(2L_b + 1)$$

$$\times (L_a L_b 0\,0|\lambda 0)^2 |u_{L_a}(R)|^2 \cdot |u_{L_b}(R)|^2, \tag{35}$$

where

$$F_s(l_a I_a l_b I_b j \lambda, R) = \frac{9\pi}{4} \frac{V_0^2}{E_a \cdot E_b} \sqrt{\frac{E_b}{E_a}} \cdot \theta_{ca}^2 \cdot \theta_{cb}^2$$

$$\times \frac{1}{(\alpha R)^6} \frac{\mathcal{R}(l_a l_b, R)^2}{R^4} \frac{Z(l_a I_a l_b I_b j \lambda)^2}{(2I_a + 1)(2I_b + 1)}, \tag{36}$$

and

$$u_L(r) \equiv kr \cdot v_L(r),$$

which will be compared with that of collection excitation in the next section.

In order that (35) can be compared with the excitation function of the compound nucleus process, it seems convenient to express $|u_L(R)|^2$ in terms of the transmission coefficient $T_L(E)$. For this purpose, according to Blatt and Weisskopf[16], the following abbreviations are to be introduced :

$$\left[\frac{d\{r v_L(r)\}/dr}{v_L(r)}\right]_{r=R} \equiv f_L, \qquad \left[\frac{d\{r h_L^{(1)}(kr)\}/dr}{h_L^{(1)}(kr)}\right]_{r=R} \equiv \Delta_L + i S_L,$$

$$[h_L^{(2)}(kR)/h_L^{(1)}(kR)] \equiv \exp(2i\xi_L).$$

Then,

$$\eta_L = [(f_L - \Delta_L + i S_L)/(f_L - \Delta_L - i S_L)] \cdot \exp(2i\xi_L),$$

$$v_L(R) = h_L^{(2)}(kR)(-2iS_L)/(f_L - \Delta_L - iS_L),$$

$$|u_L(R)|^2 = 4kRS_L / \{(Re f_L - \Delta_L)^2 + (Im f_L - S_L)^2\}.$$

On the other hand, the transmission coefficient $T_L(E)$, which is defined by $\sigma_c^{(L)} = (2L+1)(1/k^2) T_L(E)$, can be represented as

$$T_L(E) = -4S_L Im f_L / \{(Re f_L - \Delta_L)^2 + (Im f_L - S_L)^2\}.$$

Therefore, $|u_L(R)|^2$ can be written in terms of the transmission coefficient as

$$|u_L(R)|^2 = (kR/-Im\, f_L)\, T_L(E). \tag{37a}$$

If we adopt the continuum theory, $Im\, f_L$ is equal to $-KR$, then

$$|u_L(R)|^2 = (k/K)\, T_L(E). \qquad \text{(Continuum Theory)} \tag{37b}$$

It is to be noted that, for group excitation, the results obtained in section 3 and 4 for the single particle excitation will also hold with slight modification, provided the approximations used before are all adopted.

§ 5. Comparison with collective excitation

In this section the total and differential cross sections obtained in the preceding section are compared with those of collective excitation calculated by Hayakawa and Yoshida[3].

In the strong coupling case of Bohr and Mottelson[17], the total cross section of a direct process by rotational excitation is given by

$$\sigma_R(a, b) = F_R(I_a I_b 2, R) \sum_{L_a L_b} (2L_a+1)(2L_b+1)$$
$$\times (L_a L_b 0 0 | 2 0)^2 |u_{L_a}(R)|^2 \cdot |u_{L_b}(R)|^2, \tag{38}$$

where

$$F_R(I_a I_b 2, R) = \frac{2I_b+1}{2I_a+1} \frac{\pi}{4^2 \times 5} \frac{W^2}{E_a \cdot E_b} \cdot \sqrt{\frac{E_b}{E_a}} \left(\frac{Q_0}{3ZR^2}\right)^2 \cdot R^2. \tag{38a}$$

Here Q_0 and Z are, respectively, the intrinsic quadrupole moment and atomic number of the target nucleus and W is a potential whose magnitude is assumed to be approximately 30 Mev.

i) Selection rules

In order that single particle excitation can take place it is necessary that the excited state can be represented as a state in which only one particle state is different from that in the ground state. In other words, the excited state should be formed by a single particle jump from the ground state, the core state remaining unchanged. Then, the selection rules imposed on the single particle excitation are merely usual ones arising from the conservation of angular momentum and parity which are involved in the Racah and Clebsch-Gordan coefficients appearing in the T-matrix. When the ground and excited states can be described not exactly but only approximately by the ideal shell model, the mixing of single particle configurations should be taken into account, for which we must perform the summation over l_a and l_b and evaluate the parentage over-lap of core states.

On the other hand, it is characteristic of the collective excitation that levels of the same family have the same parity and small spin changes between neighbouring states. In the prescription of Bohr and Mottelson the strong coupling case is of special importance for the rotational state. Therefore, we restrict ourselves mainly to this case which is applicable to the intermediate and heavy nuclei except for the immediate vicinity of major closed shell. The rotational states correspond to rotations about an axis perpendicular to the nuclear symmetry axis and, for even-even nuclei, their spins are given by 2, 4, 6, $\cdots$. The odd values of I do not occur, since such states would have odd parity. For odd-A nuclei the circumstances are slightly involved. However, it holds generally that the rotational states should have the same parity as that of the ground state.

The weak coupling situation expected in the immediate vicinity of major closed shells implies that the collective excitations to low-lying states are essentially of the simple phonon character. As was calculated by Hayakawa and Yoshida, its cross section shows just the same form as that for rotational excitation in the strong coupling case.

Then, we may at least conclude that the process is not caused by collective excitations, if the large spin change or change of parity between ground and excited states are inferred from the observation of angular distribution.

ii) Shapes of angular distribution and excitation function.

As is apparent from (35) and (38) the energy dependences of total cross sections are independent of the detailed mechanisms of the surface direct processes, provided the selection rules mentioned in (i) do not come into play.

Similarly, the angular distributions are of the same form for all surface direct processes associated with two given states of the target nucleus.

Therefore, it is impossible to conclude that certain inelastic scattering is caused, for example, by rotational excitation by observing only its angular distribution. For the further details, it is necessary to check the absolute value of the cross section.

iii) Mass number dependence of the cross sections.

In Eq (35) the factor dependent directly upon the mass number A is given by

$$\sigma_s(a.\ b) \propto \frac{1}{R^6}\,\frac{\mathcal{R}(l_a l_b,\ R)^2}{R^4} \sum_{L_a L_b} (2L_a+1)\,(2L_b+1)$$

$$\times\,(L_a L_b 0\,0|\lambda 0)^2 |u_{L_a}(R)|^2 \cdot |u_{L_b}(R)|^2. \qquad (39\text{a})$$

As it may be apparent in the preceding section, the factor $\mathcal{R}(l_a l_b,\ R)^2$ is roughly proportional to R^4. Therefore $\mathcal{R}(l_a l_b,\ R)^2/R^4$ may be considered to be A-independent. On the other hand, $|u_L(R)|^2$ was shown to be proportional to $T_L(E)$. As we are interested in the general trend of A-dependence, the proportional factor $kR/Im\,f_L$ is assumed to be approximately A-independent, which is the result of continuum theory, though it depends strongly upon A in the low energy regions where the cloudy crystal ball model[18] has been successfully applied with the small imaginary part of the potential. The transmisson coefficient $T_L(E)$ is connected with the capture cross section σ_c through the following relation,

$$\sigma_c = (1/k^2) \sum_L (2L+1)\,T_L(E).$$

As σ_c is proportional to R^2, the sum,

$$\sum_{L_a L_b} (2L_a+1)\,(2L_b+1)\,(L_a L_b 0\,0|\lambda 0)^2 |u_{L_a}(R)|^2 \cdot |u_{L_b}(R)|^2,$$

may be estimated to be roughly proportional to R^x, where x is between 2 and 4. Therefore, $\sigma_S(a, b)$ is proportional to R^{-y}, where $4>y>2$. The cross section of group excitation shows equal R-dependence, provided the reduced width in the unit of sum rule limit are the same for all nuclei.

In the case of collective rotational excitation, we have

$$\sigma_R(a, b) \propto (Q_0/3ZR^2)^2 \sum_{L_a L_b} (L_a+1)\,(2L_b+1)$$

$$\times\,(L_a L_b 0\,0|2\,0)^2 |u_{L_a}(R)|^2 \cdot |u_{L_b}(R)|^2. \tag{39b}$$

According to the collective model, the intrinsic quadrupole moment Q_0 is intimately connected with the deformability of nucleus. If we adopt hydrodynamical description of nucleus, $(Q_0/3ZR^2)$ may be considered to be A-independent. Then, $\sigma_R(a, b)$ is proportional to $R^4 \sim R^6$.

iv) *Relative ratio.*

We shall discuss the relative ratio of the single particle and collective rotational excitations in the transition from $I_a=0^+$ to $I_b=2^+$.

$$\frac{\sigma_S(a, b)}{\sigma_R(a, b)} = \frac{NF_S}{F_R} = \frac{36N}{5} \cdot \left(\frac{V_0}{W}\right)^2$$

$$\times \frac{Z(l_a 0 l_b 2, j2)^2}{\alpha^6 R^6} \frac{\Theta_{ca}^2 \cdot \Theta_{cb}^2}{(Q_0/3ZR^2)^2} \frac{\mathcal{R}(l_a l_b, R)^2}{R^6}. \tag{40}$$

Though, in the single particle excitation, we have focused our attention only on a single particle state with angular momentum l_a outside the core, there may be several particles in the state which can equally contribute to the process. Therefore, the cross section $\sigma_S(a, b)$ should be multiplied by a factor N, where N is approximately equal to the number of the nucleons in the state l_a*. From the analysis of two-body data[19], V_0 and α are taken to be about 60 Mev and 1×10^{-13} cm, respectively, whereas W is assumed to be about 30 Mev. On the other hand, the intrinstic quadrupole moment for the ground state of even-even nucleus is represented, in the strong coupling case of Bohr and Mottelson, as follows :

$$Q_0 = (1/16\pi)\,(g/C)\,(3ZR^2),$$

where g is a coupling constant between single particle and collective modes and is assumed to be about 40 Mev, whereas C represents the deformability of the nucleus. Adopting the hydrodynamical description of nucleus, C is taken to be 60 Mev for intermediate and heavy nuclei. For light nuclei it decreases to $20\sim 30$ Mev[17]. Then, for intermediate and heavy nuclei,

$$Q_0 = (1/12\pi)\,(3ZR^2). \tag{41}$$

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* Strictly speaking, the factor N depends on the detailed structure of the ground and excited states under consideration. Its explicit expression can be obtained by using the anti-symmetric wave functions and performing the summation over i in Eq. (10).

Therefore, (40) becomes

$$(4N/A^{8/3})\,(36/5r_0^8)\,(12\pi\,\theta_{ca}\cdot\theta_{cb})^2 Z(l_a 0 l_b 2,\, j2)^2/(\kappa_a+\kappa_b)^2,$$

where r_0 is defined by $R=r_0 A^{1/3}\times 10^{-13}\,\mathrm{cm}$: r_0 and $(\kappa_a+\kappa_b)^{-1}$ in the above expression should be taken in unit of 10^{-13} cm. Here $\mathcal{R}(l_a l_b, R)$ is approximately replaced by $\mathcal{R}(0\,0, R)$, which leads to a slight underestimate. We may put $(36/5r_0^8)\simeq 1$ and $(\kappa_a+\kappa_b)^2\simeq 1$. The upper limit of $Z(l_a 0 l_b 2, j2)^2$ is given by 5, which is the case of $l_a=0$ and $l_b=2$. The other choice of l_a and l_b gives a smaller value. As is discussed in section 3, Θ^2 may be estimated to be of the order of 10^{-1}. Therefore, $(12\pi\,\Theta_{ca}\,\Theta_{cb})^2$ may be expected to be 10 or smaller.

Then, we can conclude that the cross section of the collective rotational excitation is much larger than that of the single particle excitation especially for intermediate and heavy nuclei.

As is well known, the hydrodynamical values of Q_0 are in general larger than the empirical ones by factor two at least. Moreover, there are strong correlations between the nuclear deformability and shell structure[17),20)]. However, the conclusion mentioned above will be unchanged qualitatively by the correct estimates of Q_0 and other parameters for intermediate and heavy nuclei, for which the rotational excitation is of special interest.

As is mentioned repeatedly, the mechanisms which work in actual nuclear reactions are governed by the natures of the target and residual nuclei. Therefore, if the surface direct process associated with the ground and an excited state is known to be caused by the rotational excitation, other mechanisms, e. g. single particle excitation, cannot, of course, occur. However, we can expect from the analysis mentioned above that the cross section of a surface direct process leading to the rotational state of a nucleus is larger than that of the process leading to a single particle state of the same nucleus, since the cross section of the single particle excitation is not changed so drastically as to alter the result obtained above by the other choice of I_b, E_b, l_a and l_b.

Such circumstances may also hold between different nuclei, provided their mass numbers are not so different.

And, conversely, we may further be allowed to expect that the discrimination of the natures of excited states, rotational and single particle, may be performed by the absolute values of the cross sections. In this case the contribution from the compound nucleus should, of course, be taken into account, which will be discussed in the next section.

§ 6. Comparison with compound nucleus process

In this section the excitation function of the surface direct process will be compared with that of the compound nucleus process. As it was shown in the preceding section that the cross section of collective excitation is much larger than that of single particle excitation for intermediate and heavy nuclei, here we shall concern ourselves mainly with the collective excitation.

The inelastic scattering leading to a low-lying excited state has been formulated by

Hauser and Feshbach[21)] using the statistical compound nucleus hypothesis:

$$\sigma_c(a,\ b) = (\pi/k_a^2)\ (2I_a+1)^{-1}\cdot\sum_{L_a j_a} T_{L_a}(E_a) \sum_J (2J+1)$$
$$\times\Big[\sum_{LjE} T_L(E)\Big]^{-1} \sum_{L_b j_b} T_{L_b}(E_b)\,. \tag{42}$$

Here J is the total angular momentum of the compound nucleus and j_a and j_b are the channel spins of the initial and final states, respectively. The summation over L, j and E are to be extended to all open channels from the compound state J, while the summation over L_a, L_b, j_a and j_b are extended to all possible values with the restriction of the conservation law of angular mementum and parity.

On the other hand, the cross section of rotational excitation can be represented as

$$\sigma_L(a,\ b) = (K_0 R/48\pi)^2 (K_0/K)^2 (\pi/k_a^2) \sum_{L_a L_b} (2L_a+1)\ (2L_b+1)\ (L_a L_b 0 0|2 0)^2$$
$$\times T_{L_a}(E_a)\, T_{L_b}(E_b)\,, \tag{43}$$

$$(\hbar^2 K_0^2/2M) = W$$

where the continuum theory, eq. (41), and the hydrodynamical value of Q_0, eq. (37b), are adopted.

Of course, the continuum theory may not be applicable in low energy region, especially in the case of the shape resonance[18)], where $Im\, f_L$ is very small compared with that of the continuum theory, i. e., $(-KR)$. On the other hand, the hydrodynamical value of Q_0 is well-known to be a considerable overestimate. Therefore the conclusions obtained in this section may be only qualitative with respect to the general trend of the energy dependence of the cross sections.

i) Comparison at low energy.

When the energy of an incident neutron is near the threshold, (42) can be easily evaluated, since there are few open channels and only small values of angular momenta L_a and L_b of incident and emitted neutrons may contribute to the reaction.

Assuming that only one channel is open except the entrance one and that $I_a = 0^+$ and $I_b = 2^+$, the explicit form of (42) is represented as follows:

$$\sigma_c(a,b) = \left(\frac{\pi}{k_a^2}\right)\Big\{ T_0(E_a)\,\frac{2}{T_0(E_a)+2T_2(E_b)}\,T_2(E_b)$$
$$+\,T_1(E_a)\left[\frac{1}{T_1(E_a)+T_1(E_b)+T_3(E_b)}\right.$$
$$\left.+\,\frac{4}{T_1(E_a)+2T_1(E_b)+2T_3(E_b)}\right][T_1(E_b)+T_3(E_b)]$$
$$+\,T_2(E_a)\,\frac{5}{T_2(E_a)+T_0(E_b)+2T_2(E_b)}\,[T_0(E_b)+2T_2(E_b)]$$
$$+\,T_3(E_a)\left[\frac{6}{T_3(E_a)+2T_1(E_b)+T_3(E_b)}\,[2T_1(E_b)+T_3(E_b)]\right.$$

$$+\frac{4}{T_3(E_a)+T_1(E_b)+2T_3(E_b)}\left[T_1(E_b)+2T_3(E_b)\right]\Big]\Big\}. \qquad (44)$$

On the other hand, (43) can be written as

$$\begin{aligned}\sigma_R(a,\, b) \simeq (KR/48\pi)^2(\pi/k_a^2)\,\{&5\,T_0(E_a)\,T_2(E_b)\\ &+T_1(E_a)\left[6\,T_1(E_b)+(9/6)\,T_3(E_b)\right]\\ &+T_2(E_a)\left[(9/7)\,T_0(E_b)+(50/7)\,T_2(E_b)\right]\\ &+T_3(E_a)\left[(9/4)\,T_1(E_b)+(28/3)\,T_3(E_b)\right]\}. \end{aligned} \qquad (45)$$

Here L_a and L_b are taken up to 3.

Comparing each coefficient of $T_{L_a}(E_a)\cdot T_{L_b}(E_b)$ in (44) with that in (45) and taking into account the fact that the collective excitation is much more likely than the single particle excitation in the case of surface direct process, it may be concluded that the compound nucleus process is much larger than the surface direct process in low energy region where only few channels are open.

ii) *Comparison at higher energy.*

When the energy of an incident neutron is sufficiently high so that many channels are energetically open, actual calculation of (42) is quite involved. Therefore, we shall present an approximate formula for higher energy region.

Following Hauser and Feshbach, the level density of final states is assumed to have $(2j+1)^2$ degeneracy, namely the number of levels of the residual nucleus at excitation energy (E_a-E) is assumed to be $(2j+1)\,\omega(E_a-E)\,dE$. One of the factors $(2j+1)$ has been already contained in (42). Then the denominator of (42) can be written as follows:

$$\sum_{jEL} T(E) \simeq \sum_{jL}(2j+1)\int_0^{E_a}\omega(E_a-E)\,T_L(E)\,dE. \qquad (45)$$

The summation over j for fixed L and J can be easily performed with the result of $(2J+1)\,(2L+1)$.
Therefore, (45) becomes

$$(2J+1)\int_0^{E_a}\sum_L(2L+1)\,\omega(E_a-E)\,T_L(E)\,dE=\frac{(2J+1)}{\pi}\boldsymbol{F}_n(E_a)$$

$$\boldsymbol{F}_n(E_a)\equiv\int_0^{E_a}k^2\sigma_c(E)\,\omega(E_a-E)\,dE.$$

Then,

$$\sigma_c(a,\, b)=\{2\,(2I_a+1)\}^{-1}\cdot(\pi^2/k_a^2)\,(1/\boldsymbol{F}_n(E_a))\sum_{L_a j_a}T_{L_a}(E_a)\,\{\sum_{J L_b j_b}T_{L_b}(E_b)\}. \qquad (46)$$

Further, if the final state b is also a compound nucleus state with the level density $(2j_b+1)\,\omega(E_a-E_b)$, we get the well-known formula of Weisskopf[16] for the energy spectrum of emitted neutrons,

$$\sigma_c(a,\, b)\, dE_b = \sigma_c(E_a) \left[\int_0^{E_a} E\sigma_c(E)\, \omega(E_a - E)\, dE \right]^{-1} \cdot E_b \sigma_c(E_b)\, \omega(E_a - E_b)\, dE_b,$$

since

$$\sum_{j_a j_a J} (2j_b + 1) = 2(I_a + 1)\,(2L_a + 1)\,(2L_b + 1).$$

Because of its approximate Maxwellian form with the maximum at nuclear temperature τ, it is apparent that the inelastic scattering cross section to a low-lying excited state is very small compared with the total inelastic cross section which is nearly independent of the incident energy. Moreover, $\sigma_c(a, b)$ decreases exponentially with increasing energy due to the increase of open channels, as may be seen by the denominator of (46).

In order to calculate $F_n(E)$, the denominator of (46), the level density formula of Blatt and Weisskopf[16] is adopted,

$$\omega(E) = \boldsymbol{C} \exp(2\sqrt{\boldsymbol{a}E}),$$

where $\boldsymbol{C}$ and $\boldsymbol{a}$ is the parameters whose magnitudes have been tabulated in their text.

Further, it may be convenient for qualitative illustration to use Fig. 6.3 of reference (16), where the nuclear radius R was assumed to be $r_0 A^{1/3}$ with $r_0 = 1.3 \times 10^{-13}$ cm.

Then the following conclusions, though qualitative, can be obtained : For intermediate and heavy nuclei bombarded by 10 Mev neutrons the surface direct process is much larger than the compound nucleus process. In particular, the latter may be negligibly small for heavy nuclei. If an incident energy is about 5 Mev, both processes may be of the comparable order for intermediate nuclei with $A = 60 \sim 100$.

§ 7. Discussions

We have discussed the surface direct processes from the view-points of the shell model and of the collective model. Though these models might seem to be of opposite natures, they should, of course, be considered from a unified point of view[17),22)] ; namely, nucleons in a nucleus are to be considered to move in a common shell model potential which is, in general, not spherical symmetric, but of deformed shape. Then, the excited states of the nucleus may be classified into two classes of collective and individual particle characters. The former may also be divided into vibrational and rotational states, while the latter may be expressed as a linear combination of the products of particle states in the potential. Due to the deviation of the potential from spherical symmetry, the single particle states are, of course, slightly different from those of the ideal shell model.

From the results obtained in the previous sections, we may be allowed to expect that the discrimination of the characters of excited states, collective or individual, may be performed by the observation of the absolute cross section of inelastic scattering together with its angular distribution. For intermediate and heavy nuclei the surface direct process leading to a certain low-lying state can be observed only when the state is of the collective character, especially when there are few nucleons in an outer single particle orbit l_a. The contributions from the compound nucleus cannot, of course, be neglected in lower energy region. However, since its angular distribution is symmetric about 90° direction and the excitation curve decreases more rapidly with increasing energy, it may easily be subtracted from the total cross section of inelastic scattering.

The success of the cloudy crystal model of Feshbach, Porter and Weisskopf[18)] implies a surprisingly long mean free path of an incident nucleon in nuclear matter in the energy region of 0–3 Mev, which might suggest the existence of volume direct process. However, this model implies also the large reflection of wave at nuclear surface. Therefore, the volume direct process may be expected to be very small, which is proved semiclassically by Hayakawa, Kawai and Kikuchi[24)].

On the other hand, in order that the statistical treatment is applicable to the compound nucleus, it should be necessary that a nuclear interaction is so strong that the incident particle readily forms a chaotic state together with the target nucleus. The long mean free path implied by the cloudy crystal ball model may also suggest "effectively" weak interactions in nuclear matter. However, it is only the behaviour of an incident particle toward the target nucleus in the "ground state", which is informed by total and elastic scattering cross sections. It is likely that the small imaginary part of the low energy optical potential is due to the effect of the Pauli principle. Then, the mean free path for highly excited nucleus may be expected to be nearly 1×10^{-13} cm. Therefore, we may not conclude from the work of Feshbach, Porter and Weisskopf that the statistical compound nucleus hypothesis is not valid for actual treatment of any nuclear reaction.

If we adopt the statistical compound nucleus hypothesis for the internal-region-interaction, the cross section is given by the formula (12) with (12b), in which the first and second terms are given by eq. (5) of reference (21) and eq. (31) of § 4, respectively.

Further, it may be natural to evaluate the transmission coefficients by the use of Feshbach, Porter and Weisskopf's potential, when the incident energy is below 3 Mev and the target nucleus is sufficiently heavy so that many levels are contained in the energy spread of the incident beam. It may be worth noting that the capture cross sections of neutrons calculated by the potential are considerably smaller than the experimental data. Such circumstances also occur in the higher energy region of $3\sim15$ Mev if we adopt the optical potential of square well shape which is chosen so as to reproduce the total cross sections in the energy region.

Strictly speaking, the potential which determines the distorted wave of the final state differs from that of the initial state, because the nucleus in the final state is not in its ground state.

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Nuclear Magnetic Relaxation in Solid Hydrogen

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A theoretical study of the mechanisms of the relaxation in solid H_2 and solid D_2 related with their nuclear magnetic resonances is given. When the self-diffusion of the molecules is not significant, the predominant nuclear relaxation mechanism is considered to result from the intramolecular interaction modulated by the intermolecular interaction which makes the orientations of the molecules change from time to time. The formula for T_1 due to this mechanism is given and the actual calculations are carried out for solid H_2 and solid D_2 at temperatures well above the transition point. The result for H_2 agrees with Bloom's ($>$11°K) and Hatton-Rollin's (4.2~10°K) experimental values.

The effect of self-diffusion on T_1 is treated both for H_2 and D_2 using the B. P. P. theory and the values of the self-diffusion time estimated from Bloom's data for T_2. This effect is small under Bloom's experimental condition (30 Mc), while it is important in Hatton-Rollin's case (5.4 Mc). This agrees with their experimental results though for the latter case the theoretical temperature dependence of T_1 is stronger than the experimental one.

Discussion is also given on the case where the self-diffusion becomes very rapid and it modulates appreciably the intermolecular forces dependent on the orientation of the molecules. Bloom's experiment in the liquid state seems to be explained by this mechanism.

The role of the spin-spin relaxation in establishing the thermal equilibrium within the whole spin system is discussed. This effect will be especially important for solid D_2.

§ 1. Introduction

The state of solid hydrogen has been studied both experimentally and theoretically by a number of authors.[1] The point of interest lies in the molecular rotation of the ortho-molecule ($J=1$) which remains in the solid state. The three-fold rotational degeneracy of the ortho-molecule is lifted by inter-molecular interactions at a certain low temperature. This rotational transition occurs at 1.6°K (for the case of 75% ortho-concentration) where the specific heat shows a λ-type anomaly[2] and the nuclear magnetic resonance line shape changes abruptly.[3],[4] By analysing this abrupt change of the resonance line width, Tomita[5] pointed out that the rotational transition or the quenching of the molecular rotation has a cooperative character. Nakamura[6] also noticed the cooperative character of the transition and calculated the tail of the anomalous specific heat by using the moment expansion method starting from all the possible intermolecular interaction

Hamiltonians. He found a good agreement with the experiment. Especially he showed that the electric quadrupole-quadrupole interaction between ortho-molecules plays a predominant role in the quenching of the molecular rotation. This calculation seems to confirm us that our knowledge of the intermolecular interactions between ortho-molecules are now well established. As the intra-molecular interaction was elucidated by the molecular beam experiments,[7] it may be said that we have now an almost thorough knowledge of the intra- and inter-molecular interactions of hydrogen molecules.

It seems, therefore, to be interesting to make a quantitative calculation of the nuclear relaxation time of solid hydrogen related with its magnetic resonance absorption starting from the fundamental Hamiltonian.

In the present paper we shall deal with this problem. We can consider the following two relaxation mechanisms:

1) Intramolecular dipole-dipole interaction and the $\boldsymbol{I}-\boldsymbol{J}$ coupling modulated by the intermolecular interaction which changes the orientations of the molecules from time to time.

2) Intermolecular dipole-dipole interaction modulated by the lattice vibrations and by the self-diffusion of the molecules.

The first mechanism was pointed out earlier by Sugihara* though he made only a rough order estimation. The second mechanism is not effective except when the self-diffusion is significant as one can infer from Waller's theory.[8] When the self-diffusion is significant one can use the Bloembergen-Purcell-Pound theory[9] or more refined Torrey's theory.[10]

The present work is chiefly concerned with the first mechanism. The problem is very similar to that of the nuclear relaxation in antiferromagnetic substances which was treated by one of the authors[11] and by Kranendonk and Bloom.[12] In antiferromagnetic substances, the magnetic interaction between the nuclear spin and the electron spins modulated by the exchange interaction is the predominant nuclear relaxation mechanism. In the solid hydrogen, the modulation of the intra-molecular interaction is supplied by the inter-molecular interaction especially by the quadrupole-quadrupole interaction, instead of the exchange interaction in antiferromagnetics. The calculation of the relaxation time in the solid hydrogen can be carried out following the same line as in the case of antiferromagnetics.

However, the ordered arrangement of the molecules in solid hydrogen below the transition point has not yet been known. Accordingly, the calculation of T_1 below the transition point seems to be difficult at the present stage. We can, however, make use of the moment expansion method well above the transition point. This method can be applied whenever the interaction Hamiltonian is known. As was stated above, the Hamiltonian of the problem is well known. This makes the calculation in this case more quantitative than in the case of antiferromagnetics where the exchange integral is an adjustable parameter and the vector coupling of spins itself has a model character.

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* Spoken at the meeting of the Physical Society of Japan at Sendai, July 1955.

In the following sections we shall at first give the formula of the relaxation time due to the first mechanism and carry out the actual calculation of it above the transition point (§ 2 and § 3). The effect of self-diffusion will be discussed in § 4 and the comparison of the theory with experiment will be made in § 5. The case of solid D_2 will also be treated in § 6 and some discussions on this problem will be given in § 7.

§ 2. Relaxation by intermolecular dipole-dipole interaction

The Hamiltonian of a hydrogen molecule placed in a static magnetic field (in the z direction) can be written as[7)]

$$\mathfrak{H}=\mathfrak{H}_{ab}+\mathfrak{H}_{cd}, \tag{2·1}$$

$$\mathfrak{H}_{ab}=-aI_z-bJ_z, \tag{2·2}$$

$$\mathfrak{H}_{cd}=-c\boldsymbol{I}\cdot\boldsymbol{J}-5d[3(\boldsymbol{I}^{(1)}\cdot\boldsymbol{r}/r)(\boldsymbol{I}^{(2)}\cdot\boldsymbol{r}/r)-\boldsymbol{I}^{(1)}\cdot\boldsymbol{I}^{(2)}], \tag{2·3}$$

where

$$h^{-1}a=h^{-1}2\mu_i H_0=4.258\,H_0 \text{ kc/sec,}$$

$$h^{-1}b=h^{-1}(\mu_J/J)H_0=0.6717\,H_0 \text{ kc/sec,}$$

$$h^{-1}c=h^{-1}2\mu_i H'=113.8 \text{ kc/sec,}$$

$$h^{-1}d=h^{-1}(4/5)\mu_i^2\langle r^{-3}\rangle=57.68 \text{ kc/sec.} \tag{2·4}$$

$\boldsymbol{I}=\boldsymbol{I}^{(1)}+\boldsymbol{I}^{(2)}$ is the total nuclear spin, $\boldsymbol{J}$ the rotational angular momentum, μ_i the magnetic moment of the proton, μ_J the rotational magnetic moment of the molecule in state J. The terms with c and d represent, respectively, the magnetic $\boldsymbol{I}-\boldsymbol{J}$ coupling and the magnetic dipole-dipole interaction between the two protons in the molecule; $\boldsymbol{r}$ is the vector along the line joining two protons.

When the external field is sufficiently strong, the nuclear spin state can be specified by the quantum number $I_z=m$ and the transitions corresponding to $\Delta m=\pm 1$ are observed. We are concerned only with the ortho-molecule ($J=1$, $I=1$) as the para-molecules do not take part in the magnetic resonance.

Since the $\boldsymbol{J}$-dependent intermolecular interaction is much stronger than the interaction (2·1) in normal solid hydrogen, it determines the rotational state. The Hamiltonian expressing the intermolecular interaction can be written as

$$\mathfrak{H}_R=\sum_i \mathfrak{H}_i+\sum_{i>k}\mathfrak{H}_{ik}, \tag{2·5}$$

where the first term, common to ortho-para and ortho-ortho interactions, comes from the valence force and dispersion force and the second term, which is proper to ortho-ortho interaction, includes the quadrupole-quadrupole interaction as a main part as well as the valence and dispersion forces.[6)] The rotational state of each molecule is determined by (2·5), and by using it the resonance pattern is obtained from (2·1)~(2·3). The qualitative discussion of the resonance pattern was given by Reif and Purcell.[3)]

The Hamiltonian $\mathfrak{H}_{cd}$ gives the shift of the resonance frequency (or the structure of the resonance line) and the thermal relaxation. This can be written, when we restrict ourselves to ortho-molecule, as follows:

$$\begin{aligned}\mathfrak{H}_{cd}=&-d\boldsymbol{J}^2\boldsymbol{I}^2-(c-3d/2)\boldsymbol{I}\cdot\boldsymbol{J}+3d(\boldsymbol{I}\cdot\boldsymbol{J})^2\\ =&-4d-(c-3d/2)I_zJ_z+3d\{I_z^2J_z^2+\tfrac{1}{4}(I_+I_-J_-J_++I_-I_+J_+J_-)\}\\ &-\tfrac{1}{2}(c-3d/2)I_+J_-+(3d/2)(I_zI_+J_zJ_-+I_+I_zJ_-J_z)\\ &-\tfrac{1}{2}(c-3d/2)I_-J_++(3d/2)(I_zI_-J_zJ_++I_-I_zJ_+J_z)\\ &+(3d/4)I_+^2J_-^2+(3d/4)I_-^2J_+^2.\end{aligned}\tag{2·6}$$

If we take the thermal average* of (2·6) with respect to the rotational state, we can obtain the structure of the nuclear resonance line.

The thermal transition probabilities between states of different $I_z=m$ can be obtained by a perturbation calculation similar to that used in obtaining the nuclear relaxation time in antiferromagnetics.[11] The results are as follows:

$$\begin{aligned}W(0\to1)=&\frac{1}{2\hbar^2}\int_{-\infty}^{\infty}dt\,e^{i\omega_{10}t}[(c-3d/2)^2\langle J_+(t)J_-\rangle\\ &+9d^2\langle J_+(t)J_z(t)J_zJ_-\rangle-3d(c-3d/2)\{\langle J_+(t)J_zJ_-\rangle\\ &+\langle J_+(t)J_z(t)J_-\rangle\}],\\ W(-1\to0)=&\frac{1}{2\hbar^2}\int_{-\infty}^{\infty}dt\,e^{i\omega_{0-1}t}[(c-3d/2)^2\langle J_+(t)J_-\rangle\\ &+9d^2\langle J_z(t)J_+(t)J_-J_z\rangle\\ &+3d(c-3d/2)\{\langle J_+(t)J_-J_z\rangle+\langle J_z(t)J_+(t)J_-\rangle\}],\\ W(-1\to1)=&\frac{1}{\hbar^2}\frac{9}{4}d^2\int_{-\infty}^{\infty}dt\,e^{i\omega_{1-1}t}\langle J_+^2(t)J_-^2\rangle,\end{aligned}\tag{2·7}$$

etc.,
where $\langle\ \ \rangle$ means the statistical average:

$$\langle Q\rangle=Tr[e^{-\mathfrak{H}_R/kT}Q]/Tr\,e^{-\mathfrak{H}_R/kT},\tag{2·8}$$

$$\boldsymbol{J}(t)=e^{it\mathfrak{H}_R/\hbar}\boldsymbol{J}e^{-it\mathfrak{H}_R/\hbar},\tag{2·9}$$

and

$$\omega_{mn}=(E_m-E_n)/\hbar,$$

E_m and E_n being the energy eigenvalues of the states $I_z=m$ and n.

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* This does not mean to take the space average; the ortho-molecules are distributed randomly in the crystal and the rotational state of them is different from position to position due to different arrangements of the neighboring ortho-molecules.

If we assume the spin system to be in internal thermal equilibrium at a spin temperature T_s, the thermal relaxation time T_1 can be defined as

$$dT_s/dt=(1/T_1)\,(T_s/T)\,(T-T_s).$$

Then, we get from (2·7) after simple calculations

$$\frac{1}{T_1}=\frac{3}{2+\alpha}[W_1+(1+\alpha)W_3]=\frac{3}{1-\alpha}(W_1-\alpha W_2)=\frac{3}{1+2\alpha}[\alpha W_2+(1+\alpha)W_3], \tag{2·10}$$

where

$$\begin{aligned} W_1&=\tfrac{1}{2}\{W(0\to1)+W(1\to0)\},\\ W_2&=\tfrac{1}{2}\{W(-1\to0)+W(0\to-1)\}, \qquad (2\cdot11)\\ W_3&=\tfrac{1}{2}\{W(1\to-1)+W(-1\to1)\},\\ \alpha&=(E_0-E_{-1})/(E_1-E_0). \qquad (2\cdot12)\end{aligned}$$

The expression (2·10) can also be obtained directly by Kubo-Tomita's general formula for T_1.[13] However, one will need the transition probabilities when one considers the saturation phenomena below the transition point, because the thermal equilibrium in the spin system will not be warranted in this case. At temperatures well above the transition point, to which we confine ourselves in the present paper, the molecular rotation can be considered as isotropic and the nuclear Zeeman levels thus become equidistant. Then we have $\alpha=1$, $W_1=W_2$ and

$$1/T_1=W_1+2W_3. \tag{2·13}$$

From (2·13), (2·11) and (2·7) we get above the transition point

$$\begin{aligned}\frac{1}{T_1}=\frac{1}{\hbar^2}\int_{-\infty}^{\infty}dt[\,&(1/4)\,(c-3d/2)^2(\langle J_+(t)J_-\rangle+\langle J_-(t)J_+\rangle)\\ &+(9/8)\,d^2(\langle J_z(t)J_-(t)J_+J_z\rangle+\langle J_-(t)J_z(t)J_zJ_+\rangle\\ &\quad+\langle J_z(t)J_+(t)J_-J_z\rangle+\langle J_+(t)J_z(t)J_zJ_-\rangle)\\ &-(3/8)\,d(c-3d/2)\,(\langle J_-(t)J_+J_z\rangle-\langle J_-(t)J_zJ_+\rangle+\langle J_z(t)J_-(t)J_+\rangle\\ &\quad-\langle J_-(t)J_z(t)J_+\rangle+\langle J_+(t)J_zJ_-\rangle-\langle J_+(t)J_-J_z\rangle\\ &\quad+\langle J_+(t)J_z(t)J_-\rangle-\langle J_z(t)J_+(t)J_-\rangle)\\ &\quad+(9/4)\,d^2(\langle J_+^2(t)J_-^2\rangle+\langle J_-^2(t)J_+^2\rangle)\,],\end{aligned} \tag{2·14}$$

where we have neglected the Larmor frequency as compared with the mean frequency of the modulation spectra. We have now only to calculate the correlation functions appearing in the above expression. For this purpose, the explicit form of the inter-molecular interaction Hamiltonian is required.

§ 3. Calculation of the correlation functions and the relaxation time above the transition point

We shall now calculate the correlation functions appeared in (2·7). They can be written

$$\langle J_+^2(t)J_-^2\rangle = Tr[e^{-\mathfrak{H}_R/kT}\, e^{it\mathfrak{H}_R/\hbar}\, J_+^2\, e^{-it\mathfrak{H}_R/\hbar}\, J_-^2]/Tr\, e^{-\mathfrak{H}_R/kT},$$

etc. Obviously it is not possible to calculate this in a closed form without making any assumptions. So we expand the correlation functions as power series of t, and get

$$\langle J_+^2(t)J_-^2\rangle = \langle J_+^2 J_-^2\rangle + \frac{it}{\hbar}\langle[\mathfrak{H}_R,\, J_+^2]J_-^2\rangle + \frac{1}{2}\left(\frac{it}{\hbar}\right)^2\langle[\mathfrak{H}_R,\, J_+^2][J_-^2,\, \mathfrak{H}_R]\rangle$$
$$+\cdots\cdots, \tag{3·1}$$

etc., where the coefficient of $t^n/n!$ is the n-th moment of the frequency spectra.

We assume the Gaussian distribution for the frequency spectra. Then the first and second moments are sufficient to determine the frequency spectra. We further confine ourselves to high temperatures and put $e^{-\mathfrak{H}_R/kT}=1$, then the first moment vanishes and the correlation function becomes

$$\langle J_+^2(t)J_-^2\rangle = (4/3)\,(1-\tfrac{1}{2}\omega_{R1}^2 t^2+\cdots) \simeq \tfrac{4}{3}e^{-\frac{1}{2}\omega_{R1}^2 t^2} \tag{3·2}$$

with

$$\omega_{R1}^2 = \frac{1}{\hbar^2}\,\frac{\langle[\mathfrak{H}_R,\, J_+^2][J_-^2,\, \mathfrak{H}_R]\rangle}{\langle J_+^2\, J_-^2\rangle}, \tag{3·3}$$

etc.

The inter-molecular interaction Hamiltonian was written by Nakamura[6] in a convenient form. We shall rewrite them as follows, restricting ourselves to the space of $J=1$:

$$\mathfrak{H}_i = -A\sum_k\sum_m p_m \mathfrak{D}_{mo}^{(i,k)} K_m^{(i)},$$
$$\mathfrak{H}_{ik} = (2/25)\sum_{mm'} a_{mm'}^{(i,k)} K_m^{(i)} K_{m'}^{(k)}, \tag{3·4}$$

where

$$a_{mm'}^{(i,k)} = p_m p_{m'}\{B\mathfrak{D}_{mo}^{(i,k)}\mathfrak{D}_{m'o}^{(i,k)} + C(\mathfrak{D}_{m1}^{(i,k)}\mathfrak{D}_{m'-1}^{(i,k)} + \mathfrak{D}_{m-1}^{(i,k)}\mathfrak{D}_{m'1}^{(i,k)})$$
$$+D(\mathfrak{D}_{m2}^{(i,k)}\mathfrak{D}_{m'-2}^{(i,k)} + \mathfrak{D}_{m-2}^{(i,k)}\mathfrak{D}_{m'2}^{(i,k)})\}, \tag{3·5}$$
$$p_0=\sqrt{2/3},\quad p_{\pm1}=\mp1,\quad p_{\pm2}=1,$$
$$K_0=3J_z^2-2,\quad K_{\pm1}=J_zJ_\pm+J_\pm J_z,\quad K_{\pm2}=J_\pm^2,$$

$\mathfrak{D}_{mm'}^{(i,k)}$ being the matrix elements of the five-dimensional irreducible representation of the rotation which rotates the z-axis into the direction of the line joining the i-th molecule to the k-th molecule, and

$$\begin{aligned} A&=5\,(4V\lambda_0\lambda_2\,e^{-r_0/\rho}-2Q/5r_0^6+4R/15r_0^6)=2.737\times10^{-16}\ \text{erg},\\ B&=6\,(3e^2\mu_Q^2/4r_0^5)+75V\lambda_2^2\,e^{-r_0/\rho}-3R/2r_0^6=18.39\times10^{-16}\ \text{erg},\\ C&=4\,(3e^2\mu_Q^2/4r_0^5)-R/r_0^6=10.51\times10^{-16}\ \text{erg},\\ D&=(3e^2\mu_Q^2/4r_0^5)-R/2r_0^6=2.435\times10^{-16}\ \text{erg}. \end{aligned} \tag{3·6}$$

The notation in the above expressions is the same as that of Nakamura; $V=2.78\,e^2/a_0$, $\rho=a_0/1.87$ (a_0: Bohr radius), $\lambda_0=\sinh\gamma/\gamma$, $\lambda_2=(1/\gamma+3/\gamma^3)\sinh\gamma-(3/\gamma^2)\cosh\gamma$, $\gamma=r_1/2\rho$ (r_1: the interatomic distance of a molecule), r_0 is the intermolecular distance, $Q=\alpha_\parallel(\alpha_\parallel-\alpha_\perp)\kappa$, $R=(\alpha_\parallel-\alpha_\perp)^2\kappa$, $\alpha_\parallel$ and $\alpha_\perp$ being respectively the polarizabilities parallel and perpendicular to the molecular axis, κ a certain constant related with the ionization potential, μ_Q the electrical quadrupole moment of the molecule. Numerical values of these quantities are tabulated in Nakamura's paper. In (3·6) the terms including V, those including Q and R, and those including μ_Q are respectively the contributions of the valence force, the van der Waals force, and the electric quadrupole interaction. The numerical values given in (3·6) are those for the nearest neighboring pairs. We shall take account of only the nearest neighbour interactions in our calculation. Then $\mathfrak{H}_i$ vanishes identically for a close-packed hexagonal lattice, which we assume for solid H_2 of any ortho-concentration.

The calculation of the correlation functions is now straightforward though lengthy and laborious. We get the following result, taking for brevity the external field along the hexagonal axis:

$$\begin{aligned} &\langle J_+^2(t)J_-^2\rangle=(4\cdot3)\,e^{-\omega_{R1}^2t^2/2}, \quad \omega_{R1}=1.12\times10^{12}\sqrt{K},\\ &\langle J_+(t)J_z(t)J_zJ_-\rangle=(2/3)\,e^{-\omega_{R2}^2t^2/2}, \quad \omega_{R2}=1.31\times10^{12}\sqrt{K},\\ &\langle J_+(t)J_-\rangle=(4/3)\,e^{-\omega_{R3}^2t^2/2}, \quad \omega_{R3}=1.45\times10^{12}\sqrt{K},\\ &\langle J_-(t)J_zJ_+\rangle=(2/3)\,e^{-\omega_{R3}^2t^2/2}, \quad \text{etc.}, \end{aligned} \tag{3·7}$$

where K is the ortho-concentration and the ortho-molecules are assumed to be distributed uniformly.

Inserting (3·7) into (2·14), we get the relaxation time at high temperatures.

$$\begin{aligned} 1/T_{1\infty}=\sqrt{2\pi}\,\hbar^{-2}[&3d^2(2/\omega_{R1}+1/\omega_{R2})\\ &+\{2(c-3d/2)^2/3+2d(c-3d/2)\}/\omega_{R3}]. \end{aligned} \tag{3·8}$$

Using the values of (2·4), we get

$$T_{1\infty}=0.361\sqrt{K}\ \text{sec.} \tag{3·9}$$

For the normal hydrogen (75% ortho-concentration)

$$T_{1\infty}=0.31\ \text{sec.}$$

In the above calculation we assumed the uniform distribution of the ortho-molecules. Actually, the distribution will be random and the nuclear relaxation time T_1 will be different from place to place according to different local arrangements of the ortho-molecules. However, at temperatures well above the transition point the intermolecular dipole-dipole interaction will provide a thermal equilibrium of the whole spin system, because the Larmor frequencies are equal for all the ortho-molecules and the spin-spin relaxation time in this case will be far shorter than the spin-lattice relaxation time. Therefore, a single relaxation time will be observed at high temperatures, whereas a distribution of T_1 such as that found experimentally by Sugawara[14)] may be expected at low temperatures. T_1 at high temperatures will be proportional to the square root of the ortho-concentration. This relation will be valid except at very low concentrations.

We shall remark finally that among the intermolecular interactions the electric quadrupole-quadrupole interaction makes the overwhelming contribution to the relaxation. T_1 calculated by taking only the quadrupole interaction for $\mathfrak{H}_R$ becomes

$$(T_{1\infty})_q = 0.357\sqrt{K} \text{ sec.} \tag{3·10}$$

This differs from the correct value (3·9) by only one percent.

§ 4. Effect of self-diffusion

When the self-diffusion of the molecules becomes appreciable, the intermolecular dipole-dipole interaction becomes important as a relaxation mechanism. The intermolecular dipole-dipole interaction is smaller than the intra-molecular one by a factor of $\sim 10^{-2}$. Accordingly, the relaxation rate by the inter-molecular interaction will be smaller than that by the intra-molecular one by a factor of $\sim 10^{-4}$ if the spectra of the modulation frequencies are equal. The mean modulation frequency by the inter-molecular interaction calculated in the preceding section is $\sim 10^{12}$ cps and is much higher than the usual Larmor frequency so that the Fourier component of the local field spectra corresponding to the Larmor frequency is small. If the mean frequency of the self-diffusion is near to the Larmor frequency, the contribution of the self-diffusion to the nuclear relaxation can overcome the contribution of the intra-molecular interaction.

Recently Rollin and Watson[15)] and Bloom[16)] measured the line width or T_2 and found that it becomes appreciably narrowed above about 10°K. This narrowing is considered to be caused by the self-diffusion of the molecules. If the self-diffusion is characterized by a single correlation time τ, τ can be estimated from the line width data by using the theory of motional narrowing. The B. P. P. formulas for T_2 and T_1[9)] modified by Kubo and Tomita[13)] are written as

$$1/T_2 = 1/T_2' + 1/T_1', \tag{4·1}$$

$$(1/T_2')^2 = (2/\pi)(1/T_2'')^2 \tan^{-1}(\pi\tau/(4\log 2)T_2'), \tag{4·2}$$

$$1/T_1' = \frac{1}{6\log 2}(1/T_2'')^2\left(\frac{5\tau}{1+\omega^2\tau^2} + \frac{2\tau}{1+4\omega^2\tau^2}\right), \tag{4·3}$$

$$1/T_1 = \frac{1}{3\log 2}(1/T_2'')^2\left(\frac{\tau}{1+\omega^2\tau^2} + \frac{4\tau}{1+4\omega^2\tau^2}\right), \tag{4·4}$$

where $1/T_2'$ and $1/T_1'$ represent respectively the secular and the nonsecular line widths and $1/T_2''$ the line width for the rigid lattice. Since $1/T_2''$ is proportional to $\sqrt{K}$, T_1 is inversely proportional to K.

For $T_2 \gg \tau \gg \omega^{-1}$, T_2 and T_1 can be approximated by

$$T_2 = (2 \log 2)\,(T_2'')^2/\tau\,, \tag{4·5}$$

$$T_1 = (3 \log 2)\,(T_2'')^2 \omega^2 \tau/2\,. \tag{4·6}$$

Rollin and Watson estimated τ from their own data as $\tau = \tau_0 \exp(E_a/kT)$ with $E_a = 350$ calories/mole, and $\tau = 3 \times 10^{-6}$ sec at 10°K, $\tau = 1.6 \times 10^{-7}$ sec at 12°K, etc. Bloom found that T_2 is proportional to $\exp(E_a/kT)$ above about 10°K, where $E_a = 380 \pm 30$ calories/mole. The correlation times estimated from Bloom's data using (4·5) are shown in Fig. 1 and will hereafter be used.

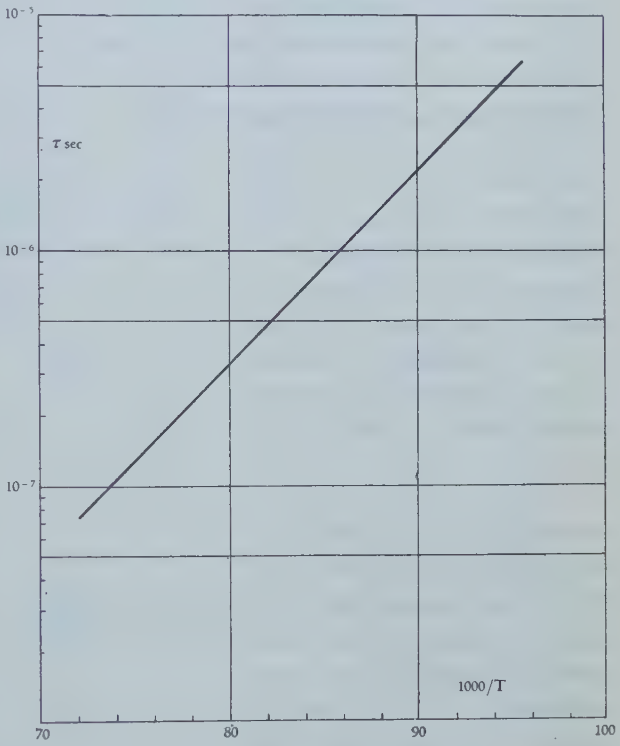

Fig. 1. The correlation time characterizing the self-diffusion estimated from Bloom's experimental values of T_2.

The spin-lattice relaxation time T_1 due to the self-diffusion can then be calculated by the use of (4·4). The results for 5.4 Mc (used by Hatton and Rollin[17]) and 30 Mc (used by Bloom) and for several ortho-concentrations are shown in Fig. 2.* As for the value of T_2'' Bloom's data 1.4×10^{-5} sec for normal concentration was adopted.

As one sees from (4·4) T_1 depends rather sensitively on the frequency. We shall be able to find a minimum of T_1 at a certain temperature by a proper choice of the resonance frequency. The minimum value is

$$(T_1)_{min.} = 1.6\,(T_2'')^2 \omega, \text{ corresponding to } \omega\tau = 0.58. \tag{4·7}$$

For the frequency of 5.4 Mc

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* A more refined treatment of T_1 due to the translational diffusion was given by Torrey.[10] However, the formula for T_2 by the same model has not yet been given. Although we calculate T_1 adopting Torrey's formula instead of the B. P. P. formula, the result will not essentially be altered. Torrey's formula is quantitatively not correct because it is based on the B. P. P. formula in its original form which has to be corrected following Kubo-Tomita's quantum mechanical theory.

$$(T_1)_{\text{min.}}=0.01 \text{ sec } (75\% \text{ ortho conc.}) \tag{4·8}$$

and for 30 Mc

$$(T_1)_{\text{min.}}=0.06 \text{ sec } (75\% \text{ ortho conc.}). \tag{4·9}$$

We shall also be able to depress this mechanism by choosing such resonance frequencies as to satisfy $\omega \gg \tau^{-1}$. This condition is almost satisfied in Bloom's experiment. In this case the contribution of self-diffusion to $1/T_1$ will be overwhelmed by the contribution treated in the preceding sections, and accordingly T_1 will not depend much on temperature.

The contribution of the intermolecular dipole-dipole interaction to the relaxation rate treated in this section and that of the intramolecular interaction treated in the preceding sections will be additive so long as $\tau \gg 1/\omega_R$.

When $\tau \lesssim 1/\omega_R$, the contribution of the intramolecular interaction becomes predominant. In this case, however, the modulation spectrum of the intramolecular interaction due to the intermolecular forces will be modified by the translational motion. When $\tau \ll 1/\omega_R$, the modulation spectrum will appreciably be narrowed and

$$T_1 \sim (T_1)_{\text{rigid lattice}}\, \omega_R \tau .$$

Thus T_1 will become shorter and its concentration dependence will be linear. Bloom's measurement of T_1 in the liquid state[16)] seems to indicate that this situation is approximately realized in the liquid state.*

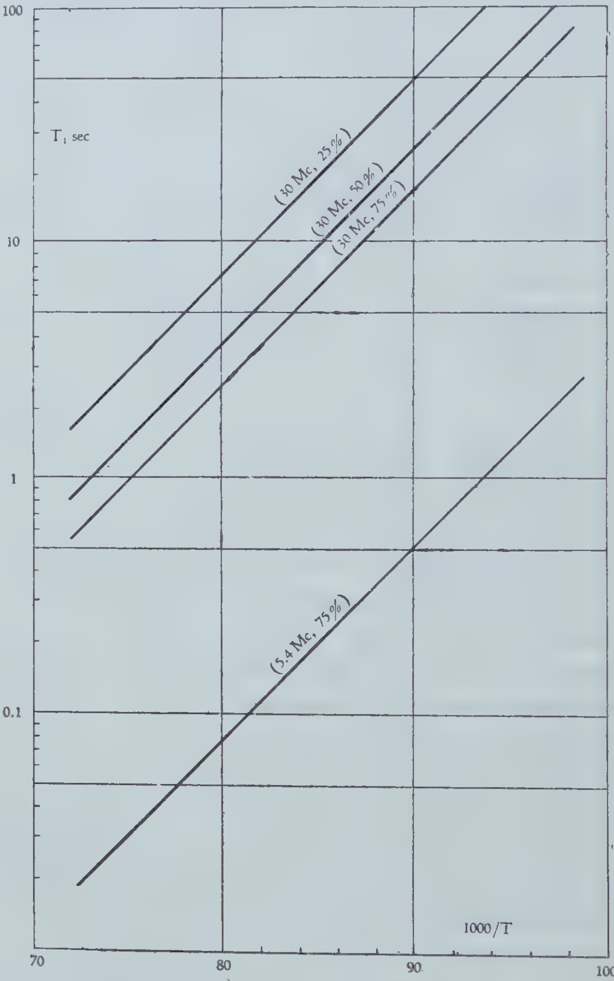

Fig. 2. The spin-lattice relaxation time due to the intermolecular dipole-dipole interactions modulated by self-diffusion.

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* The value of τ estimated from the viscosity data ($\eta=1.35\times10^{-4}$ poise at 20°K, 2.17×10^{-4} poise at 15°K) is $\tau \simeq 10^{-12} \sim 10^{-13}$ sec. Therefore, $\tau \lesssim \omega_R^{-1}$ and accordingly the shortening of T_1 is weak and the concentration dependence of T_1 will be between K and $\sqrt{K}$. These seem to be consistent with Bloom's measurements. As for the temperature dependence, T_1 will become longer as temperature is lowered since τ (which is proportional to η/T) becomes longer as temperature is lowered. This agrees qualitatively with Bloom's data.

§ 5. Comparison with experiment

The available experimental data of the relaxation time T_1 are those given by Hatton and Rollin,[17] those by Sugawara[14] and those by Bloom.[16] Sugawara's data are confined to low temperatures ($T \leq 4.2°K$) and our calculation cannot directly be compared with them. Sugawara observed an anomalous saturation behaviour of the absorption intensity and attributed it to the distribution of the relaxation time. Hatton and Rollin measured T_1 at the temperature range from 1.1°K to the melting point and further in the liquid state using r. f. field of 5.4 Mc. Their data at low temperatures do not agree with Sugawara's data. Bloom measured T_1 between 11°K and the melting point using r. f. field of 30 Mc. He found that T_1 is 0.225 sec and is independent of temperature within accuracy of measurements ($\sim$10%) in the temperature range studied. His measurements cover also liquid and gaseous states.

The theoretical values of T_1 calculated by taking account of all the contributions treated in the present paper are shown in Fig. 3. In the same figure Bloom's and Hatton-Rollin's experimental points are shown. Their data should be compared respectively with the curves for 30 Mc and 5.4 Mc.

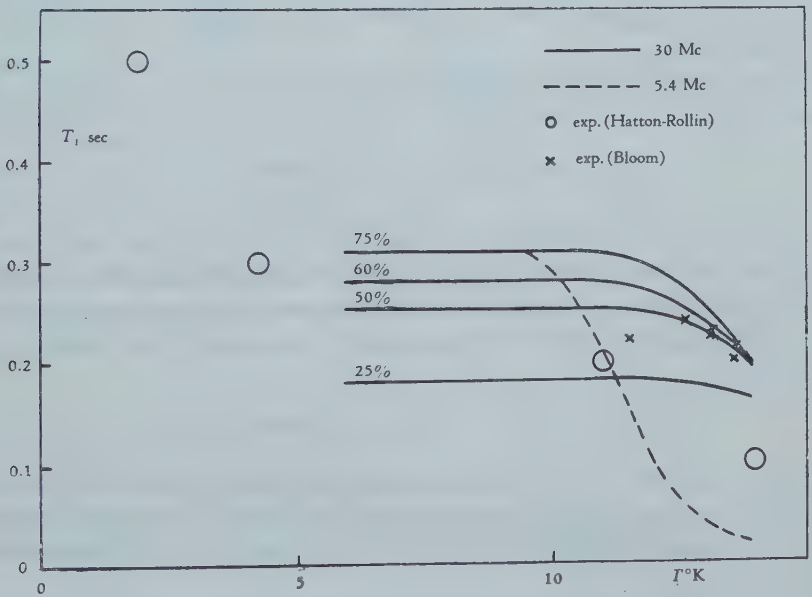

Fig. 3. The theoretical and the experimental values of T_1.

For Bloom's case the agreement between theory and experiment seems reasonable, though the ortho-concentration is not definite in the experiment ($\sim$60%). However, Hatton-Rollin's data show weaker temperature dependence than the theoretical one, though the values 0.3$\sim$0.2 sec at 4.2$\sim$11°K agree with the theoretical value, 0.3 sec, without the effect of self-diffusion.

For a more decisive test of the theory, more detailed measurements are hoped, especially of the frequency dependence and of the concentration dependence.

§ 6. Case of solid D_2

In this section we shall discuss the nuclear relaxation in solid D_2. The molecular wave function of the D_2 molecule must be symmetric under interchange of the two nuclei since the spin of a deuteron is 1. The normal concentrations of the ortho- and para-molecules (the room temperature equilibrium values) are as follows:

$$\text{ortho-D}_2 \quad J=0, \quad \begin{cases} I=0 & 1/9, \\ I=2 & 5/9, \end{cases}$$

$$\text{para-D}_2 \quad J=1, \quad I=1 \quad 1/3.$$

The ortho-molecules will give rise to a resonance line without fine structure since the state $J=0$ is spherically symmetric while the para-molecule should give rise to the same line shape as that of ortho-H_2 since the Hamiltonian for the D_2 molecule has the same form as that for the H_2 molecule. At high temperatures where the state of the para-molecule should effectively be spherically symmetric, the resonance lines of ortho-D_2 and para-D_2 will coincide.

The Hamiltonian for the D_2 molecule is given by (2·1), (2·2) and (2·3) replacing the constant values of a, b, c and d times h^{-1} by

$$h^{-1}a=0.6536\,H, \quad h^{-1}b=0.3368\,H,$$

$$h^{-1}c=8.783, \qquad h^{-1}d=25.24 \ \text{(kcps)}. \tag{6·1}$$

a, b and c have the same significance as those in the case of H_2, while d includes not only the contribution of the dipole-dipole interaction between the two nuclei of a molecule but also the contribution of the electric quadrupole interaction of the nuclei. For the para-molecule we have

$$d=(2/5)\mu_D^2\langle r^{-3}\rangle+(1/10)\,eQ\partial^2V^e/\partial Z_0^2,$$

where μ_D and Q are the magnetic dipole and electric quadrupole moments of the deuteron, and $\partial^2V^e/\partial Z_0^2$ the electric field gradient at the nucleus along the direction combining the two nuclei. The numerical values for the first and second terms are respectively $2.73\times10^3\,h$ and $22.51\times10^3\,h$.

In considering the thermal relaxation mechanisms, we shall at first treat the para- and ortho-molecules separately. Next we shall discuss the effect of the spin-spin relaxation which may provide the thermal contact between the systems of the para- and ortho-molecules.

(A) *para-molecule*

We can consider in this case the same mechanisms as those for ortho-H_2, i. e. the intra-molecular interaction modulated by the $\boldsymbol{J}$-dependent inter-molecular interaction and the effect of self-diffusion. The calculation can be carried out following the same line as that in § 2~§ 4.

1) *Intra-molecular interaction*

This mechanism is quite the same as that treated in § 2 and § 3 except that the intra-molecular interaction $\mathfrak{H}_{cd}$ is in this case dominated by the contribution of the nuclear quadrupole interaction.

If we assume that the inter-molecular interaction between two para-D_2 is equal to that between two ortho-H_2, and the inter-molecular distance of solid D_2 is equal to that of solid H_2, we can calculate T_1 for para-D_2 by inserting the numerical values of (6·1) into (2·22). The result is

$$T_{1\infty}=2.38\sqrt{K_p}\ \text{sec}, \tag{6·2}$$

where K_p is the concentration of para-D_2. For the normal D_2 (para-concentration$=1/3$) we get

$$T_{1\infty}=1.37\ \text{sec}. \tag{6·3}$$

The discussions given in § 3 for the ortho-H_2 are also applicable here.

2) *Effect of self-diffusion*

This effect is also similar to that in solid H_2. The difference is that there are ortho-molecules ($J=0$, $I=2$) which can take part in this mechanism. We shall again use the B. P. P. theory. The ratio of the relaxation times due to this mechanism for D_2 and H_2 may be written as

$$(T_1)_{D_2}/(T_1)_{H_2}=(\gamma_H/\gamma_D)^4 2K/\{2K_p+(5/6)6K_0\}, \tag{6·4}$$

for the same value of $\omega\tau$. Here K is the ortho-concentration of the solid H_2, K_p and K_0 the para- and the ortho-concentrations of the solid D_2 and γ_H and γ_D the gyromagnetic ratios of the proton and the deuteron. Substituting (γ_H/γ_D) by its numerical value 6.514, we get

$$(T_1)_{D_2}/(T_1)_{H_2}=1.80\times10^3\,K/(K_p+5K_0/2). \tag{6·5}$$

For the normal concentration

$$(T_1)_{D_2}/(T_1)_{H_2}=675. \tag{6·6}$$

The minimum value of $(T_1)_{D_2}$ can be obtained by inserting (4·7) into (6·5). We get

$$(T_1)_{D_2\,\text{min}}=2.2\times10^3(T_2'')^2\omega/(K_p+5K_0/2), \tag{6·7}$$

where T_2'' represents the value for solid H_2 of normal concentration. For the normal concentration

$$(T_1)_{D_2\,\text{min}}=2\times10^{-7}\omega\ \text{sec}. \tag{6·8}$$

Comparing (6·8) with (6·3) we may say that when the resonance frequency is higher than ~10 Mc, the contribution of the self-diffusion is at least less than ~1/10 times that of the intra-molecular interaction. Thus we can expect that T_1 will depend very little on temperature at temperatures where the quenching of the molecular rotation scarcely occurs.

(B) *Ortho-molecule*

For the ortho-molecule there is no intra-molecular interaction. Therefore, all the relaxation mechanisms are due to the inter-molecular interactions. As the inter-molecular interaction we can consider at first the magnetic dipole-dipole interaction between the nuclear moments of two molecules and that between the nuclear moment of the ortho-molecule and the rotational magnetic moment of the para-molecule. Another interaction is the electric quadrupole interaction of the nuclei of a molecule with the electric field gradient produced by the surrounding molecules. This will give a relaxation of the same order of magnitude as the dipolar interactions do. We shall here discuss the following two mechanisms:

1) *Relaxation by the dipolar field coming from the rotational magnetic moment of the para-molecules modulated by the inter-molecular interactions among the para-molecules.*

The dipole-dipole interaction between the nuclear magnetic moment of an ortho-molecule and the rotational magnetic moment of the nearest neighboring para-molecule is smaller than the intra-molecular interaction $\mathfrak{H}_{cd}$ of the para-molecule by a factor of $\sim 10^{-3}$. Therefore the relaxation time of the ortho-molecule due to this mechanism will be $\sim 10^6$ times as long as that of the para-molecule due to the first mechanism since the modulation mechanism is the same for these two cases. The relaxation time due to this mechanism will be as long as $\sim 10^6$ sec.

2) *Effect of self-diffusion*

This mechanism will give the same relaxation time both for the ortho- and for the para-molecules.

Another relaxation mechanism we can consider is the effect of lattice vibrations. However, the effect will be very small as we stated in the case of ortho-H_2.

(C) *Effect of spin-spin relaxation*

According to the above calculation the relaxation time T_1 for the ortho-D_2 is far longer than that for the para-D_2 except when the self-diffusion of the molecules is significant. However, the dipole-dipole interaction between the nuclear moments of the ortho- and para-molecules will provide the thermal contact between these two systems. Accordingly the apparent T_1 will be a certain average of the individual values.

Now we shall assume that the effect of the self-diffusion is negligible and the state of the para-molecule is effectively spherically symmetric, i. e. the resonance lines for the ortho- and the para-molecules are equal. The latter assumption will be justified at temperatures where the quenching of the molecular rotation scarcely occurs. The spin-spin interactions can then establish the thermal equilibrium of the whole nuclear spin system. The time of the establishment of the thermal equilibrium may be estimated from the resonance line width. This is $10^{-3} \sim 10^{-4}$ sec and is much shorter than T_1. We may thus have the apparent relaxation time,

$$1/T_1=[C_0/(T_1)_0+C_p/(T_1)_p]/(C_0+C_p), \tag{6·9}$$

where C_0 and C_p are respectively the magnetic heat capacities of the ortho- and para-molecules, and $(T_1)_0$ and $(T_1)_p$ their relaxation times. The heat capacities can be written as follows:

$$\begin{aligned} C_0&=N(5K_0/6)\,(6\gamma_D{}^2\hbar^2/3k)\,H^2/T^2, \\ C_p&=NK_p(2\gamma_D{}^2\hbar^2/3k)\,H^2/T^2, \end{aligned} \tag{6·10}$$

where N is the total number of the molecules. Inserting (6·10) into (6·9) we get

$$1/T_1=[5K_0/(T_1)_0+2K_p/(T_1)_p]/(5K_0+2K_p). \tag{6·11}$$

For the normal concentration this gives

$$1/T_1=[5/(T_1)_0+1/(T_1)_p]/6, \tag{6·12}$$

and

$$T_1\simeq 6(T_1)_p\simeq 8 \text{ sec.} \tag{6·13}$$

At the lower temperatures, where the quenching of the rotational angular momentum is significant, the para-molecules will show a fine structure. Then the thermal equilibrium of the whole system is not rapidly established. The state of affairs becomes very much complicated and we shall not here get into the detail.

(D) *Comparison with experiment*

The only experimental data we have are those due to Hatton and Rollin, which give $T_1 \sim 5$ sec at 1.1°K for the mixture of 90% D_2 and 10% H_2. Anyway, experimental data are too scanty to test our theory.

§ 7. Discussion

The treatment given in the present paper is restricted to the temperature range well above the rotational transition point. Though the formula (2·7) will be valid at any temperatures, we approximated the density by 1, which means that we neglected the effect of the short-range order.

We may go a step further by expanding the density in inverse powers of T. This gives the moments of the local field spectra in the form of an expansion in powers of $1/T$. If we assume the Gaussian random modulation, the relaxation time T_1 can be obtained as a function of T. However, the lowering of temperature gives a change not only of the moments but also of the shape of the frequency spectra due to growing short-range order. So, even if we calculate the temperature dependence of T_1 by this method, taking a few terms of the series, it would be weak as in the case of antiferromagnetic substances.11) At the present stage, we know no reasonable ways of taking account of the short range order in the problem of the magnetic resonance near the transition point.

The probable random distribution of the ortho- and para-molecules gives a further complication to the problem. As was stated in § 3, the effect of the local inhomogeneity will appear at lower temperatures, where the short range order of the molecular rotation grows to a considerable extent and the resonance line becomes broadened.

Below the transition point, the long-range order sets in. We have at first to determine the ordered orientation of the molecular axes. This problem, however, has yet been solved neither theoretically nor experimentally. We cannot proceed to the relaxation problem before the state of solid H_2 below the transition point is studied. The state of solid H_2 at very low temperatures may be described by "rotation wave" picture since low energy excitations of the rotational state in the crystal will have a wave-like character similar to spin waves in ferro- and antiferro-magnetics. If the rotation waves could be obtained, we are able to calculate the relaxation time at low temperatures in the same way as in the case of antiferromagnetics. The local inhomogeneity of the ortho-para distribution will make the problem much complicated also in this case.

The comparison of our theory with experiments was made only for solid H_2. The values at high temperatures calculated without the effect of the self-diffusion agree with Hatton-Rollin's data at 4.2∼10°K as well as with Bloom's data above 11°K. According to the measurements of T_2 given by Rollin and Watson and by Bloom, self-diffusion occurs above about 10°K. The theoretical calculation of the relaxation rate due to the intermolecular dipole interactions modulated by the self-diffusion was given by using the correlation time of the self-diffusion estimated from Bloom's data for T_2. This contribution is small under Bloom's experimental condition (30 Mc), while it is important in Hatton-Rollin's case (5.4 Mc). This agrees with the measurements though for the latter case the theory gives stronger temperature dependence than the experimental data.

In the liquid state Bloom's experiment shows that T_1 is shorter than in the solid and it becomes shorter as temperature is raised. The concentration dependence of T_1 is between K and $\sqrt{K}$. These results may be explained by the first mechanism (§ 2 and § 3) modified by the very rapid diffusional motion, as was stated at the end of § 4.

It is hoped that more detailed experiments on solid H_2 as well as solid D_2 are carried out.

In conclusion the writers would like to express their sincere thanks to Professor T. Nagamiya for his continual interest and dissussions. They are also indebted to Dr. M. Bloom for informing them of his experimental data prior to publication.

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On the Theory of the Unstable Particle in Lee's Model

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In Lee's model of the renormalizable field some investigations are made on the possibility of choosing the mass m_V of a physical V-particle to be larger than the sum of the masses of an N- and a θ-particle (namely, the case $m_V > m_N + \mu$). If the mass m_0 of a bare V-particle is chosen to be larger than some particular value, it seems to be possible to describe an unstable V-particle by Lee's model without any inconsistency, and to give a prescription for the renormalization of such an unstable particle. In this argument a method is used which is somewhat similar to that used in the theory of α-decay. Finally a brief discussion is made on the anomalous states.

Introduction

The renormalizable field proposed by Lee[1] is quite interesting and is instructive in making investigations of the theory of actual quantized fields. One of the most remarkable points of Lee's theory is the occurrence of the so-called ghost state[2] when the coupling constant becomes larger than the critical one g_c. The similar situation may be expected to occur concerning the relationship between the renormalized masses of the V-, N- and θ-particles.

Recently Araki *et al.*[3] have examined how to renormalize the mass of the V-particle when that mass is larger than the sum of the masses of N and θ $(m_N+\mu)$ in Lee's model. However, it does not seem so obvious as one expects whether we can make the mass m_V of the V-particle simply increase without causing new troubles. In fact, we shall show in the case of $m_V > m_N + \mu$ the existence of a new state of the V-particle in which the energy is lower than $m_N+\mu$ provided the system is enclosed in a box with a finite volume V.* It will be shown that such a state can be excluded in the limit $V \to \infty$, though there still remains some trouble concerning a property of the S-matrix.

In the present paper, we shall give a prescription for determining the mass renormalization of V, in the case of $m_V > m_N + \mu$, and for giving the breadth of the energy-level (mass-level) of the V-particle (in § 2). In connection with the above investigation, a property of the S-matrix for N and θ scattering is examined. This result seems to suggest that the theory of the S-matrix is inapplicable to the case in which we take the unstable state as the initial or final one. In the final section a brief discussion will be given for the case of the coupling larger than g_c.

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* Recently Glaser and Källén have investigated the similar problem. Prof. R. Oppenheimer kindly informed us the abstract of their article which will be published in Nuclear Physics. The author would like to express his sincere gratitude to Prof. R. Oppenheimer for his kindness.

§ 1. Preliminary considerations

For the sake of simplicity, let us consider a Lee system enclosed in a cubic box with a finite volume V. The Hamiltonian* is

$$H=m_0N^2\sum_p\psi_V^*(\boldsymbol{p})\psi_V(\boldsymbol{p})+m_N\sum_p\psi_N^*(\boldsymbol{p})\psi_N(\boldsymbol{p})+\sum_k\omega_K a^*(\boldsymbol{k})a(\boldsymbol{k})$$
$$+\sum_{p\cdot k}\frac{gf(\omega_k)}{\sqrt{2V\omega_k}}(\psi_V^*(\boldsymbol{p})\psi_N(\boldsymbol{p}-\boldsymbol{k})a(\boldsymbol{k})+\psi_N^*(\boldsymbol{p}-\boldsymbol{k})\psi_V(\boldsymbol{p})a^*(\boldsymbol{k})) \tag{1·1}$$

with $\omega_k=\sqrt{k^2+\mu^2}$, where all the quantities are renormalized ones except for the bare V-particle mass m_0. The field operators $\psi_V(\boldsymbol{p})$, $\psi_N(\boldsymbol{p})$ and $a(\boldsymbol{k})$ satisfy the well-known commutation relations, for example

$$\{\psi_V^*(\boldsymbol{p}),\ \psi_V(\boldsymbol{p}')\}=N^{-2}\delta_{p,p'}.$$

In the following argument, the constants N^2, m_0, m_N, μ and g are assumed to have some arbitrarily chosen values, while the physical mass of the V-particle, $m_V=m_0-\delta m$, will be discussed in a later section.

Let us consider an eigenstate of the Hamiltonian of the form

$$|z\rangle=(\alpha N\psi_V^*(\boldsymbol{p})+\sum_k\chi(\boldsymbol{k})\psi_N^*(\boldsymbol{p}-\boldsymbol{k})a^*(\boldsymbol{k}))|0\rangle. \tag{1·2}$$

The eigenvalue E of this state is given by

$$m_0\alpha+\sum_k\frac{g}{N}\frac{f(\omega_K)}{\sqrt{2V\omega_k}}\chi(\boldsymbol{k})=E\alpha \tag{1·3}$$

and

$$(m_N+\omega_k)\chi(\boldsymbol{k})+\frac{g}{N}\frac{f(\omega_k)}{\sqrt{2V\omega_k}}\alpha=E\chi(\boldsymbol{k}), \tag{1·4}$$

or equivalently

$$m_0-y(E)=0, \tag{1·5}$$

where

$$y(E)\equiv E+\sum_k\frac{g^2}{N^2}\frac{f^2(\omega_k)}{2V\omega_k}\frac{1}{m_N+\omega_k-E}. \tag{1·6}$$

As Lee has pointed out, the character of the equation (1·5) depends on the sign of N^2, namely, if $N^2<0$, the so-called ghost state occurs and makes our argument somewhat complicated. In the present and the succeeding section the case of $N^2>0$ will be discussed.

The behaviour of $y(E)$ as a function of E is illustrated in Fig. 1. Namely, $y(E)$ monotonously increases from $-\infty$ to $+\infty$ in the region $(-\infty<E<m_N+\mu)$, because of $dy(E)/dE>1$ and

$$y(E)\rightarrow\pm\infty\ \text{ for }\ E\rightarrow\begin{cases}m_N+\mu\\-\infty.\end{cases}$$

---

* In the present paper, the notations have the same meaning as those used by Källén and Pauli.

Accordingly, in the region of E mentioned above, there exists one and only one real root $E(m_0)$ without regard to the magnitude of m_0. The wave function $\chi(x)$ of the θ-particle is shown by some calculations to vanish more rapidly than $1/r$ for large r. In other words, this state may be interpreted to describe either a bound state of N- and θ-particles or a stable V-state of a physical V-particle.

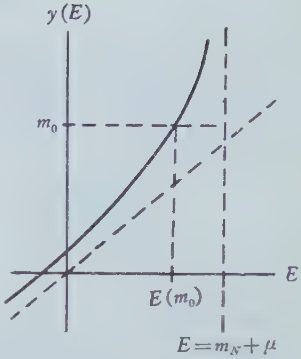

Fig. 1. Graph of $y(E)$
(finite volume)

In the current theory, this particular state has been assumed to represent a stable physical V-particle and the eigenvalue $E(m_0)$ has been identified with the physical mass m_V. On the contrary, if m_V is assumed to be larger than $m_N+\mu$, it seems quite obscure how to interpret this particular state $(E(m_0)<m_N+\mu)$, because there may be no definite criterion in distinguishing the V-particle state from a bound state of N- and θ-particles. The arguments we have stated so far depend essentially on the assumption of the finite size of the volume. We may expect that the above conclusion will drastically be changed in the limit $V\to\infty$.

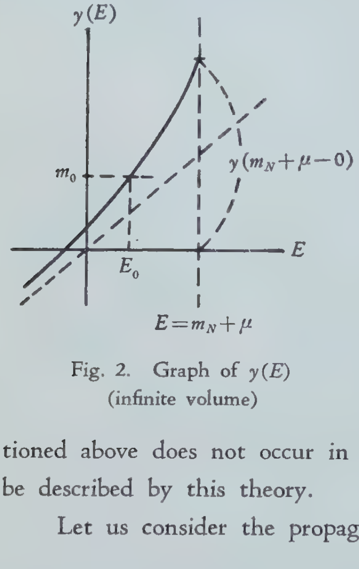

Fig. 2. Graph of $y(E)$
(infinite volume)

In the limit $V\to\infty$, the integral in (1·6) has a well-defined meaning if $E<m_N+\mu$, and has a finite value at the limit $E\to m_N+\mu-0$ owing to the $\boldsymbol{k}$-integration (cf. Fig. 2). If $m_0<y(m_N+\mu-0)$, we have a state with the energy $E_0<m_N+\mu$ which is usually regarded as to correspond to the stable V-particle. On the contrary, if $m_0>y(m_N+\mu-0)$, the state mentioned above does not occur in the limit $V\to\infty$, and the current stable V-particle cannot be described by this theory.

Let us consider the propagator of the V-particle :

$$S_V'(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iEt}\left(E+\frac{1}{(2\pi)^3}\frac{g^2}{N^2}\int d^3k\,\frac{f^2(\omega_k)}{2\omega_k}\,\frac{1}{m_N+\omega_k-E-i\epsilon}-m_0+i\epsilon\right)^{-1}$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iEt}\,(y(E+i\epsilon)-m_0)^{-1}. \tag{1·7}$$

The path of the E-integration in (1·7) can be replaced with that shown in Fig. 3, where E_0 is the root of (1·5) in the case of $m_0<y(m_N+\mu-0)$. Corresponding to these new pathes, C_1 and C_2, the integration leads to

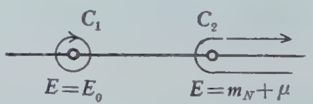

Fig. 3. The path in (1,8)

$$S_V'(t)=S_1(t)+S_2(t),$$

where $S_1(t)$ and $S_2(t)$ stand respectively for the integrations over the paths C_1 and C_2, and can be written as follows:

$$S_1(t)=\begin{cases}\theta(t)\,e^{-iE_0t}\left(1+\dfrac{1}{(2\pi)^3}\dfrac{g^2}{N^2}\displaystyle\int d^3k\,\dfrac{f^2(\omega_k)}{2\omega_k}\dfrac{1}{(m_N+\omega_k-E_0)^2}\right)^{-1} & \text{for } m_0<y(m_N+\mu-0)\\ 0 & \text{for } m_0>y(m_N+\mu-0)\end{cases}$$

$$\text{and}\quad S_2(t)=\frac{\theta(t)}{2\pi i}\int_{C_2}dE\,e^{-iEt}\left(E+\frac{1}{(2\pi)^3}\frac{g^2}{N^2}\int d^3k\,\frac{f^2(\omega_k)}{2\omega_k}\frac{1}{m_N+\omega_k-E}-m_0\right)^{-1}. \tag{1·8}$$

Since $S_2(t)$ vanishes in the limit $t\to\infty$ as is seen from (1·8), the wave of the V-particle cannot be propagated to the infinitely distant future in the case of $m_0>y(m_N+\mu-0)$. This situation suggests us to interpret the V-particle with $m_0>y(m_N+\mu-0)$ as an unstable one.

§ 2. Unstable V-particle

In the present section, we shall consider a stationary state where an N-particle and a θ-particle are being combined in an unstable V-particle and at the same time the V-particle is decaying into a pair of N and θ. To get this stationary state, let us solve eqs. (1·3) and (1·4) under the boundary condition which is a superposed wave of the outgoing and the incoming spherical waves of N and θ. Namely, the boundary condition does not contain any incoming plane wave contrary to the case of the familiar scattering problems. Such a treatment is somewhat similar to that used in the α-decay problem.

From (1·4), we have

$$\chi(\boldsymbol{k})=-\frac{g}{N}\frac{f(\omega_k)}{\sqrt{2(2\pi)^3\omega_k}}\left(\mathrm{P}\frac{1}{m_N+\omega_k-E}+\lambda\delta(m_N+\omega_k-E)\right)\alpha, \tag{2·1}$$

where the constant λ will be determined later. Inserting (2·1) into (1·3), we get

$$m_0=E+\frac{1}{(2\pi)^3}\frac{g^2}{N^2}\int d^3k\,\frac{f^2(\omega_k)}{2\omega_k}\left(\mathrm{P}\frac{1}{m_N+\omega_k-E}+\lambda\delta(m_N+\omega_k-E)\right). \tag{2·2}$$

From (2·2), we can easily see that λ is real. A short calculation leads us to

$$\chi(x)\equiv\frac{1}{(2\pi)^{3/2}}\int d^3k\,e^{ikx}\chi(\boldsymbol{k})$$

$$=\frac{ig\alpha}{(2\pi)^2Nr}\int_{-\infty}^{\infty}dk\,e^{ikr}\,\frac{kf(\omega_k)}{\sqrt{2\omega_k}}\left(\mathrm{P}\frac{1}{m_N+\omega_k-E}+\lambda\delta(m_N+\omega_k-E)\right). \tag{2·3}$$

Introducing a path of C in the complex k-plane (cf. Fig. 4), we can transform (2·3) into the sum of $\chi_0(x)$ and $\chi_1(x)$. Here $\chi_0(x)$ is

$$\chi_0(x) = \frac{ig\alpha}{(2\pi)^2 Nr} \int_C dk\, e^{ikr} \frac{kf(\omega_k)}{\sqrt{2\omega_k}} \frac{1}{m_N+\omega_k-E} \tag{2·4}$$

and $\chi_1(x)$ is the sum of the contributions from the λ-term in (2·3) and from the two poles shown in Fig. 4 and has the expression

$$\chi_1(x) = -\frac{g\alpha f(\omega_0)}{4\pi Nr} \sqrt{\frac{\omega_0}{2}} \left(e^{ik_0 r}\left(1-\frac{i\lambda}{\pi}\right) + e^{-ik_0 r}\left(1+\frac{i\lambda}{\pi}\right)\right) \tag{2·5}$$

with $k_0 = \sqrt{(E-m_N)^2-\mu^2}$ and $\omega_0 = \sqrt{k_0^2+\mu^2}$.

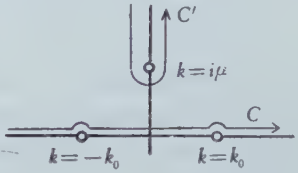

Fig. 4. The path in (2,4)

By replacing the path C with another one C', shown in Fig. 4, it is seen that $\chi_0(x)$ vanishes more rapidly than $1/r$ for the large r provided that the cut-off function $f(\omega_k)$ is regular in the upper half plane of k.

If this state-vector describes the state mentioned in the beginning of this section, $|\chi_1|$ must be much smaller than $|\chi_0|$ for $E > m_N+\mu$, because χ_0 represents a wave packet located in the vicinity of the origin while the χ_1 represents the outgoing and incoming waves. The former may be interpreted to describe the unstable V-particle which is impossible to observe at remote places whereas the latter describe the θ-meson which is directly observable. We may expect from the above interpretation that in the neighbourhood of the origin the packet $|\chi_0|$ may be larger than $|\chi_1|$. Therefore, it seems reasonable to determine λ in such a way as to make $|\chi_1|$ minimum. The energy E is a function of λ as is easily seen from (2·2) when m_0 is fixed. In order to make the lifetime of the unstable V-particle relatively long, it will be seen later that $|dE/d\lambda|$ should be small compared with E. From this assumption $|\chi_0|$ can be considered to be unchanged while the parameter λ is varied around a particular λ_0 which makes $|\chi_1|$ minimum. This particular value λ_0 is zero as we can easily see from the expression (2·5) when E is assumed to be fixed. In this case the relationship between E and the bare mass m_0 turns out*

$$m_0 = E + \frac{1}{(2\pi)^3} \frac{g^2}{N^2} \mathrm{P} \int d^3k \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{m_N+\omega_k-E}. \tag{2·6}$$

The magnitude of the mass renormalization δm can be determined by putting E in (2·6) equal to the observed mass $m_V (> m_N+\mu)$ of the V-particle, and also by substituting $m_V+\delta m$ for m_0. Now, we shall investigate the level-breadth (or the uncertainty of the energy of the V-particle) Γ. Eq. (2·2) gives a relationship between λ and E, accordingly the magnitude of $|\chi_0|/|\chi_1|$ near the origin is a function of E. Conversely, the change of magnitude of $|\chi_0|/|\chi_1|$ gives rise to a change of magnitude of E by $\Delta E = \partial E/\partial(|\chi_0|/|\chi_1|)\Delta(|\chi_0|/|\chi_1|)$. As $|\chi_0|/|\chi_1|$ is regarded as a function of the parameter λ, ΔE

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* The expressions like (2·6), (2·7) and (2·8) were derived by several authors from different viewpoints. (cf. foot-note 1 and reference 3)

is rewritten as $\Delta E = \partial E/\partial\lambda \cdot \Delta\lambda$. If $\Delta(|\chi_0|/|\chi_1|)$ is put equal to $\frac{1}{2}|\chi_0|/|\chi_1|$ at $E = m_V$ the above introduced ΔE turns out to be the level-breadth. Therefore by taking the derivative of Eq. $(2\cdot2)$ with respect to λ (m_0 fixed) we get

$$\Gamma \backsim \frac{\frac{1}{(2\pi)^2}\frac{g^2}{N^2} f^2(m_V - m_N)\sqrt{(m_V-m_N)^2-\mu^2}}{1+\frac{\partial}{\partial m_V}\frac{1}{(2\pi)^3}\frac{g^2}{N^2}\mathrm{P}\int d^3k\,\frac{f^2(\omega_k)}{2\omega_k}\frac{1}{m_N+\omega_k-m_V}}. \tag{2·7}$$

The denominator of $(2\cdot7)$ is the renormalization constant, that is

$$N^{-2} = 1+\frac{1}{(2\pi)^3}\frac{g^2}{N^2}\frac{\partial}{\partial m_V}\mathrm{P}\int d^3k\,\frac{f^2(\omega_k)}{2\omega_k}\frac{1}{m_N+\omega_k-m_V}. \tag{2·8}$$

Following the current definition of the critical value g_c of the coupling constant, we have

$$g_c^2 = \left(\frac{1}{(2\pi)^3}\frac{\partial}{\partial m_V}\mathrm{P}\int d^3k\,\frac{f^2(\omega_k)}{2\omega_k}\frac{1}{m_N+\omega_k-m_V}\right)^{-1}.$$

In the usual case, namely, when $m_V < m_N + \mu$, g_c^2 is positive definite. In our present case, however, the sign of g_c^2 depends on the type of the cut-off function. The case of the unstable particle seems to prohibit the current interpretation of the renormalization constant as representing the probability for finding a bare state in a physical particle. We shall make a detailed discussion in a succeeding paper.

In connection with the above argument, let us investigate a property of the S-matrix. From $(1\cdot3)$ and $(1\cdot4)$, the element of the S-matrix for $N-\theta$ scattering is written as

$$\langle N\theta_k|S|N\theta_k{'}\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')$$
$$+\frac{\frac{i}{(2\pi)^2}\frac{g^2}{N^2}\frac{f^2(\omega_k)}{2\omega_k}\delta(\omega_k-\omega_{k'})}{m_0-m_N-\omega_k-\frac{1}{(2\pi)^3}\frac{g^2}{N^2}\int d^3k''\,\frac{f^2(\omega_k{''})}{2\omega_k{''}}\left(\mathrm{P}\frac{1}{\omega_k{''}-\omega_k}+i\pi\delta(\omega_k{''}-\omega_k)\right)}. \tag{2·9}$$

If the energy E is equal to m_V, the unitarity relation leads to

$$\int d^3k''\langle N\theta_k|S^*|N\theta_k{''}\rangle\langle N\theta_k{''}|S|N\theta_k{'}\rangle + \langle N\theta_k|S^*|V\rangle\langle V|S|N\theta_k{'}\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}'),$$

where $|V\rangle$ and $|N\theta_k\rangle$ are the bare states. However, a direct calculation gives

$$\int d^3k''\langle N\theta_k|S^*|N\theta_k{''}\rangle\langle N\theta_k{''}|S|N\theta_k\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}').$$

This result suggests us that either the unitarity of the S-matrix is violated in the present case, or

$$\langle V|S|N\theta_k\rangle = 0.$$

In view of the Hermiticity of the Hamiltonian, we must exclude the conjecture that the

S-matrix is not unitary. In fact, the latter expression is directly derived from Dyson's S-matrix.

Therefore, we see that the theory of S-matrix is applicable only when we take a stable state as the initial or final state, but not an unstable one. At first sight, this argument seems to be curious. We shall, however, be able to accept this argument, if we note that the unstable particle cannot be found in the infinite future. We are planning to investigate this problem in more detail in a succeeding paper.

§ 3. Ghost state

In this section, we shall consider how the conclusion of the previous two sections will be changed in the case of negative N^2. As was discussed by Källén and Pauli, we must in this case, introduce the indefinite metric and interpret the asterisk to indicate the adjoint instead of the Hermite conjugate. These procedures, however, do not give any change to the expressions $(1\cdot5)$ and $(1\cdot6)$. For simplicity, we use the following cut-off function :

$$f^2(\omega_k)>0 \quad \text{for} \quad \mu\leqq\omega_k\leqq\omega_C$$

and

$$f^2(\omega_k)=0 \quad \text{for} \quad \omega_C<\omega_k.$$

In the first place, let us consider the case of the finite volume V. The behaviour of $y(E)$ is illustrated in Fig. 5. It must be noted that $y(E)$ has a maximum at the point $E_1\,(E_1<m_N+\mu)$ and a minimum at the point $E_2\,(m_N+\omega_C<E_2)$. The norm of an eigenvector $\Psi(E_0)$ with an eigenvalue E_0 is

$$(\Psi^*(E_0)\,\eta\Psi(E_0))\equiv\langle E_0|\eta|E_0\rangle=-|\alpha|^2dy(E)/dE\Big)_{E=E_0} \tag{3·1}$$

The following four cases are examined separately.

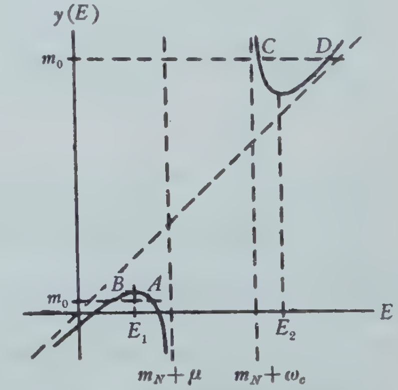

Fig. 5. Graph of $y(E)$ $(N^2<0)$

a) $m_0<y(E_1)$. There are two real roots of $(1\cdot5)$ for $E<m_N+\mu$. The state A in Fig. 5 has a positive norm, while the state B is the ghost because of its negative norm. In the current theory, namely, in the case of $m_V<m_N+\mu$, the state A is regarded to describe the stable V-particle by putting $E_A=m_V$.

b) $m_0=y(E_1)$ or $y(E_2)$. In both cases, we have a single root E_1 or E_2 respectively, both corresponding to the states with a zero-norm.

c) $y(E_2)>m_0>y(E_1)$. A conjugate pair of complex eigenvalues appears.[4] The norm of each eigenstate vanishes, whereas the inner product of the conjugate eigenstates does not vanish (cf. Appendix).

d) $m_0>y(E_2)$. There are two real roots for $E>m_N+\omega_C$. The state C is a real state while the state D is a ghost state (Fig. 5).

In the limit $V\rightarrow\infty$, $y(m_N+\mu-0)$ becomes finite. Therefore, if $m_0<y(m_N+\mu-0)$,

there exists only one ghost state and the real state mentioned above disappears. No further discussion is made about this case.

Acknowledgements

The author would like to express his cordial thanks to Prof. R. Utiyama for his guidance and encouragement. Thanks are also due to Messrs S. Sunakawa, T. Imamura, T. Goto and Dr. S. Nakai for their valuable discussions.

Appendix

We shall study the possibility of the appearance of the complex eigenvalue. Consider an eigenstate with an energy $E+i\Gamma$ (E, Γ real). The real and imaginary parts of (1·5) now give the relations

$$m_0-E-\sum_k\frac{g^2}{N^2}\frac{f^2(\omega_k)}{2V\omega_k}\frac{m_N+\omega_k-E}{(m_N+\omega_k-E)^2+\Gamma^2}=0 \qquad (\mathrm{A}\cdot 1)$$

and

$$Im\,y(E+i\Gamma)/\Gamma=1+\sum_k\frac{g^2}{N^2}\frac{f^2(\omega_k)}{2V\omega_k}\frac{1}{(m_N+\omega_k-E)^2+\Gamma^2}=0. \qquad (\mathrm{A}\cdot 2)$$

Now we have the following relation :

$$Im\,y(E+i\Gamma)/\Gamma\geq(Im\,y(E+i\Gamma)/\Gamma)_{\Gamma=0}=\left(\frac{dy(E+i\Gamma)}{dE}\right)_{\Gamma=0}. \qquad (\mathrm{A}\cdot 3)$$

If (A·2) is valid for some value of E, it is seen from (A·3) that $(dy(E+i\Gamma)/dE)_{\Gamma=0}$ should be non-positive for this value of E. This fact gives a domain of $E\,(E_1\leq E\leq E_2)$ where (A·2) is valid. Strictly speaking, some exceptional cases can be expected to occur in the case of the finite volume V. Namely, in that case, dy/dE is not always non-positive for those E' lying on the domain, $m_N+\mu<E<m_N+\omega_C$.* Since, for any E of the domain (E_1, E_2), eq. (A·2) gives a definite value to Γ^2, we can define a function $Z(E)$

$$Z(E)=E+\sum_k\frac{g^2}{N^2}\frac{f^2(\omega_k)}{2V\omega_k}\frac{m_N+\omega_k-E}{(m_N+\omega_k-E)^2+\Gamma^2}\quad \text{for } E_1\leq E\leq E_2 \qquad (\mathrm{A}\cdot 4)$$

by inserting (A·2) into (A·1). By taking into account eq. (A·2) and its derivative with respect to E, it is seen that

$$\frac{dZ(E)}{dE}=-\left(2\Gamma^2+\frac{1}{2}\left(\frac{d\Gamma^2}{dE}\right)^2\right)A(E)>0 \qquad (\mathrm{A}\cdot 4)$$

where

$$A(E)\equiv\sum_k\frac{g^2}{N^2}\frac{f^2(\omega_k)}{2V\omega_k}\frac{1}{((m_N+\omega_k-E)^2+\Gamma^2)^2}.$$

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* In this case, in place of the state with a complex eigenvalue, there occur an additional real state and a ghost state.

Namely, $Z(E)$ is a monotonously increasing function in the domain of $E_1 \leq E \leq E_2$, and its maximum and minimum values are $y(E_2)$ and $y(E_1)$, respectively.

The foregoing argument will lead to the following conclusion. In the cases of a), b) and d), the complex energy state does not appear; but only in the case of c), one pair of the complex energy states appears (note that $E - i\Gamma$ is also an eigenvalue). We can easily confirm that norms of these states are zero by using (A·2), but the inner product of such a state with its conjugate does not vanish.

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Letters to the Editor

The opinions exressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30cm), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Ratio of Hyperons Produced by Cosmic Rays

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April 13, 1957

Using Landau-Belenkiis' multiple production theory[1)], we could account for the hyperon production ratio of cosmic rays, which Leighton et al. observed by cloud chambers during three and half years.[2)]

At first sight it may seem to be strange that Landau-Belenkiis' (L-B) theory can account for such low energy data. But at least particle ratios among π mesons and hyperons, and various hyperons given by the L-B theory are not inconsistent with the experimental results at rather low energies,[3)] while other aspects of the L-B theory are justified only at much higher energies.[3)]

In the followings we discuss the experimental results of Leighton et al. and the estimated production ratios of hyperons, and then show that these production ratios are not inconsistent with those given by L-B theory.

1) The Observed Number of Hyperons. The number of hyperons which Leighton, Trilling et al. observed using 48 inch magnetic cloud chambers during three and half years at altitude 210 m,[4)] is shown in Table 1.[2)] Here Λ^0 is identified by $V^0 \to \pi^- + p$, and Ξ^- is identified by $V^- \to \pi^- + V^0$, $V^0 \to \pi^- + p$. $V^\pm$ denotes those, whose lifetimes are very short, and direct measurement of their momentum is impossible. There are few $K^\pm$ mesons in $V^\pm$. In V^- there are some Ξ^-, and about 2% of K^-. (estimated by Trilling from the lifetimes of V^- and K^-)[5)]

Table 1. The number of hyperons observed by Leighton et al. There are some Ξ^- in V^-, and few $K^\pm$ mesons in $V^\pm$.

Sort	Number
$\Lambda^0(+\Sigma^0)$	230～490
V^-	90～100
V^+	about 50
Ξ^-	6

2) Ratio of Ξ^- to Σ^-. If all Λ^0 decay according to the mode $\Lambda^0 \to \pi^- + p$, the number of Ξ^- contained in V^- is 6～7, according to the estimation of Trilling.[5)] On the other hand, because $\rho_\Lambda > 0.5$,[6)7)] ($\rho_\Lambda \equiv$ the number of $\Lambda^0 \to \pi^- + p/$ the number of Λ^0) the total number of Ξ^- should be taken as 10～20, and the number ratio of Ξ^- to Σ^- is 10～20/80～90. Then if the lifetimes of Ξ^- and Σ^- are about the same, the produced ratio of Ξ^- to Σ^- is 10～20/80～90.

3) Ratio of Σ^+ to Σ^-. Because the lifetime of $\Sigma^+ (\tau_{\Sigma^+})$ is half of τ_{Σ^-}, it is not strange that the observed number of V^+ is about half of the observed num-

ber of V^-.

4) Ratio of Λ^0 to Σ. Because $\tau_{\Lambda^0} \simeq 2\tau_{\Sigma^-}$, in considering the ratio of Λ^0 to Σ^-, we should double (or more) the observed number of Σ^-. Then the number of Σ^- is 150～200. Now because it is reasonable to regard the number of $\Sigma^- \simeq$ the number of Σ^0,[8] we can justify the following subtraction

$$\begin{aligned}&(\text{the observed number of } \Lambda^0+\Sigma^0)\\ &\quad-(\text{the number of } \Sigma^0)\\ &\quad=(230\sim490)-(150\sim200)\times\rho_\Lambda d\\ &\quad=(230\sim490)-(60\sim130)\\ &\quad=(100\sim430).\end{aligned}$$

Here we took $\rho_\Lambda=0.7$,[7] $d=0.6\sim0.8$,[9] where d is the detection probability. Therefore the production ratio of Λ^0 to Σ^0 is to be taken as 60～130/100～430.

5) Comparison with the Calculation. Now let us compare the above ratios with the calculated ratios. Using the method shown in the previous letter,[10] we got the ratios of hyperons as functions of the critical temperature. (See Table 3. Here we took the spins of all hyperons as ½.)

Table 2. The lifetimes of hyperons

Sort	lifetime (sec)
Λ^0	3.6×10^{-10}
Σ^-	1.8×10^{-10}
Σ^+	0.9×10^{-10}
Ξ^-	$\sim2.0\times10^{-10}$?

Table 3. The calculated ratios of hyperons as the function of the critical temperature. Here the spins of all hyperons are taken as ½.

$T(m_\pi)$	n_{Ξ^-}/n_{Σ^-}	$n_{\Sigma^0}/n_{\Lambda^0}$
0.5	0.07	0.28
1.0	0.17	0.50
1.2	0.20	0.55
3.0	0.31	0.82
ratios estimated from the experimental data	$\dfrac{10\sim20}{80\sim90}$	$\dfrac{60\sim130}{100\sim430}$

As it can be seen from this table the agreement is not bad at $T\simeq m_\pi$, if we regard the spins of all hyperons as ½.

Moreover the result is not so changed, even if we vary the value of ρ_Λ from 0.6 to 0.8. Thus the production ratio of hyperons by cosmic rays is not inconsistent with the result given by the L-B theory. Although it seems for us that the number of Λ^0 is too much, this may be due to the thickness is production matter (which tends to increase the number of Λ^0 relative to Σ) and the too low estimation of the number of Σ.

There may occur many questions concerning the applicability of the L-B theory, the nucleon-nucleus collision at low energies, the thickness of the plates in the cloud chambers and others. These poins will be clarified in the following papers.

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2) G. H. Trilling and R. B. Leighton, Phys. Rev. **104** (1956), 1703; G. H. Trilling and G. Neugebauer, Phys. Rev. **104** (1956), 1688.

3) About these we shall elucidate later in another place. For the brief explanation see S. H. Hsieh, In this issue.

4) G. H. Trilling and R. B. Leighton, Phys. Rev. **100** (1955), 1468; R. B. Leighton et al, Phys. Rev. **100** (1955), 1457.

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6) G. H. Trilling and R. B. Leighton, Phys. Rev. **104** (1956), 1703.

7) Recently Steinberger obtained $\rho_\Lambda=0.66\pm0.03$. See J. Steinberger, to be published.

8) This can be justified for cosmic rays. About it we shall explain in another place.

9) Trilling showed $d=0.4\sim0.85$.

10) S. H. Hsieh, In this issue.

On the Observability of a New Hyperon

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May 2, 1957

Today one may think all new particles have been found up to now. However it will be still premature to conclude that. Perhaps the following consideration may give some insight on this question.

In cosmic rays there has been found one example of anomalous event,[1)2)] which seems to be interpreted as

$$\mathfrak{J}^- \to K^- + \Lambda^0 \qquad m_{\mathfrak{J}^-} \simeq 3160\, m_e \qquad (1)$$

or

$$\mathfrak{J}^- \to K^- + n \qquad m_{\mathfrak{J}^-} \simeq 2830\, m_e\,. \qquad (2)$$

If $\mathfrak{J}^-$ really exists, perhaps only the former is observable, because in the latter case the strangeness of $\mathfrak{J}^-$ is -2, and the transition $\mathfrak{J}^- \to \Xi^- + \gamma$ occurs quickly for this case.

In this note we estimated whether $\mathfrak{J}^-$ shown in (1) exists or not, by calculating the relative production ratio of $\mathfrak{J}^-$ to Ξ^- by cosmic rays. To calculate it we used Landau and Belenkiis' multiple production theory[3)4)] (L-B theory) and the energy spectra of cosmic rays. We made use of the L-B theory, because it can explain many phenomena.[5)] It is true that the L-B theory is not well established in lower energy regions. However only the hyperon production ratio[6)] or the production ratio of hyperons to π mesons[7)] given by the L-B theory is not bad at very low energies. Therefore we may expect that it can be used as the estimation of order for our case.

According to the L-B theory, the number density of baryons produced in nucleon-nucleon or nucleon-nucleus collision is given by

$$n_i = \frac{g_i}{2\pi^2}\left(\frac{kT}{c\hbar}\right)^3 Z_i^3 \int_0^\infty \frac{x^2\,dx}{e^{-y+Z_i\sqrt{1+x^3}}+1} \qquad (3)$$

where g_i is the statistical factor, $Z_i = m_i c^2/kT$, T is the critical temperature given by $kT \simeq m_\pi c^2$, and $y = \mu/kT$. (μ is the chemical potential.)

In our case $Z_i \gg 1$ ($m_{\Xi^-} \simeq 9.5\, m_\pi$), $|y| \ll Z_i$, and (3) may be reduced to[8)]

$$n_i = g_i (kT m_i / 2\pi\hbar^2)\, e^{-m_i c^2/kT + y}. \qquad (4)$$

Then using the relation $N_i = f(g_i, m_i, T, A, E)$ given by the L-B theory and the proton energy spectra[9)10)] $(0.7 \text{ Bev} + E)^{-2.8}$, we found that the required ratio (the production ratio of $\mathfrak{J}^-$ to Ξ^- by cosmic rays) can be in good approximation

$$g_{\mathfrak{J}}/g_{\Xi}\,(m_{\mathfrak{J}^-}/m_{\Xi^-})^{3/2}\, e^{(m_{\Xi^-} - m_{\mathfrak{J}^-})c^2/kT} \cdot G(E_{\mathfrak{J}^-}, E_{\Xi^-}) \qquad (5)$$

both for cloud chamber and emulsion. Here we made the assumption that $\mathfrak{J}^-$ and Ξ^- are always produced above certain energies $E_{\mathfrak{J}^-}$ and E_{Ξ^-}. (More detailed explanation will be given afterward.) It is to be remarked that the calculated result is scarcely changed by the ambiguity of the energy dependence of the production number (for example $E^{1/3}$ or $E^{1/2}$), because the energy spectra of nucleon is very steep.

From (5) we see that for known values of mass $m_{\mathfrak{J}^-}$, m_{Ξ^-}, and assumed values of spins $j_{\mathfrak{J}}$, j_{Ξ}, the required ratio becomes a function of $E_{\mathfrak{J}^-}$, E_{Ξ^-} and T. We took xE_{th} in place of $E_{\mathfrak{J}^-}$ and E_{Ξ^-}, and varied x between 1.0 and 2.0. E_{th} is the threshold energy to produce $\Xi^- + 2K$ for Ξ^-, and to produce $\mathfrak{J}^-(3160\, m_e) + 3K$ for $\mathfrak{J}^-$ (according to the conservation of the strangeness). As for the critical temperature T we varied it in the range $0.6\, m_\pi c^2 < kT < 1.5\, m_\pi c^2$. Although we varied x, the final result is not dependent on x due to cancelation.

The calculated ratios are given in Table 1 for various values of spins. These ratios are to be regarded as the ratios of production, when we do not take into account the thickness of the production matter. If the production matter is thin, and the lifetimes of $\mathfrak{J}^-$ and Ξ^- are nearly equal, these ratios may correspond to the ratios of observation. (The angular and momentum distribution of various hyperons seem to be not so different both in production and average.) The thickness of matter makes the observed ratio smaller, generally speaking.

Table 1. Ratio of $\mathfrak{J}^-(m_{\mathfrak{J}^-} \simeq 3160\, m_e)$ to Ξ^- to be produced by cosmic rays as the function of the critical temperature T and spin j.

$j_{\mathfrak{J}}$	1/2	1/2	3/2
j_{Ξ}	3/2	1/2	1/2
$T=0.6 m_\pi c^2$	0.01	0.02	0.04
$T=1.5 m_\pi c^2$	0.07	0.14	0.28

Up to now more than twenty examples of Ξ^- have been found in cosmic rays, and one example which may be interpreted as $\mathfrak{J}^-$, though not very reliable. From the above results we would like to say that $\mathfrak{J}^-$ shown in (1) may really exists. Perhaps more than fifty or one hundred examples of Ξ^- should be found to ascertain whether $\mathfrak{J}^-$ corresponding to (1) exists or not.

It is interesting to note that the above method of analysis can be used in the investigation of various phenomena of cosmic rays and new particles.

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1) Y. Eisenberg, Phys. Rev. **96** (1954), 541.
2) also see W. F. Fry et al. Phys. Rev. **97** (1955), 1189.
3) L. D. Landau, Ezv. Ann. **17** (1953), 51; S. Belenkii, Dok. Aka, Na. **99** (1955), 523; S. Belenkii and L. D. Landau, Uspe. Phys. Na. **56** (1955), 309.
4) Analysis of cosmic ray data using the L-B theory was previously done by Z. Koba. See Z. Koba, Prog. Theor. Phys. **51** (1956), 461.
5) E. L. Rosental and D. S. Chernabskii, Uspe. Phys. Na. **52** (1954), 185; Y. Fujimoto, Lecture at the Annual Meeting of the Physical Society of Japan, 1955, Oct.; Z. Koba, Prog. Theor. Phys. **15** (1956), 461.
6) S. H. Hsieh, To be published.
7) S. H. Hsieh, To be published.
8) S. Belenkii, Jou. Ezpe. Theor. Phys. **28** (1955), 111; S. Belenkii and L. D. Landau, Uspe. Phys. Na. **56** (1955), 309.
9) We assumed that the neutron energy spectra is similar to that of proton in shape. Nucleon is the main component which produces new particles, and we can expect very similar results will appear in other nuclear interactions.
10) See, for example, S. Hayakawa, The Cosmic Rays. (Iwanami Gendai Butsurigaku) 1955. Note that, the shape of proton energy spectra is scarcely changed by altitude.

On the Heavy Meson Production in the π^--Proton Reaction

Takeshi Ogimoto* and Tetsuzi Shimizu**

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June 17, 1957

In this letter, the angular distributions of the heavy mesons produced by the π^--proton reaction:

$$\pi^- + P \rightarrow \Lambda^0 + \theta^0 \tag{1}$$

$$\rightarrow \Sigma^0 + \theta^0 \tag{2}$$

$$\rightarrow \Sigma^- + \theta^+ \tag{3}$$

are examined by means of the lowest order perturbation theory.

Experimentally, the θ^0 meson is emitted predominantly in the forward, whereas the θ^+ meson preferentially in the backward in the center-of-mass system[1]. We shall show that, by assuming the preferable spins and parities for Λ^0, Σ and θ, these aspects of the π^--proton reaction are fairly well explained by the straightforward perturbation theory, without introducing the interaction of the type $KK\pi$ which was expected from the forward production of the θ meson[2]. Furthermore, the parities of Λ^0, Σ and θ will be automatically determined from these results.

We set the following assumptions

(1) far the spins and parities of Λ^0, Σ and θ:

	Λ^0	Σ	
Model (a)	$\frac{1}{2}^+$ (or $\frac{1}{2}^-$)	$\frac{1}{2}^+$ (or $\frac{1}{2}^-$)	0^+ (or 0^-)
Model (b)	$\frac{1}{2}^+$ (or $\frac{1}{2}^-$)	$\frac{1}{2}^-$ (or $\frac{1}{2}^+$)	
Model (c)	$\frac{1}{2}^-$ (or $\frac{1}{2}^+$)	$\frac{1}{2}^+$ (or $\frac{1}{2}^-$)	
Model (d)	$\frac{1}{2}^-$ (or $\frac{1}{2}^+$)	$\frac{1}{2}^-$ (or $\frac{1}{2}^+$)	

(2) for the interaction Hamiltonian densities:

$$H(NN\pi) = g_0 \overline{\psi}_N O \ \tau_\alpha \psi_N \phi_{\pi\alpha} + \text{h.c.},$$

$$H(\Sigma\Sigma\pi) = g_1 \overline{\psi}_\Sigma O \ I_\alpha \psi_\Sigma \phi_{\pi\alpha} + \text{h.c.},$$

$$H(\Lambda\Sigma\pi) = g_2 \overline{\psi}_\Lambda O \ \psi_{\Sigma\alpha} \phi_{\pi\alpha} + \text{h.c.},$$

$$H(\Lambda N\theta) = g_3 \overline{\psi}_N O \ \psi_\Lambda \varphi_\theta + \text{h.c.},$$

$$H(\Sigma N\theta) = g_4 \overline{\psi}_N O \ \tau_\alpha \psi_{\Sigma\alpha} \varphi_\theta + \text{h.c.},$$

where $O=1$ or $O=i\gamma_5$.

The coupling constants $g_1^2/4\pi = f_1^2$ and $g_2^2/4\pi = f_2^2$ are so estimated as to fit in with the experimental data: the total cross section for the processes (1), (2) and (3) is 1 mb at 1.3 Bev π^- meson energy and the branching ratio is given by

$$\frac{\sigma(\pi^- p|\Lambda^0\theta^0) + \sigma(\pi^- p|\Sigma^0\theta^0)}{\sigma(\pi^- p|\Sigma^-\theta^+)} \sim 2^{1)}. \tag{4}$$

We have $f_1^2 \sim 1.4$ and $f_2^2 \sim 6.3$ for Model (a), using $f_0^2 \sim 10$, $f_3^2 \sim 1$ and $f_4^2 \sim 0.3$;[3] we obtain $f_1^2 \sim 1.9$ and $f_2^2 \sim 2.3$ for Model (d), using $f_0^2 \sim 10$, $f_3^2 \sim 0.4$ and $f_4^2 \sim 0.2$. For other types of model the values of the coupling constant which fit in with the experimental data cannot be found.

Therefore, we may consider only the case of model (a) and (d) for the spins and parities Λ^0, Σ and θ. The theoretical angular distributions for the production processes (1), (2) and (3) are respectively given in Fig. 1 for these models. For comparison, we have also plotted the experimental ones according to Steinberger et al.[1)]

It is seen from this figure that the angular distribution of the heavy meson for the process (2) is different from that for the process (3). This difference may be interpreted as the process having Λ^0 in the intermediate state contributes to the latter, whereas the process does not appear in the former.

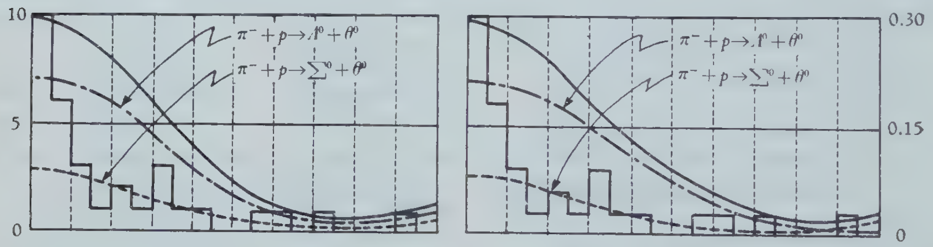
Needless to say, the experimental statistics is quite poor: moreover it is quite doubtful whether the perturbation calculation is meaningful for such a strong interaction. However, it should be remarked that, as is seen from Fig. 1, if we postulate the model (a) or (d) for the spins and parities of Λ^0, Σ and θ, the experimental aspects of the hyperon production in the π^--proton reaction seems to be explained fairly well by the perturbation calculation[4)], without taking into account an interaction of the type $KK\pi$. Note that Λ^0 and Σ must have the same parities.

We wish to express our cordial thanks to Dr. S. Minami and Dr. T. Nakano for their helpful comments.

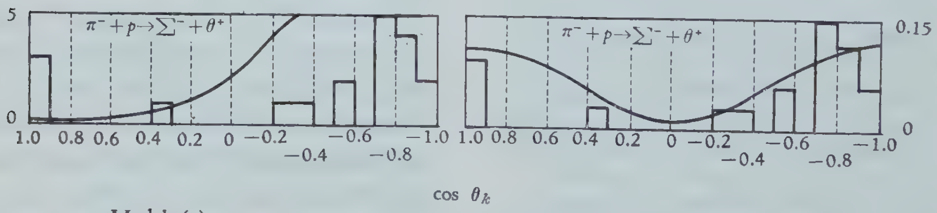
1) R. Budde, M. Chretein, J. Leitner, N. P. Samios M. Schwartz and J. Steinberger, Phys. Rev. **103** (1956), 1827.
2) J. Schwinger, Phys. Rev. **104** (1956), 1164.
3) S. Iwao, Soryushiron Kenkyu **10** (1956), 192 (in Japanese).
4) The similar results for model (d) only were also obtained by S. Goto. S. Goto, Soryushiron Kenkyu **13** (1956), 569.

Fig. 1.

The production angular distributions for the π^--proton reactions at 1.3 Bev π^--meson energy.



The center-of-mass distributions for the events which are either $\Lambda^0-\theta^0$ or $\Sigma^0-\theta^0$ production are presented together in this Figure.



Model (a)

Model (d)

θ_k is the center-of-mass angle of the heavy mesons in each case.

Progress of Theoretical Physics 投稿規定

A 原稿

I. 一般的注意

(1) 用紙は普通のタイプ用紙を用い，正副２通作って下さい。なお著者の手元にも写しを必ず保存願います。**正副いずれの原稿にも文字の指定を入れて下さい。**指定のしかたは別項及び見本を参照して下さい。

(2) 数式と特殊タイプの文字の他はすべてタイプ印書して下さい。

(3) 各頁とも，上下 2cm 左右それぞれ 1cm あけ，行間は１行ずつあけて下さい。この規定に従うと，１行が大体74字詰くらいになります。但し，脚註もこの中に含めますが，短い脚註は行間なしにつめて下さい。本文と脚註の境の棒は本文の１行分をとって下さい。

(4) 題名，氏名，研究場所，アブストラクト，本文の順に，各１行ずつあけて夫々行を改めて下さい。アブストラクトは１頁以内，氏はキャピタルで記し，名は略さないで全部記して下さい。(例：Kôtarô HONDA)

(5) 節 (§) の標題の上下の行間は，本文の２行分だけ余分にあけて下さい。

(6) 原稿は充分推敲の上清書して（**鉛筆やペンの書込みは(2)項以外は差控えて下さい**），全部をまとめて上の中央部を紙よりでとじ，各紙毎に右肩に頁を入れて下さい。

II. 使用文字

(1) 指定がなければ，本文中の文字はすべて立体，数式中の文字はすべてイタリックで組みます。従って本文中のイタリック，数式中の立体文字（例：log, lim, Trace, sin, *etc*.）とすべてのゴチックは指定願います。(指定記号は見本参照)

(2) ギリシャ文字，ドイツ文字は特に指定して下さい。ベクトルは矢を用いずに，ゴチックで表わして下さい。ギリシャ文字のゴチック体は現在次のものしかありません。

$$\boldsymbol{\Pi}, \boldsymbol{\Delta}, \boldsymbol{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\phi}, \boldsymbol{\Phi}, \boldsymbol{\pi}, \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\omega}$$

(3) **まぎれやすい文字，大文字と小文字の区別の困難な文字は，はっきり書いて下さい。**

i) まぎれやすい文字

0 (ゼロ)と O (オー)，　a (エィ)と α (アルファ)，　v (ヴィ)と ν (ニュー)，
1 (イチ)と l (エル)，　e (イー)と ε (イプシロン)，　x (エックス)と χ (カイ)，
k (ケー)と κ (カッパ)，　r (アール)と γ (ガンマ)，　w (ダブリュー)と ω (オメガ)

ii) 大文字と小文字の区別が困難な文字

C, K, O, P, S, V, W, X, (または $\mathcal{G}, \mathcal{M}, \mathcal{N}, \mathcal{Y}$ の場合)

(4) 添字ははっきり上下のルビにして下さい (H^{α} H_{α} etc.)。まぎらわしい時には(上つき)(下つき) と指定して下さい。複雑な添字はなるべくさけるよう表現に工夫して下さい。**二重の添字 (添字の添字)，あるいは添字のゴチックは殆んど組版できません。**

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⑴ 数式の１行は本文の３行に相当するだけあけて，ていねいに書いて下さい。特に長い式は印刷したとき１行に入る長さに切って，何行かに分けて書いて下さい。

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⑴ 本文中の引用番号は 1)，2～5)，6) の如く右カッコを用いて下さい。

⑵ 引用文献は通し番号をつけて論文の終りにまとめ，**次の様式に統一して下さい。**

H. Yukawa, Rev. Mod. Phys. **21** (1949), 474.

⑶ 単行本の引用は書名をイタリックで記し，発行された年号，発行所を書いておいて下さい。文献以外のものは番号をつけず，＊印等の印で脚註にして下さい。

⑷ 引用文献の著者名は上記⑵のように名の initial letter を必ず入れて下さい。

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原稿の書き方見本

§5. Wave forms of normal vibration

Let us next investigate the spatial wave forms of the normal vibrations for Case M. First consider a regular lattice. In this case the equation determining the allowed initial vector is, in the representation in which T is diagonalized,

$$\left\{(I+\Lambda)^{N}-I\right\}x_1'=\begin{pmatrix} e^{2Ni\beta}-1 & 0 \\ 0 & e^{-2Ni\beta}-1 \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=0, \tag{5.1}$$

where $x_1'=(x_1',\ y_1')^{T}=U^{-1}x_1$. Since for the eigenvalues $\beta=n\pi/N$ the matrix in the above formula is null, the initial vector is here arbitrary, except for $\beta=\pi/2$, when U^{-1} does not exist. Let it be $(x_1', y_1')^{T}$, then the state of the $(n+1)$th atom is given by

$$\begin{pmatrix} e^{2ni\beta} & 0 \\ 0 & e^{-2ni\beta} \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=\begin{pmatrix} e^{2ni\beta}x_1' \\ e^{-2ni\beta}y_1' \end{pmatrix}. \tag{5.2}$$

If we return to the original representation, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} \{\cos(2n\beta)+\tan\beta\sin(2n\beta)\}x_1+\sin(2n\beta)y_1/\sin2\beta \\ -2\tan\beta\sin(2n\beta)x_1+\{\cos(2n\beta)-\tan\beta\sin(2n\beta)\}y_1 \end{pmatrix}. \tag{5.3}$$

文字指定記号		校正記号			
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小文字	㊇又は小文字	小文字	㊇又は小文字	下げる	⊤
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ゴチックイタリック	⌒	行間あける	> <	左へ寄せる	⊢⊣
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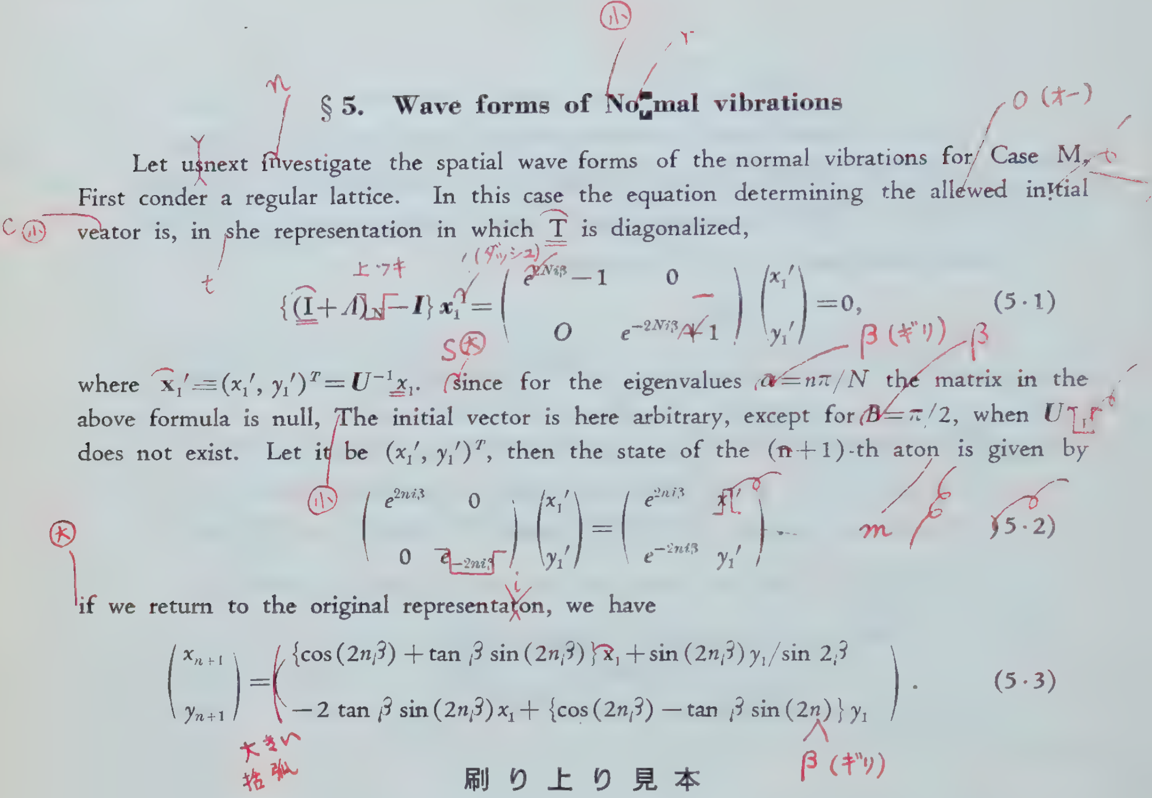
校正の仕方見本



§ 5. Wave forms of No■mal vibrations

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where $\mathrm{x}_1'\equiv(x_1', y_1')^T=U^{-1}x_1$. since for the eigenvalues $\alpha=n\pi/N$ the matrix in the above formula is null, The initial vector is here arbitrary, except for $B=\pi/2$, when U does not exist. Let it be $(x_1', y_1')^T$, then the state of the $(n+1)$-th aton is given by

$$\begin{pmatrix} e^{2ni\beta} & 0 \\ 0 & e_{-2ni\beta} \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=\begin{pmatrix} e^{2ni\beta} & x_1' \\ e^{-2ni\beta} & y_1' \end{pmatrix}\cdots \qquad 5\cdot 2)$$

if we return to the original representaton, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} \{\cos(2n\beta)+\tan\beta\sin(2n\beta)\}\mathrm{x}_1+\sin(2n\beta)y_1/\sin 2\beta \\ -2\tan\beta\sin(2n\beta)x_1+\{\cos(2n\beta)-\tan\beta\sin(2n)\}y_1 \end{pmatrix}. \qquad (5\cdot 3)$$

刷り上り見本

— 上掲の校正を終えて印刷したもの —

§ 5. Wave forms of normal vibrations

Let us next investigate the spatial wave forms of the normal vibrations for Case M. First consider a regular lattice. In this case the equation determining the allowed initial vector is, in the representation in which $\boldsymbol{T}$ is diagonalized,

$$\{(\boldsymbol{I}+\Lambda)^N-\boldsymbol{I}\}\,\boldsymbol{x}_1'=\begin{pmatrix} e^{2Ni\beta}-1 & 0 \\ 0 & e^{-2Ni\beta}-1 \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=0, \qquad (5\cdot 1)$$

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$$\begin{pmatrix} e^{2ni\beta} & 0 \\ 0 & e^{-2ni\beta} \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=\begin{pmatrix} e^{2ni\beta}\,x_1' \\ e^{-2ni\beta}\,y_1' \end{pmatrix}. \qquad (5\cdot 2)$$

If we return to the original representation, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} \{\cos(2n\beta)+\tan\beta\sin(2n\beta)\}x_1+\sin(2n\beta)y_1/\sin 2\beta \\ -2\tan\beta\sin(2n\beta)x_1+\{\cos(2n\beta)-\tan\beta\sin(2n\beta)\}y_1 \end{pmatrix}. \qquad (5\cdot 3)$$

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THE SUPPLEMENT OF THE PROGRESS OF THEORETICAL PHYSICS

No. 4. Relativistic Hydrodynamics of the Dirac Matter

Part I. General Theory

by

T. TAKABAYASI

Dear Sir:

The publication of progress of Theoretical Physics, SUPPLEMENT, "Theory of Beta Disintegration" has been already announced as No. 4 but we regret to inform you that it will be postponed owing to the circumstances on the author's part.

Instead, we are now preparing for the following issue, No. 4, which will contain Dr. T. Takabayasi's work, "**Relativistic Hydrodynamics of the Dirac Matter,**" The author is Assistant Professor at Nagoya University, now, visiting Poincaré Institute in Paris.

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Date of issue: in September, 1957.

Progress of Theoretical Physics, Vol. 18, No. 3, September 1957

On the Magnetic Properties of Cubic Cerium

Tsuyoshi MURAO* and Takeo MATSUBARA

Department of Physics, Hokkaido University, Sapporo
and
Research Institute for Fundamental Physics, Kyoto University, Kyoto

(Received May 28, 1957)

In terms of the level splitting of the electronic state $4f^1(^2F_{5/2})$ due to the crystalline and exchange field, the magnetic susceptibility and specific heat of cubic cerium are calculated theoretically and compared with experiments. The intensity of the crystalline field is also estimated using a simplified model of the metalic state. The agreement between theory and observation is fairly good.

§ 1. Introduction

Since the mass separation of rare earth metals has become possible by means of the ion exchange technique,[1] the magnetic properties of these metals have been extensively studied experimentally. It is now known that many of these elements are ferro- or anti-ferromagnetic and their magnetic behaviors are quite different from those of the iron group. The carriers of the magnetic moment in rare earth metals are the electrons occupying $4f$ orbitals incompletely, which seem to localize almost as if they were in the free atomic state, and hence keep their orbital magnetic moments without much interference from the electric crystalline field. The mechanism of exchange coupling between magnetic moments in rare earth metals is therefore expected to be somewhat different from the case of ordinary magnetic metals in that conduction electrons would play an important role in aligning the spin magnetic moments of the ion core, each of which is strongly coupled to the orbital moment through the spin-orbit interaction. This is Zener's picture of exchange.[2] A similar idea was applied by Pauling[3] to Gd and more recently investigated in detail by Kasuya[4] in connection with the electric and magnetic properties of transition metals. As was pointed out by Kasuya,[5] the strange feature of magnetism in rare earth metals is believed to come from the fact that, beside the extraordinary exchange mechanism mentioned above, there exist interactions between an orbital moment and crystalline field or between an orbital moment and other orbital moments, which will give rise to complicated effects, varying with the number of $4f$ electrons and with the temperature.

The object of this paper is to present an atomistic theory of the thermal and magnetic properties of metallic cerium. This element is the simplest one in the sense that the incomplete shell has only one 4*f*-electron, and therefore provides a good starting point for theoretical study.

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The cerium metal has a cubic close-packed and hexagonal close-packed structure, but here we restrict our interest only to the former and the latter is reserved to a later paper. For the cubic close-packed cerium metal, the measurements on the specific heat[5], magnetic susceptibility,[6] electrical resistivity,[7] Hall effect[8] and dilatometry[9] reveal that there exists a kind of phase change accompanied with large temperature hysteresis extending over about 50~200°K, and from the remarkable contraction observed on cooling[10] Pauling[11] suggested that this phase change is caused by the promotion of 4*f*-electrons to the 5*d* valence band. According to the specific heat measurement there appears another anomaly at about 12°K and Lock[6] showed in his recent experiments on susceptibility that this temperature corresponds to an antiferromagnetic Néel point. He also showed that the transition to the contracted phase, called the α-phase, depends largely on the history of the sample: repeating the measurements more than one hundred times the transition did not take place, but in the first cycle half of the 4*f*-electrons seemed to be promoted to the valence band. For the older observations we must therefore consider the cycle of experiment when we compare them with theoretical results.

§ 2. The state of the 4*f*-electrons in metals with cubic close-packed structure

The same kind of problem has already been considered by Penney and Schlapp[12] for paramagnetic rare earth salts in which there was no exchange field. We here treat the exchange field in the molecular field approximation, and hence the problem can be reduced to the one ion problem. In a cubic crystal the Hamiltonian has the form

$$\mathcal{H}=D(x^4+y^4+z^4-\tfrac{3}{5}r^4)+H_0J_z, \tag{1}$$

where D is a constant which expresses the intensity of the crystalline field and $H_0=g\mu\cdot(H_{ex}+H_z)$. g is the Lande g-factor, μ Bohr magneton and H_{ex} and H_z mean respectively the effective exchange field which acts on each 4*f*-electron and the applied external magnetic field. As the spin-orbit coupling is strong, the energy separation between the ground state, ${}^2F_{5/2}$, and the next excited state, ${}^2F_{7/2}$, is about 2200 cm^{-1},[14] and we can regard $\boldsymbol{J}^2=$constant. In the subspace of $J=5/2$ the Hamiltonian (1) can be expressed in terms of equivalent operators:

$$\mathcal{H}=qJ_z^2\left(7J_z^2-\frac{190}{4}\right)+\frac{q}{2}(J_+^4+J_-^4)-H_0J_z, \tag{2}$$

where $q=D\beta\overline{r^4}/4$, and the notations β and $\overline{r^4}$ are due to Stevens.[13),14] Here we have dropped constant terms.

The energy eigenvalues and their eigenfunctions are easily obtained: Putting $x=H_0/q$, the eigenvalues are

$$W_1=\frac{q}{32}\{-1518+16x-(1327104+98304x+4096x^2)^{1/2}\},$$

$$W_2=\frac{q}{32}\{-1518-16x-(1327104-98304x+4096x^2)^{1/2}\},$$

$$W_3=\frac{q}{32}\{-1518+16x+(1327104+98304x+4096x^2)^{1/2}\},$$

$$W_4=\frac{q}{32}\{-1518-16x+(1327104-98304x+4096x^2)^{1/2}\},$$

$$W_5=q\left\{-\frac{183}{16}-\frac{x}{2}\right\},$$

$$W_6=q\left\{-\frac{183}{16}+\frac{x}{2}\right\}, \tag{3}$$

and their corresponding eigenfunctions are

$$\begin{aligned} &\Psi_1=N_1(\psi_{3/2}+c_1\psi_{-5/2}), && \Psi_2=N_2(\psi_{-3/2}+c_2\psi_{5/2}), \\ &\Psi_3=N_3(\psi_{3/2}+c_3\psi_{-5/2}), && \Psi_4=N_4(\psi_{-3/2}+c_4\psi_{5/2}), \\ &\Psi_5=\psi_{1/2}, && \Psi_6=\psi_{-1/2}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \left.\begin{matrix} c_1 \\ c_2 \end{matrix}\right\} &= \frac{1}{384\sqrt{5}}\{768\pm 64x-(1327104\pm 98304x+4096x^2)^{1/2}\}, \\ \left.\begin{matrix} c_3 \\ c_4 \end{matrix}\right\} &= \frac{1}{384\sqrt{5}}\{768\pm 64x+(1327104\pm 98304x+4096x^2)^{1/2}\}. \end{aligned} \tag{5}$$

In eqs. (5), plus signs are taken for c_1 and c_3 and minus signs for c_2 and c_4. N_i in eqs. (4) are normalization constants, namely $(1+|c_i|^2)^{-1/2}$, and ψ_{J_z} is the normalized eigenfunction of J_z. For $x=0$ they group into a doublet (1 and 2) and a quartet (3, 4, 5 and 6), and their energy separation $\varDelta$ equals $72q$. In § 5, where we calculate the magnitude and sign of D, it is shown that $q>0$, and therefore the doublet is lower. When $x\neq 0$, all of the degeneracies are removed completely and the ground level is the state 1. This is illustrated in Fig. 1. In terms of these energy levels, we can calculate the magnetic susceptibility and the specific heat, from which the parameter q can be determined.

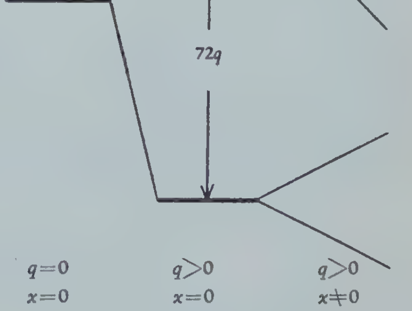

Fig. 1.

At sufficiently low temperature, $T\ll\varDelta/k$, where the lower levels play a dominant role in determining the physical properties, the quasi-angular momentum defined for the lower levels is expected to be replaced by an effective spin, $\boldsymbol{s}$, which will have the magnitude

$1/2$ in the present instance, for the ground state is a doublet and must be symmetrical with respect to the x, y and z directions. The wave functions for these two levels are given by

$$\Phi_1=\sqrt{\frac{5}{6}}\left(\psi_{3/2}-\frac{1}{\sqrt{5}}\psi_{-5/2}\right),\quad \Phi_2=\sqrt{\frac{5}{6}}\left(\psi_{-3/2}-\frac{1}{\sqrt{5}}\psi_{5/2}\right). \tag{6}$$

Taking Φ_1 and Φ_2 as a basis, the matrix elements of $\boldsymbol{J}$ may be calculated and expressed as

$$\boldsymbol{J}=-\frac{5}{3}\boldsymbol{s}\,, \tag{7}$$

where $\boldsymbol{s}$ is the usual spin operator with $s=1/2$. This means that the problem can be reduced to the case in which the magnitude of the spin is $1/2$ but the Lande g-factor is modified to $g'=-(5/3)g$.

§ 3. Specific heat

For the moment we shall forget the antiferromagnetism below $T_N=12°K$ in which case we can easily calculate the anomalous part of the specific heat due to Stark splitting, and by making comparison with observation it will be possible to determine the parameter q. Without considering the lattice vibration and other small contributions, the specific heat is given by

$$C=\frac{2Nk\Delta^2\beta^2e^{-\beta\Delta}}{(1+2e^{-\beta\Delta})^2}\,, \tag{8}$$

where $\beta=1/kT$ and N is the number of $4f$-electrons. If we take

$$\Delta/k=240°K, \tag{9}$$

the maximum of the general trend of the calculated specific heat coincides with that part of the experimental curve which is believed to be due to the crystalline splitting. The experimental values are however far lower than the theoretical curve if we take N as one electron per atom. In Fig. 2 we show the observed and theoretical specific heat curves calculated for two typical cases: (a) one $4f$-electron per atom and (b) $1/2$ per atom on the average. From the figure we see that experimental values observed by Parkinson et al. corresponds nearly to the case (b).

Actually, this can be confirmed still further when we consider the specific heat below the Néel point. The entropy of the anomalous part below the Néel point estimated by Parkinson et al. is about 0.6, which is close to $(R/2)\ln 2=0.688$. Parkinson and Roberts[15)] made a more detailed experiment on the specific heat and found that the entropy under the highest peak up to $20°K$ is only a little less than $R\ln 2$ for the 50th cycle of measurement in which case there should be almost no phase transition to α-phase: But for the first cycle the anomalous part of specific heat is very nearly equal to half of that of the 50th cycle at every temperature they observed. On the other hand, the magnetic entropy seems to tend to the value $R\ln(2J+1)$ at 300°K,[5] and this leads us to consider the α-phase as that half the atoms having a trivalent configuration and the other half a quadrivalent configuration are in some kind of ordered state. The magnetic moments of the former group are to have an antiferromagnetic arrangement below the Néel point.

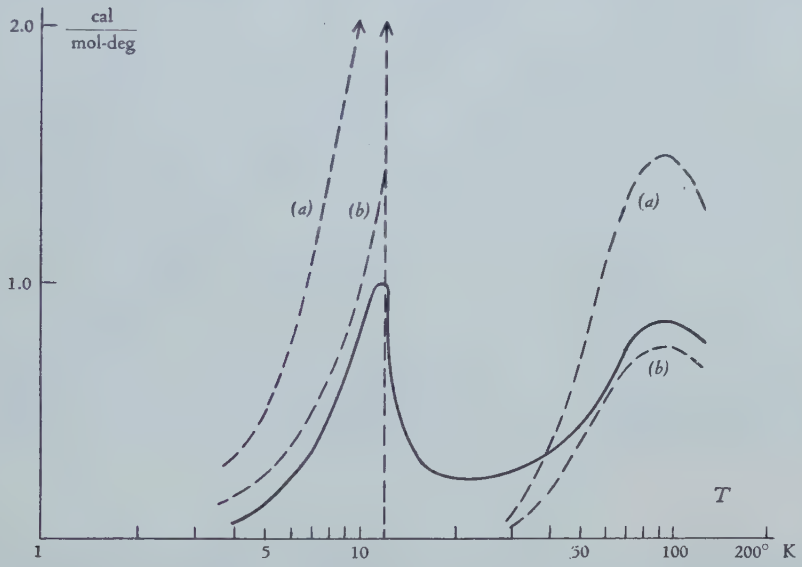

Fig. 2. Specific heat of cubic cerium
——— observed by Parkinson, Simon and Spedding
(Proc. Roy. Soc. **A207** (1951), 137)
- - - - (a) calculated assuming one 4f-electron per atom
- - - - (b) calculated assuming 1/2 4f-electron per atom

Using the molecular field approximation, the specific heat below the Néel point can be expressed as

$$C=-\tfrac{1}{2}NkT_N\cdot\partial\sigma^2/\partial T, \qquad (10)$$

where σ is the reduced spontaneous magnetization of each sublattice. As was already mentioned in § 2, the reduced magnetization of spin 1/2 can be used for σ. Taking $T_N=12$°K, the calculated curves are shown in Fig. 2 as (a) and (b), corresponding to the two cases mentioned above. The tail immediately above the Néel point is thought to be due to short-range order.

§ 4. Magnetic susceptibility

The susceptibility was observed by La Blanchetais[6] for pure samples down to 80°K and it was found that the susceptibility of the γ-phase, (high temperature phase of cubic close-packed structure) and of the hexagonal β-phase almost coincided with each other. Lock[6] also observed the susceptibility in successive cycles of measurements and found nearly the same result qualitatively, but he found that the Néel temperatures of each cycle were almost the same and that the successive cycles inhibited the transition to the α-phase. The phase obtained after many cycles seems to correspond to the γ-phase, although X-ray analysis shows traces of β-phase in addition to the γ-phase. On the other hand, the Curie constant estimated for the first cycle was nearly half of that of the γ-phase. This again confirms the supposition that half of the $4f$-electrons are promoted to the valence band.

We now calculate the susceptibility with the same value of $\varDelta$ as obtained in § 3. The value $\varDelta/k=240$°K is, however, thought to be due to 3.5 valence electrons per atom on the average in the α-phase. Hence we use $\varDelta/k=240\times(3/3.5)=206$°K for the γ-phase, because in our approximation $\varDelta$ is proportional to the valency (see § 5). The statistical mean value of the magnetic moment per atom, $\langle m_z\rangle$, is generally given by

$$\langle m_z\rangle=\frac{1}{Z_0}\sum_n\sum_{n'}\frac{1-e^{-\beta\Delta_{nn'}}}{\varDelta_{nn'}}e^{-\beta E_{n'}}|m_{znn'}|^2(H_{ex}+H_z), \tag{11}$$

where E_n is the energy eigenvalue of the n-th state and $\varDelta_{nn'}=E_n-E_{n'}$, $Z_0=\sum_n \exp(-\beta E_n)$, and $m_{znn'}$ is the matrix element of the operator $g\mu J_z$ between the state n and n'. The summation is performed over all states. Calculating the matrix elements by (4) and (5), we can express the magnetization $M=N\langle m_z\rangle$ as

$$M=\frac{C}{T}F(T)\,(H_{ex}+H_z), \tag{12}$$

where $C=N(g\mu)^2J(J+1)/3k$ is the Curie constant for sufficiently high temperatures and $F(T)$ is a correction factor due to crystalline field given by

$$F(T)=\frac{4}{7}\left(\frac{1}{2+e^{\Delta/kT}}\right)\left[\frac{1}{12}(26+5e^{\Delta/kT})+\frac{8}{3}\frac{kT}{\varDelta}(e^{\Delta/kT}-1)\right]. \tag{13}$$

As it should be, $F(T)$ tends to unity as $T\rightarrow\infty$. In the molecular field approximation H_{ex} can be assumed proportional to M, and hence we may put

$$H_{ex}=-(\theta/C)M. \tag{14}$$

From (12) and (14) susceptibility $\chi=M/H$ is obtained as

$$\chi=\frac{CF(T)}{T+\theta F(T)}. \tag{15}$$

We calculate χ as a function of temperature by adjusting the parameter θ in such a way that theoretical value coincides with observed one at T_N. The results are shown in Fig. 3 together with experimental values measured by Lock. The assumed values of $\varDelta$, θ, etc., are as follows;

	cycle no. in Lock's exp.	$\varDelta/k$	θ	N
curve (b)	1	240°K	10°K	1/2 $4f$-electron per atom
curve (a)	102	206°K	12°K	one $4f$-electron per atom

The susceptibility below the Néel point depends on the nature of magnetic superstructure and anisotropy energy, and since little is known experimentally about both properties we cannot go into the detailed discussion. As was seen in § 2, however, cerium at low temperatures can be regarded as an assembly of spin 1/2 with the Lande factor $g' = -(5/3)g$, and magnetic anisotropy will be very small.16) Therefore the susceptibility below the Néel point is constant in the first approximation as shown in Fig. 3.

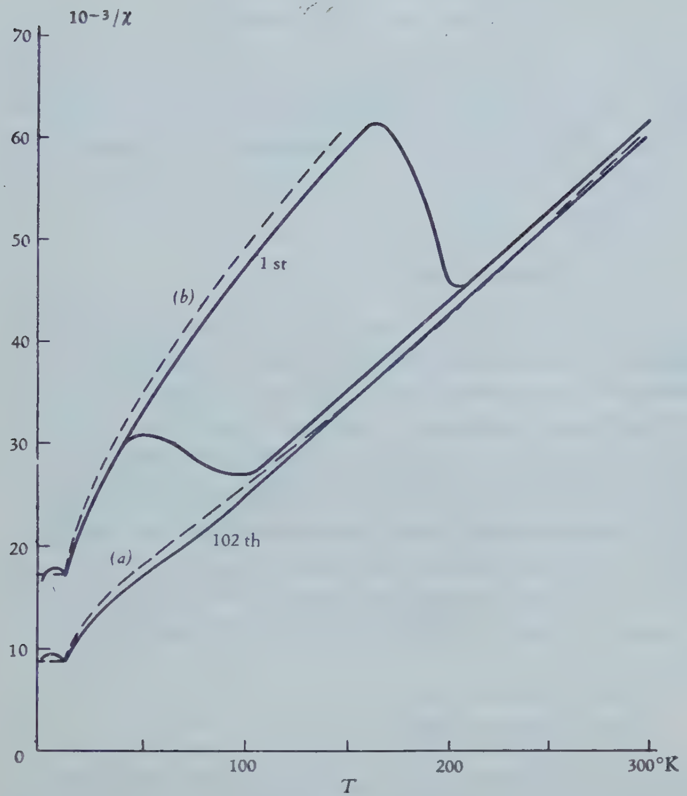

Fig. 3 Calculated and observed susceptibility of cerium

§ 5. The intensity of the crystalline field

In order to verify that the empirically derived crystalline field intensity used in the earlier sections is consistent, we calculate it using a reasonable model: The valence electrons are smeared out uniformly over the metal and the point (or spherical) charges of ion cores are located at each lattice site. The uniform minus charge does not contribute to the crystalline field except for the Madelung term and may be omitted here. The D in (1) is then given by

$$D = -\tfrac{5}{2}\sum_{n}\cdot e^2v/R_n^5 \tag{16}$$

where v is the valency and R_n means the distance of the n-th atom from the origin. We adopt the direct sum method, and actually the summation is performed within the cube of edge length $12a$ including the origin. Here $2a$ equals the lattice constant. The result is

$$D = +0.680\,(ve^2/a^5).$$

Hence we get

$$q = \frac{D\beta\overline{r^4}}{4} = +170\,\beta\frac{ve^2}{a}\frac{\overline{r^4}}{a^4}. \tag{17}$$

Taking $a = 2.57\times10^{-8}$cm,10) $\beta = 2/7.45$,14) and $q = 2.86k$ (§ 3, § 4) for the γ-phase, we obtain

$$v(\overline{r^4}/d^4) = 4.08 \times 10^{-2}.$$

If we use the Slater-function for the $4f$-orbital to determine $\overline{r^4}$ in terms of Z^*, the effective nuclear charge, v equals 2.5 for $Z^* = 12$. This value of Z^* is also obtained from the Slater-rule, although it is not very reliable for $4f$-electrons. The result here obtained is nevertheless quite reasonable. That the value of v is smaller than 3 is thought to be due to the screening by conduction electrons.

§ 6. Conclusions

The anomalous behavior of cerium metal at low temperatures can be partly explained by taking account of the crystalline field as well as exchange field. The magnetic properties of the α-phase are interpreted by assuming that half of the $4f$-electrons are promoted to the valence band. The detailed structure of this phase can be revealed only by neutron diffraction experiments however. The order of magnitude of the exchange coupling between nearest neighbours can be given by $k\theta$. If the exchange force affects more than nearest neighbours, as is the case of Zener's mechanism, the small value of the ratio $\theta F(T_N)/T_N$, which is 0.264 for the α-phase and 0.312 for the γ-phase, suggests that the exchange interaction changes its sign with distance.[4),17)]

Analysis of hexagonal rare earths is now in progress and will be discussed in another paper.

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The Electric-Octupole Transitions of Nuclei

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The j-forbidden electric octupole transitions of nuclei are explained on the basis of configuration mixing. The main contributions to the matrix element of the j-forbidden electric octupole transitions come from a proton transition for odd proton nuclei, and from both proton and surfon transitions for odd neutron nuclei.

In the case of a proton transition, the allowed matrix element is of the first order for odd proton nuclei. For odd neutron nuclei, the mixing of proton configuration must be taken into account. The matrix element for a proton transition is of the second order.

The allowed matrix element for a surfon transition is also of the second order.

The calculations are based upon the perturbation theory. The agreement between the calculated and observed lifetimes are qualitatively good.

§ 1. Introduction

Much important information about stationary states of the nucleus can be obtained from the study of the interaction of the nucleus with the electromagnetic field. The gamma-transition probability between two states depends, in a sensitive manner, on the wave functions of these states, hence the knowledge of nuclear wave functions can be obtained by comparing the experimental transition probability with the theoretical value calculated on some specific model. Among the various nuclear models proposed, the shell model seems to be most suitable to explain the nuclear gamma-transitions, especially the existence of the " island of isomerism ".[1] There are, however, some details with which the shell model fails to obtain quantitative agreement. One of these is the j-forbidden electric octupole ($E3$) transitions of nuclei which belong to the first island in the region before the closed shell at 50. These have been identified as $E3$-transitions of the $(7/2+)\longleftrightarrow(1/2-)$ type and occur for the nucleon numbers 43, 45, and 47. The $(7/2+)$ states have been classified on the basis of the shell model as $(p_{1/2})^2(0)\,(g_{9/2})^{3,5,7}(7/2)$ and the $(1/2-)$ as $(p_{1/2})^1(1/2)\,(g_{9/2})^{4,6,8}(0)$.[2),3)] The nucleons in the $g_{9/2}$ orbit are of abnormal coupling in the former configurations and of normal coupling in the latter configurations. Such a transition between the two states cannot occur, according to shell model, assuming a symmetric single-particle transition operator. It shows evidently the deviation from the pure configuration by the shell model.

On the other hand, there is an additional difficulty for odd neutron nuclei. The ratio of the probability for a single neutron transition to that for a single proton transition should be of the order $(Z/A^L)^2$ because of the recoil of the core in the former case;

i. e., of the order of 10^{-9} for $L=3$, $A=100$, $Z=50$. The transition probabilities determined from the experimental data have similar values in odd neutron nuclei and also in odd proton nuclei.

It may be expected that these difficulties are removed by the surface coupling in the collective model.[4)]

In general, the interaction potential[4),14)] is given as

$$V=\sum_{i} V(\alpha, \boldsymbol{r}_i)+\sum_{i<k} V_{\mathrm{res}}(\boldsymbol{r}_i\, \boldsymbol{r}_k). \tag{1}$$

The first term is surface coupling interaction. The surface coupling will admix particle states and, furthermore, the coupling with the deformation of order three produces an $E3$-moment which contributes to the surfon transition.

Now, the assignment of a configuration given by the simple shell model is assumed to be correct in zero-th order. We can define the mixing between various configurations due to the interaction (1) as configurations of higher orders. According to this idea, the transition matrix element is also decomposed into those of zero-th order and higher orders.

In the case of the surface coupling only, the allowed matrix element for a particle transition is of second order and that for a surfon transition is of third order, because the $(7/2+)$ state is of abnormal coupling as mentioned above and the surface interaction operator is the sum of single particle ones.

The residual interaction in (1) is composed of two-body interaction operators so that the states of two particles can be changed simultaneously.

The purpose of this paper is to show that the mixing of configurations by the interaction (1) gives the main contribution to the $E3$-transition. The allowed matrix element for a proton transition is of first order for odd proton nuclei. For odd neutron nuclei, the mixing of proton configuration must be taken into account. The matrix element for a proton transition is of second order. On the other hand, there is a matrix element for the surfon transition between the states with mixed configurations due to the two-body interaction and that due to the surface coupling interaction. They are also of second order. Therefore, for odd neutron nuclei, the main contributions to the matrix element come from both proton and surfon transitions.

The ratio of the unfavoured factors for a single proton transition $g_{7/2}\longleftrightarrow p_{1/2}$ in the odd neutron nuclei to that in the odd proton nuclei lies in a region of one to 10^{-2}. This gives strong support to the above considerations.

Flowers[5)] has shown that within the limits of the pure j-j coupling approximation the $(7/2+)$ level of the configurations $(g_{9/2})^{3,5,7}$ is never the ground state and would normally be expected to lie above the $(9/2+)$ level of seniority one. There exist, however, two low-lying states: $(7/2+)$ and $g_{9/2}$. In more than half of the cases the $(7/2+)$ state is lower than the $g_{9/2}$ state. We can imagine as usual that the observed lowering of the $(7/2+)$ state is brought about by small perturbing forces representing, for example, configurational interaction. Flowers[5)] has suggested its possibility for the configurational interaction due to the two-body force with finite range. But, we do not discuss this problem here.

Other evidence of configurational mixing is given by magnetic moments,[6] quadrupole moments,[7] l-forbidden magnetic dipole transitions,[8] and deuteron stripping reactions.[9] These have been calculated, for simplicity, with the two-body interaction force of the δ-function type. We assume, however, that the radial dependence of the two-body interaction is of finite range, because of the reason as suggested by Flowers.

§ 2. Transition probabilities

The shell model matrix elements vanish for the j-forbidden transitions. Non-vanishing elements appear as the effect of configurational mixing, and the dominant contributions come from the interaction (1).

The unfavoured factors in the odd-proton nuclei for a $g_{7/2} \longleftrightarrow p_{1/2}$ single proton transition lie in a region of 10^{-2} to 10^{-3}. This fact indicates that the extent of the admixture of $g_{7/2}$ orbit is of the order of one percent. Therefore, the calculations in the following are based upon the perturbation method.

We shall make a quantitative comparison between theoretical and experimental values of the lifetimes in § 4. Before that, we shall derive the matrix elements for the transitions by the consideration of the configurational mixing.

The radiative transition probability due to an electric 2^L-pole moment is given by

$$W_\gamma = \frac{1}{\tau_\gamma} = \frac{8\pi(L+1)}{L[(2L+1)!!]^2} \frac{e'^2}{\hbar} \left(\frac{E_\gamma}{\hbar c}\right)^{2L+1} \frac{\mathfrak{M}^2}{2j_i+1}. \tag{2}$$

where τ_γ is a mean lifetime, and $\mathfrak{M}^2$ is given by eqs. (3) and (3′). e' is the effective charge.

For a proton transition, $\mathfrak{M}^2$ is

$$\mathfrak{M}^2 = \sum_{m_i m_f M} |(j_f m_f | \sum_p r_p{}^L Y_{LM}(\theta_p \varphi_p) | j_i m_i)|^2 = (j_f \| \sum_p r_p{}^L Y_{L0}(\theta_p \varphi_p) \| j_i)^2. \tag{3}$$

In eq. (3) $\sum_p$ means a sum over individual nucleons in the nucleus. j_i is the angular momentum of the initial state and j_f is that of the final state, m_i and m_f being their z-components.

For a surfon transition, $\mathfrak{M}^2$ is

$$\mathfrak{M}^2 = (j_f \| 3Z/4\pi \cdot R_0{}^L \alpha_L{}^* \| j_i)^2 \tag{3'}$$

where Z is the proton number in nucleus and α_L the coordinates which describe the deformation of the nuclear surface.

Let us denote the initial and the final state by $\Psi(j_i m_i)$ and $\Psi(j_f m_f)$. We adopt as the zero-th order configuration the shell model wave function $\Psi_0(jm)$, and let $\Psi_n^{(1)}(jm)$ and $\Psi_n^{(2)}(jm)$ be typical excited configurations of the first order and the second order which mix with the zero-th order configurations.

If the admixture is small, initial and final wave functions are

$$\Psi(j_i m_i) = \Psi_0(j_i m_i) + \sum_n \alpha_n^{(1)} \Psi_n^{(1)}(j_i m_i) + \sum_n \beta_n^{(2)} \Psi_n^{(2)}(j_i m_i) \tag{4}$$

and

$$\Psi(j_f m_f) = \Psi_0(j_f m_f) + \sum_{n'} \alpha_{n'}^{(1)} \Psi_{n'}^{(1)}(j_f m_f) + \sum_{n'} \beta_{n'}^{(2)} \Psi_{n'}^{(2)}(j_f m_f). \tag{5}$$

There α's and β's are given by perturbation theory in terms of non-diagonal elements of the energy matrix, and are of the first order and the second order in V, respectively. By using wave functions (4) and (5), we get the reduced matrix element in eqs. (3) and (3′).

For j-forbidden transitions, the element between the initial and final zero-th order configurations vanishes, and the leading terms of the element are of first order for odd proton nuclei and are of second order for odd neutron nuclei. We neglect the effect of higher order contributions.

(i) *The case of odd proton nuclei*

The $(7/2+)$ states are classified as $(p_{1/2})^2(0)\,(g_{9/2})^{3,5,7}(7/2)$ on the basis of the shell model and the $(1/2-)$ states as $(p_{1/2})^1(1/2)\,(g_{9/2})^{4,6,8}(0)$. The experiment indicates that the $(1/2-)$ state has not only the configuration of zero-th order $(p_{1/2})^1(1/2)\,(g_{9/2})^n(0)$, but also that of first order which contributes to the $E3$-transition. Such configuration must be $(p_{1/2})^1(1/2)\,\{(g_{9/2})^{n-1}(7/2)\,g_{7/2}\}\,(0)$. Any other mixing of the configurations in the $(7/2+)$ state and those in the $(1/2-)$ state give rise to higher contributions to the transition matrix element.

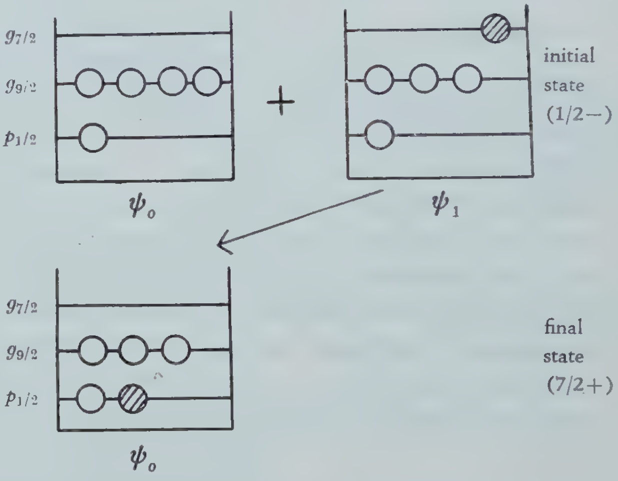

Fig. 1. A schematic picture of the transition, $g_{7/2} \to p_{1/2}$.

We represent the zero-th order configurations of the initial and the final state of the transitions as

$$\Psi_0(j_i m_i) = \Psi(j^{2j}(j)\, j_1^{\,n}(0)\,;\, j_i = j) \tag{6}$$

and

$$\Psi_0(j_f m_f) = \Psi(j^{2j+1}(0)\, j_1^{\,n-1}(j_2)\,;\, j_f = j_2) \tag{7}$$

where nucleons in $j_1 = l_1 + 1/2$, $j_2 = l_1 - 1/2$, and $j = l - 1/2$ are like and n is even number.

For a $j_2 \to j$ transition, we need only exchange the roles of the initial and the final state of the transition in the following calculation, since the relation

$$(j_2 \| \sum_p r_p^{\,L} Y_{L0}(\theta_p \varphi_p) \| j) = (-)^{j_2 - j} (j \| \sum_p r_p^{\,L} Y_{L0}(\theta_p \varphi_p) \| j_2) \tag{8}$$

Therefore, we can assume as eqs. (6) and (7) the initial and the final configuration without loss of generality.

Then, an excited state

$$\Psi^{(1)}(j^{2j}(j)\{j_1^{n-1}(j_2)j_2\}(0);\ j) \tag{9}$$

can interact with the zero-th order configuration in the initial state (6). The decay by the electric octupole transition from this to a zero-th order configuration of the final state (7) appears to be allowed because configurations (7) and (9) are the same except for one orbit and the different orbits j and j_2 satisfy the $\Delta l=3$ for orbital angular momenta.

We can get the reduced matrix element for the electric octupole transition due to the mixing of the excited configuration (9). The result is

$$\mathfrak{M}=(-)^{L+1+j_2-j}\alpha'\cdot\frac{F(j_2j;\ L)}{\sqrt{2j_2+1}} \tag{10}$$

where

$$F(j_2j;\ L)=(-)^{l_2-s+j}\frac{1}{\sqrt{4\pi}}[(2l_2+1)(2_2j+1)(2l+1)(2j+1)]^{1/2}$$

$$\times(-)^{g-l}W(l_2j_2lj;\ sL)(ll_200|L0)\cdot\int R_l(r)r^L R_{l_2}(r)\,dr,\quad 2g=l+l_2+L. \tag{11}$$

s is the intrinsic spin of the proton, and $R_l(r)$ the radial wave function. α is the amplitude of mixed configuration (9).

(ii) *The case of odd neutron nuclei*

The calculation of transition probabilities for odd neutron nuclei is similar to the calculation for odd proton nuclei. However, the electric 2^L-pole transition between single neutron states has a probability of only $(Z/A^L)^2$ in comparison with that in the single proton case, because in the former case the recoil of the core is small. For the electric octupole transition, this contribution to the transition probability is negligibly small. Therefore, for a particle transition, we must take into account the mixing of configurations for the proton core.

The matrix element which gives a main contribution to a proton transition is of second order. The contributions come from the transitions between the first order configurations, and between the zero-th order and second order configurations.

In the former case, such configuration in the $(7/2+)$ state must be $\{(p_{3/2})^3(3/2)g_{9/2}\}_P(J)[p_{1/2}\{(g_{9/2})^{n-1}(7/2)g_{7/2}\}(0)]_N(1/2)$, and that in the $(1/2-)$ state $\{(p_{3/2})^4(0)\}_P[p_{1/2}\{(g_{9/2})^{n-1}(7/2)g_{7/2}\}(0)]_N(1/2)$.

In the latter case, the $(7/2+)$ and $(1/2-)$ states would have also the second order configurations $\{(p_{3/2})^3(3/2)g_{9/2}\}_P(J)$, $\{p_{1/2}(g_{9/2})^n(0)\}_N(1/2)$, and $\{(p_{3/2})^3(3/2)g_{9/2}\}_P(J)\{(p_{1/2})^2(0)(g_{9/2})^{n-1}(7/2)\}_N(7/2)$, which mix with the zero-th order configurations.

The transitions between these configurations correspond to the single proton transitions between the $g_{9/2}$ state and the $p_{3/2}$ state.

We represent the first order configurations of the initial state and the final state of the transition as

$$\Psi^{(1)}(\{j'^{2j'+1}(0)\}_P(0)[j^{2j}(j)\{j_1^{n-1}(j_2)j_2\}(0)]_N(j);\ j) \tag{12}$$

and

$$\Psi^{(1)}(\{j'^{2j'}(j')j_1\}_P(J),\ [j^{2j}(j)\{j_1^{n-1}(j_2)j_2\}(0)]_N(j)\,;\,j_2). \tag{13}$$

where $j'=l+1/2$. Only the state with $J=3$ can decay by the electric octupole transition as is readily seen from the coupling scheme of angular momenta in (12) and (13). Then, the reduced transition matrix element is given by

$$\mathfrak{M}_{NI}=\alpha\alpha'(-)^{j'-j-j_1+j_2}\sqrt{\frac{2j_2+1}{2J+1}}\,F(j'j_1\,;\,L). \tag{14}$$

The second order configuration in the initial state, which contributes to the transition into the zero-th order configuration in the final state, is expressed as

$$\Psi^{(2)}(\{j'^{2j'}(j')j_1\}_P(J),\ \{j^{2j+1}(0)j_1^{n-1}(j_2)\}_N(j_2)\,;\,j). \tag{15}$$

The reduced transition matrix element is given by

$$\mathfrak{M}_{NII}=\beta\sqrt{\frac{2j+1}{2J+1}}\,F(j_1j'\,;\,L). \tag{16}$$

The contribution due to similar excitation in the final state:

$$\Psi^{(2)}(\{j'^{2j'}(j')j_1\}_P(J),\ \{j^{2j}(j)j_1^{n}(0)\}_N(j)\,;\,j_2) \tag{17}$$

can be obtained. The result is that $\mathfrak{M}$ is given by the same formula as $\mathfrak{M}_{NI}$, except for the coefficient of mixing.

$$\mathfrak{M}_{NIII}=\beta'\cdot(-)^{j'+j_2-j-j_1}\sqrt{\frac{2j_2+1}{2J+1}}\,F(j'j_1\,;\,L). \tag{18}$$

(iii) *Surfon transition*

The radiation emitted by the freely oscillating nuclear surface is of the electric multipole type of the same order L as the surface deformation. Due to the surface coupling interaction, the $(7/2+)$ state has a mixed configuration that a particle and a surfon excite simultaneously,

$$\Psi^{(1)}(p_{1/2}\{(g_{9/2})^{n-1}(7/2)g_{7/2}\}(0)\,;\,N=1,\ R=3\,;\,7/2). \tag{19}$$

N and R show the number of surfon and its angular momentum respectively. The transition between such configuration and that given by eq. (9) corresponds to a surfon transition.

The reduced transition matrix element is given by

$$\mathfrak{M}_S=\alpha\cdot\gamma\frac{3Z}{4\pi}R_0^3\sqrt{2j_2+1}\cdot\sqrt{\frac{\hbar\omega_3}{2C_3}} \tag{20}$$

where R_0 is the radius of the nucleus in its spherical equilibrium shape and γ the amplitude of mixed configuration. ω_3 and C_3 are the frequency and deformability of the 3rd surface mode respectively.[4)]

§ 3. Matrix elements of configuration interaction

(i) *Configuration interaction due to the two-body force*

In order to determine the coefficients of mixing of configurations which appeared in the preceding section, it is necessary to calculate the matrix elements of configuration interaction. The methods of calculation of the interaction matrix element have been shown by many authors.10),11),12) Therefore, only the more important points need be given here.

We assume that the forces of interaction are of the central form. The matrix element of interaction between the configuration of eq. (6) and that of eq. (9) is

$$\begin{aligned}(j^{2j}(j)\,j_1^{n}(0)\,;j|V|j^{2j}(j)\,\{j_1^{n-1}(j_2)\,j_2\}\,(0)\,;j)\\ =-\sqrt{\frac{2n(n-1)(n-2)}{(2j_1-1)(2j_1+1)}}\sum_I\sqrt{2I+1}\,(j_1j_1\,;I|V_{12}|j_1j_2\,;I)\\ \times\langle j_1^{n-2}(\nu=2,\,I)\,j_1|\}\,j_1^{n-1}(\nu=3)\,j_2\rangle.\end{aligned}\tag{21}$$

$\langle j_1^{n-2}(\nu=2,\,I)\,j_1|\}\,j_1^{n-1}(\nu=3)\,j_2\rangle$ is the fractional parentage coefficient and is reduced by Racah's method10) as

$$\langle j_1^{n-2}(\nu=2,\,I)\,j_1|\}\,j_1^{n-1}(\nu=3)\,j_2\rangle=\sqrt{\frac{3(2j_1+1-n)}{(n-1)(2j_1-3)}}\langle j_1^{2}(\nu=2,\,I)\,j_1|\}\,j_1^{3}(\nu=3)\,j_2\rangle\,.\tag{22}$$

The values of $\langle j_1^{2}(\nu=2,\,I)\,j_1|\}\,j_1^{3}(\nu=3)\,j_2\rangle$ for the $g_{9/2}$ configurations are tabulated by Flowers.5) $(j_1j_1\,;I|V_{12}|j_1j_2\,;I)$ is the two-body interaction matrix element. For the radial dependence of the interaction between two particles $V(r_{12})$, we can expand

$$V(r_{12})=\sum_k V_k(r_1,\,r_2)\,P_k(\cos\omega)\,.$$

Two-body interaction matrix element of the Wigner type is given by

$$\begin{aligned}&(j_1j_2,\,IM|V(r_{12})|j_3j_4,\,IM)\\ &=\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}\sum_{SL}(2S+1)(2L+1)\,U\begin{pmatrix}1/2 & l_1 & j_1\\ 1/2 & l_2 & j_2\\ S & L & I\end{pmatrix}U\begin{pmatrix}1/2 & l_3 & j_3\\ 1/2 & l_4 & j_4\\ S & L & I\end{pmatrix}\\ &\quad\times\sum_k(-)^{l_1+l_4-L}W(l_1l_2l_3l_4\,;Lk)\cdot(l_1\|C^{(k)}\|l_3)\,(l_2\|C^{(k)}\|l_4)\,F^{(k)}\end{aligned}\tag{23}$$

where

$$\begin{aligned}(l\|C^{(k)}\|l')=(-)^{g-l}\Big[(2l+1)(2l'+1)\,\frac{(l+l'-k)\,!\,(l+k-l')\,!\,(l'+k-l)\,!}{(l+l'+k+1)\,!}\Big]^{1/2}\\ \times\frac{g\,!}{(g-l)\,!\,(g-l')\,!\,(g-k)\,!}\,,\qquad(l+l'+k=2g:\ \text{even}).\end{aligned}$$

The Slater integrals $F^{(k)}$ are then defined by

$$F^{(k)}=\frac{2k+1}{2}\iint V_k(r_1,\,r_2)\,R_{l_1}(r_1)\,R_{l_2}(r_2)\,R_{l_3}(r_1)\,R_{l_4}(r_2)\,d^3r_1d^3r_2\,.$$

As the calculation for other matrix element of the configuration interaction is quite similar to the above mentioned case, it is not repeated here.

(ii) *Configuration interaction due to the surface coupling*

The surface coupling interaction $V(\alpha, \mathbf{r})$ is, according to A. Bohr,[4)]

$$V(\alpha, \mathbf{r}) = -k(r) \sum_{\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta\varphi). \tag{24}$$

$k(r)$ has the form of a delta function at the surface, $\delta(r-R_0)$. The matrix elements of $k(r)$ are approximately constant and are of the order of 40 Mev, assuming a kinetic energy inside the nucleus of 25 Mev. In the following, we assume that k is a constant.

The matrix element of interaction between the configuration of eq. (9) and that of eq. (19) is

$$\begin{aligned}
&(j^{2j+1}(0), j_1^{n-1}(j_2) : N=0, R=0\,;\, j_2 | \sum_i V(\alpha, \mathbf{r}_i) | j^{2j}(j) \\
&\qquad\qquad \times \{j_1^{n-1}(j_2) j_2\}(0) : N=1, R=3\,;\, j_2) \\
&= -k\sqrt{\frac{\hbar\omega_3}{2C_3}} \cdot \sqrt{\frac{7(2j+1)}{4\pi(2j_2+1)}} \cdot (l \| C^{(3)} \| l_1) W(l j l_1 j_2\,;\, \tfrac{1}{2}\, 3) \sum_I (2I+1) W(j j_2 j_2 j\,;\, I 3), \\
&\qquad\qquad (j_2 = l_1 - 1/2).
\end{aligned} \tag{25}$$

§ 4. Comparison with experimental values

In order to compare the theoretical values for matrix elements with the experimental ones, it becomes necessary to have more knowledge about the quantities which appeared in § 2 and § 3.

(I) *The interaction between two particles*

We assume that the interaction between two particles is the Serber potential with the Gaussian type and that the wave function of the nucleon is a harmonic oscillator one. The depth of interaction potential V_0 and the force range r_0 are taken from the data of the nucleon-nucleon scattering for low energies.[13)] These lead to $V_t = -70.8$ Mev, $V_s = -46.0$ Mev and $r_0 = 1.50 \times 10^{-13}$ cm.

(II) *The energy denominators*

The energy difference of the denominator in the perturbation theory consists of two parts. One is the difference between the single-particle levels in which the ground and excited configurations differ, and the other is that between the energies caused by the interaction between nucleons in each configuration. The estimate of the latter is complicated and it will be ignored in the actual calculation. Even if we restrict ourselves only to the energy differences of the single-particle levels, it is not easy to determine completely on the basis of conventional potentials, spin-orbit interactions, and the observed values of nuclear spins. The oscillator potential gives strongly degenerate states with different principal and azimuthal quantum numbers, and the square well potential with finite or infinite well depth does not always provide the order of levels which, together with plausible doublet splittings, would yield the observed values of the nuclear spins.

The energy differences between $g_{9/2}$ and $g_{7/2}$, $p_{3/2}$ and $g_{9/2}$, $p_{1/2}$ and $g_{7/2}$, are respectively taken to be 2.5 Mev, 2.5 Mev, and 3.5 Mev from the empirical data.[7)]

The values of energy of a surfon $\hbar\omega_3$ and deformability C_3 depend upon the mass number. However, we assume that these are constants and are about 5 Mev and 200 Mev respectively.

(III) *The amplitude of the mixed configurations*

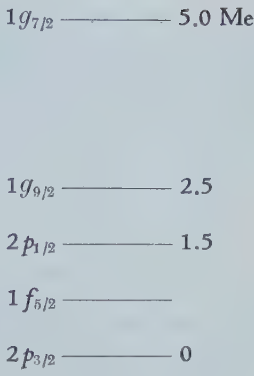

Fig. 2. Level schemes of single particle.

There are two difficulties in the determination of the amplitude of the second order configurations. One of these is that there are a number of configurations in the intermediate state, so that the calculation of the matrix element becomes involved. The other is that such a fractional parentage coefficient as $\langle j_1^{n-1}(\nu=3,\ J_0)j_1|\}j_1^{n}(\nu=2)J\rangle$ for the $g_{9/2}$ configurations is not available, because it has not been calculated up to now.

These facts give rise to the difficulties in our calculation of the amplitude of the second order configurations.

For the sake of simplicity, we assume that the neutrons in the $g_{9/2}$ and $p_{1/2}$-orbits excite only to the $g_{7/2}$-orbit in the intermediate state, and the protons in the $p_{3/2}$ orbit only to the $g_{9/2}$-orbit. These restrictions of the configurations in the intermediate state are not sufficiently accurate. Therefore, we must consider the obtained results as the qualitative ones.

The extent of the admixture is estimated to be of the order of a few percent for

Table I. The amplitude of the mixed configuration of the first order $(p_{1/2})^1(1/2)\{(g_{9/2})^{n-1}(7/2)g_{7/2}\}(0)$.

Nucleus	α	Nucleus	α
Se^{77}	0.103		
Se^{79}	0.141	Rh^{103}	0.114
Se^{81}	0.138	Rh^{105}	0.112
Kr^{79}	0.100	Ag^{107}	0.110
Kr^{81}	0.138	Ag^{109}	0.109
Kr^{83}	0.136		

The amplitude of the mixed configuration of the first order $\{(p_{3/2})^3(3/2)g_{9/2}\}_P(3)\,[(p_{1/2})^1(1/2)\{(g_{9/2})^{n-1}(7/2)g_{7/2}\}(0)]_N(1/2)$.

A	α'	A	α'
77	0.1887	81	0.1777
79	0.1833	83	0.1727

Table II. The amplitude of the mixed configuration of the second order
$\{(p_{3/2})^3(3/2)\,g_{9/2}\}_P(3)$, $\{(p_{1/2})^2(0)\,(g_{9/2})^{n-1}(7/2)\}_N(7/2)$.

Nucleus	β	Nucleus	β
Se^{77}	−0.0171	Kr^{79}	−0.0156
Se^{79}	−0.0220	Kr^{81}	−0.0209
Se^{81}	−0.0209	Kr^{83}	−0.0199

The amplitude of the mixed configuration of the second order
$\{(p_{3/2})^3(3/2)\,g_{9/2}\}_P(3)\{(p_{1/2})^1(1/2)\,(g_{9/2})^n(0)\}_N(1/2)$.

Nucleus	β'	Nucleus	β'
Se^{77}	0.0139	Kr^{79}	0.0131
Se^{79}	0.0185	Kr^{81}	0.0176
Se^{81}	0.0176	Kr^{83}	0.0168

the first order configurations. The calculated values for the coefficients of mixing of the configurations are listed in Tables I and II. Our calculated values are, in general, somewhat larger than those for the δ-type interaction, which is used in the calculations of magnetic moments, quadrupole moments, and l-forbidden magnetic dipole transitions.

The value of the amplitude of mixed configuration due to the surface coupling interaction is $\gamma=-0.156$.

(IV) *Comparison with experimental values*

Comparison of the theoretical value with the experimental one[2)] is shown in Table III. For odd neutron nuclei, we take into account the effects from both proton and surfon transitions. The contribution from the neglected mode of excitation in the intermediate state may be of the same order as that from the adopted mode here. Then, the theoretical value for odd neutron nuclei may vary in magnitude to some extent. We

Table III. j-forbidden electric octupole transitions.
$(7/2+)\longleftrightarrow(1/2-)$

Nucleus	E_γ (kev)	$\log \tau_\gamma$ (sec)	j_i	j_f	$\mathfrak{M}^2_{\mathrm{exp}}/\mathfrak{M}^2_{\mathrm{cal}}$
${}_{34}Se^{77}_{43}$	162	1.68	7/2+	1/2−	13.96
${}_{34}Se^{79}_{45}$	80	3.89	1/2−	7/2+	1.74
${}_{34}Se^{81}_{47}$	98	4.64	7/2+	1/2−	0.34
${}_{36}Kr^{79}_{43}$	127	2.47	1/2−	7/2+	0.41
${}_{36}Kr^{81}_{45}$	187	1.45	1/2−	7/2+	1.65
${}_{36}Kr^{83}_{47}$	32.2	~7.39	1/2−	7/2+	0.49
${}_{45}Rh^{103}_{58}$	40	~6.95	7/2+	1/2−	0.38
${}_{45}Rh^{105}_{60}$	130	2.55	1/2−	7/2+	0.62
${}_{47}Ag^{107}_{60}$	93.9	3.03	7/2+	1/2−	7.88
${}_{47}Ag^{109}_{62}$	89	3.04	7/2+	1/2−	11.16

cannot estimate it, because of the reason mentioned above. The interaction between two particles in a nucleus may differ from the force observed in the two particle system. For the sake of convenience, it is taken from the data of nucleon-nucleon scattering for low energies. In fact, the values of V_0, $\varDelta E$ and $\hbar\omega_3$ may fluctuate from nucleus to nucleus. These must be taken from the values which fit in nuclear energy level. However, it will be considered that the result in Table III shows a qualitative tendency.

§ 5. Conclusion

There are two major problems in the explanation of the electric octupole transitions. One of these is that the transitions are forbiddenness, because of the transitions between the state of normal coupling and the state of abnormal coupling of only protons (or neutrons) involving a change of orbit for one particle. This evidently shows the presence of mixing configurations in addition to the one assigned by the shell model. The other is that the transition probabilities determined from the experimental data have similar values in odd neutron nuclei and also in odd proton nuclei. It may be expected that these difficulties are removed by the surfon transition. However, we conclude in this paper that the main contributions to the matrix element come from a proton transition for odd proton nuclei, and from both proton and surfon transitions for odd neutron nuclei. The allowed matrix element is of first order for odd proton nuclei and is of second order for odd neutron nuclei. The agreement between the calculated and observed lifetimes are good, except for some cases. Some of the theoretical values deviate considerably from the experimental one. It may be attributed to an uncertainty of $\varDelta E$ and V_0, and also to the restriction of the configurations in the intermediate state. It should be noted that the estimate of the energy denominators, and also the restriction of the configurations in the intermediate state are not sufficiently accurate.

It is hoped to investigate the energy levels of those nuclei based on the consideration of configuration interaction.

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A Theory of Elementary Particles

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A theory of elementary particles formulated without using space-time co-ordinates is given. The causality principle in our theory is given by the relation of the results of two measurements in the ordered observation (O_1, O_2). Our causality principle contains an undetermined constant (a four-vector) $C(O_1, O_2)$. Our space-time is derived when the undetermined constants in the ordered observation $(O_1, O_2, O_3 \cdots)$ are given values by the law of probability. The field theory is valid only for "the local ideal system" which has no fluctuation of space-time. The limits of validity of the theory of relativity and the quantum mechanics based on Schrödinger equation are given in view of our theory. The difficulty of nonintegrability of the non-local field theories which aim at avoiding the divergence difficulty disappears into the fluctuation of space-time in our theory. A view is mentioned that the spatial extension of the proton core found in the high energy electron scattering is possibly an observed effect of the fluctuation of the "space-time of the system" in our theory.

§ 1. Introduction

When Einstein had renounced the mechanical theory of light ether and regarded the Maxwell equation of electro-magnetic field as the equation of motion of vacuum, the space-time was considered not only as the frame to describe the natural phenomena but as the fundamental physical substance. Relativistic quantum field theory has considered the Dirac equation of electron and the field equations of other particles as the equations of motion of vacuum and excellently described elementary particles and their interactions as the quanta and their interactions resulted from the quantization of the equation of motion of vacuum containing interactions. Owing to the success of the field theory, the fundamental theory of physics has been governed by the thought that the space-time is the fundamental substance which contains all the natural laws.

The field theory, however, has the divergence difficulty which is related intimately to the infinite numbers of points in space-time. What curbes various attempts to avoid the divergence difficulty is the assumption of existence of the infinitely exact clock and rule in the field theory. None of us knows at present in what way the divergent quantities will be replaced by the finite quantities in the future theory. We must await the future experiments. But at the present stage, if we can conquer the yoke of space-time by reconsiderations about its concept and extend the limit of applicability of the theory of elementary particles, the difficulty will be replaced by the ignorance.

Following these ideas, the author discussed in the previous paper[1)] the possibility to formulate the fundamental law of nature without using the space-time co-ordinates, and to derive the space-time as a statistical concept such as the temperature in the kinetic theory of gases. Our theory started from the existence and the description of the elementary particles, and attempted to formulate the quantum mechanics of the system of elementary particles without using the space-time co-ordinates. The essential point of our theory was the introduction of an undetermined constant $C(O_1, O_2)$ into the fundamental law which replaced the Schrödinger equation. At this point the theory differs essentially from the theory of second quantization starting from the particle aspect.

It is the purpose of this paper to formulate the theory more completely following the program given at the end of the previous paper.[1)] The author believes that our theory gives a new concept of space-time and has the possibility of inclusion of finite theory and suggests that the structure of elementary particles must be discussed in the co-ordinate spaces independent of the space-time.

§ 2. Existence and degrees of freedom of elementary particles

Our theory assumes the existence of elementary particles and starts from the description of their degrees of freedom in the co-ordinate spaces independent of the space-time. We assume the following (A_1), (A_2), (A_3) and (A_4).

(A_1)* Elementary particles have intrinsic quantities m, σ and are divided into kinds $k(m, \sigma)$ which are distinguished by intrinsic quantities, (m, σ).

Intrinsic quantities are quantities which we believe from observations to be intrinsic of the elementary particles. Intrinsic quantities will increase and may change in the progress of physics. At present, mass, magnitude of spin and isospin, etc., are considered to be intrinsic quantities. For convenience, we denote mass by m and other intrinsic quantities by $\sigma=(\sigma^{(1)}, \sigma^{(2)},\cdots)$. m is the mechanical mass (We mention the reason at the end of the next section).

(A_2) Elementary particles have degrees of freedoms which are characterized by the intrinsic quantities m, σ. We describe the freedoms in the co-ordinate spaces $\sum^{(m)}$, $\sum^{(1)}$, $\sum^{(2)}$, $\cdots \sum^{(f-1)}$. The freedom of momentum of elementary particles are described in four dimentional p_μ-space which we denote by $\sum^{(m)}$. In this space, the momentum of the particle of mass m lies on the surface:

$$\sum\nolimits^{(m)}(m)=\{p_\mu\,;\,p_1^2+p_2^2+p_3^2+p_4^2=-m^2\},$$

$$p_4=ip_0, \quad p_0,\ p_1,\ p_2,\ p_3 \text{ are all real,}$$

which are restricted by mass m.

The freedom of spin is described in the space of discrete points $s^{(1)}$ which we denote by $\sum^{(1)}$. In this space, the freedom of spin of the particle of spin $\sigma^{(1)}$ is described by the assembly of points:

$$\sum\nolimits^{(1)}(\sigma^{(1)})=\{s^{(1)}\,;\,|s^{(1)}|\leq\sigma^{(1)}\}$$

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* The alphabetical letters designating the main assumptions and postulates keep the correspondence to those in the previous paper. (1)

which are restricted by $\sigma^{(1)}$.

Other freedoms are described similarly by $\sum^{(i)}(\sigma^{(i)})$'s which are characterized and restricted by $\sigma^{(i)}$'s. That the degree of freedom of elementary particles is characterized by intrinsic quantity restricting its freedom is the criterion for the degree of freedom of the elementary particle in our theory. Space-time localization or orbital angular momentum are not considered to be the degree of freedom of elementary particles because they are not characterized by intrinsic quantities. This is a motive of our attempt to formulate the fundamental law without using the space-time co-ordinates.

(A_3) To a state of the system of elementary particles corresponds the assembly of integers (containing zero) $n_k(p, s)$ which specify the state of the system. $n_k(p, s)$'s are the variables of our theory. We call $n_k(p, s)$ as the numbers of the elementary particles of the kind k and of the co-ordinates which are specified by the points p and $s^{(i)}$ in the co-ordinate spaces $\sum^{(m)}$ and $\sum^{(i)}$.

(A_4) Elementary particles are divided into two groups of Fermi and Bose particles and the possible values of $n_k(p, s)$'s are restricted as

$$n_k(p, s) = 0,\ 1 \qquad \text{for Fermi particles,}$$

$$n_k(p, s) = 0,\ 1,\ 2,\ \ldots \quad \text{for Bose particles.}$$

$n_k(p, s)$'s are not the numbers of observed particles, but the numbers of bare particles. It is a central problem in the Tomonaga-Schwinger theory to discuss the true vacuum and observed particles. In relation to the idea of renormalization two standpoints are possible at present. One anticipates that the future theory is described in terms of only observed quantities, and has tried to formulate the theory in the renormalized quantities. The other expects that we can calculate unrenormalized quantities from observed quantities. This standpoint considers the renormalization technique as the phenomenological theory which calculates the observed quantities which are insensible to the detailed structure of the elementary particles. The attempts which have been tried to calculate the neutron-proton mass difference stand on the latter view point. We take the latter standpoint and this is the reason why we choose the numbers of bare particles as variables. The problems of the true vacuum or physical particles are the problems of the observables discussed in the next section. (In our theory, we can choose in principle the representation in which the numbers of physical particles are diagonal from the transformation theory mentioned in § 5.)

§ 3. Quantum mechanics of the system of elementary particles

We start from the quantum theory, not from the classical theory. Correspondence principle must be merely the heuristic method in our theory. We can postulate the kinematical part of the quantum mechanics, but we cannot use the dynamical part, because the Schrödinger equation assumes the existence of the exact clock and rule. We have no clock or rule at the beginning. Therefore we must have different description of dynamics.

1. *Hilbert space*

The state of the system of elementary particles is described by the vector in a Hilbert space spanned by the functions of the form,

$$\text{(A)} \qquad \psi[n_k'(p,\ s)].$$

ψ has the physical meaning of the probability amplitude of the system, i.e. the square of its norm gives the probability of finding the system in the state specified by the assembly of $n_k'(p,\ s)$, and it defines the physical meaning of the superposition principle.

2. *Observable*

We introduce the matrices $a_k(p,\ s)$ and $a_k^*(p,\ s)$,

$$a_k(p,\ s)=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad a_k^*(p,\ s)=\begin{pmatrix}0&0\\1&0\end{pmatrix}\qquad \text{for Fermi particles,}$$

$$a_k(p,\ s)=\begin{pmatrix}0&\sqrt{1}&0&0&\cdots\\0&0&\sqrt{2}&0&\cdots\\0&0&0&\sqrt{3}&\cdots\\ \cdot&\cdot&\cdot&\cdot&\cdots\end{pmatrix},\quad a_k^*(p,\ s)=\begin{pmatrix}0&0&0&\cdots\\ \sqrt{1}&0&0&\cdots\\0&\sqrt{2}&0&\cdots\\0&0&\sqrt{3}&\cdots\\ \cdot&\cdot&\cdot&\cdots\end{pmatrix}$$

for Bose particles.

We can get all the operators which move the state vector in the space spanned by the functions $\psi[n_k'(p,\ s)]$'s, making the linear combinations of the product of $a_k(p,\ s)$ and $a_k^*(p,\ s)$, and we call the hermitians in these operators as observables which we denote by Ω.

We assume that every physical quantity corresponds to a observable. The most part of theoretical physics concerns with the problem of finding observables which correspond to physical quantities. The field theory utilizes the correspondence principle to find observables. Our theory has a correspondence to the field theory as we mention in § 6, so that our theory has also a correspondence to the classical theory. But, in fact, the elementary particles have many quantities which do not correspond to classical quantities. And even in the field theory, almost all the physical quantities are not given their observables explicitly or exactly, because the vacuum or physical particles and interactions between them are not simple. Generally, at present, we must rely on the result of observations and the physical intuition to get observables which correspond to physical quantities.

We assume the existence of the observables which are important in our theory:

$$\text{(C)} \qquad \mathfrak{H}_\lambda=\sum p_\lambda a_k^*(p,\ s)\,a_k(p,\ s)+\mathfrak{H}_\lambda^I,$$

$$\lambda=1,\ 2,\ 3,\ 4,$$

which correspond to the total energy and momentum of the system we observe and we assume that $\mathfrak{H}_\lambda$ is a four vector in $\sum^{(m)}$. $\mathfrak{H}_\lambda^I$ will have the form

(D) $$\mathfrak{H}_\lambda^I=\sum\mathfrak{A}_\lambda G$$

where $\mathfrak{A}_\lambda$ is a product of $a_k(p, s)$ and $a_k^*(p, s)$ and G is a function of co-ordinates which appear in $\mathfrak{A}$. The form of G is considered to be determined by the future observations and future theoretical considerations on $\sum^{(m)}$ and $\sum^{(\sigma)}$, but at present we are free to presume the form of G to get a finite theory. This is the possibility of our theory for the inclusion of finite theories.

3. *Observation*

Our theory gives the fundamental law in the relation between the results of two measurements. Our fundamental law is based on the following assumptions.

(E_1) We have the measurement which is characterized by the observable Ω and the system we measure which has a four vector $\mathfrak{H}_\lambda$ in $\sum^{(m)}$.

(E_2) We have measurements. Measurements of the same system are able to be ordered. We denote the order of measurements on the same system as O_1, O_2, O_3, $\cdots$, and call the ordered measurements on the same system as the ordered observation $(O_1, O_2, O_3, \cdots)$. The concept of the ordering of measurements is more primitive* than the concept of space-time in physics.

(E_3) When we measure a system, and the observable Ω is expressed for the system as the direct sum

$$\Omega=\Omega_1\dot{+}\Omega_2\dot{+}\cdots,$$

we call the measurement of Ω as the simultaneous measurements of Ω_1, Ω_2, $\cdots$. Simultaneous measurements are assumed to belong to the same O, and to be measured independently.

We have often the system for which an observable is approximately expressed as direct sum of other observables. This is the reason why we find that physicists discuss the structure of a nucleus in the same way wherever we go in the world. However, we have no observable which can be expressed as the direct sum of more than two observables exactly. This is the reason, as we mention in § 4, why we cannot have in principle an exact clock and rule.

4. *Results of measurements* We postulate:

(E_4) When the system is the state represented by ψ, the expectation value of the result of the measurement of the quantity which corresponds to the observable Ω is given by $\langle\psi|\Omega|\psi\rangle$. ($\langle|$ and $|\rangle$ are bra and ket of Dirac.)

(E_5) The result of measurement of Ω is one of eigenvalues of Ω.

5. *The principle of causality* in our theory is given by the relation between the results of an ordered observation (O_1, O_2). We postulate:

(F) When the result of Ω_1 in O_1 is ω, then the expectation value of the result of measurement of Ω_2 in O_2 is $\langle\psi'|\Omega_2|\psi'\rangle$, where

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* Exact clock and rule can order measurements, but the ordering of measurements cannot imply the exact clock and rule.

(G) $$\psi'=\exp(i\sum_{\lambda}\mathfrak{H}_{\lambda}C_{\lambda}(O_1, O_2))\psi_{\omega} \text{ and } \Omega_1\psi_{\omega}=\omega\psi_{\omega}.$$

$C(O_1, O_2)$ is a constant four vector in $\sum^{(m)}$. $C(O_1, O_2)$ is undetermined except that it is indepent of Ω_1, Ω_2 and $\mathfrak{H}$, and depends only on O_1 and O_2. Hence, if the measurement of Ω_2 is the simultaneous measurements (E_3) of $\Omega_{21}\dot{+}\Omega_{22}\dot{+}\cdots$, the expectation values of Ω_{21}, Ω_{22}, $\cdots$ in O_2,

$$\langle\psi'|\Omega_{21}|\psi_1'\rangle,\quad \langle\psi'|\Omega_{22}|\psi'\rangle,\ \cdots$$

have the same $C(O_1, O_2)$, but the value of $C(O_1, O_2)$ is undetermined.

Our fundamental law (G) replaces the Schrödinger equation containing time. We have got the fundamental law which contains no space-time co-ordinate. We have conquered the space-time by the introduction of undetermined constant $C(O_1, O_2)$ into the fundamental law.

Our theory insists that we cannot generally predict the expectation values of the result of measurement of O_2 even when we know the result of measurement of O_1 no more than the values containing undetermined quantity $C(O_1, O_2)$. $C(O_1, O_2)$ reveals a fundamental unknowable. The meaning of $C(O_1, O_2)$ and our intention in what way we want to describe the nature will be explained by the figurative illustration of the "rope-dancing". Now we imagine that the many equal ropes are extended through the air. On each rope a rope-dancer is performing his rope-dance. They are tireless but fall down due to uncontrollable causes. We have no watches. Rope-dancers are classified into A, B, C, $\cdots$ according to their abilities. If the abilities of classes is given only in the ratio $a_A : a_B : a_C : \cdots$, our knowledge contains an undetermined constant C. If rope-dancers correspond to elementary particles, C will correspond to $C(O_1, O_2)$ in our theory. In what way of description can we predict the fate of a rope-dancer B if we have no concept of time? The way we can find is as follows. When there are many rope-dancers of the same class A, the probability of surviving is α when the ratio of the number of survivers and the original numbers of A-rope-dancers is β The value of β is the statistical quantity having fluctuations. When the number of A is very large, $\{\beta\}$ will play a rôle of a graduation of time.

§ 4. Space-time

1. *Space-time of the system determined by* $\{\Omega\}$

When the system allows or approximately allows a number of simultaneous measurements of the observables $\{\Omega\}$ which do not commute with the total energy momentum operators of the system, we can, by the law of probability, give the values to the $C(O_i, O_j)$ in the ordered observation $O_1, O_2, \cdots$. The aggregate of the values of $C(O_i, O_j)$ determined thus statistically in all the possible ordered observations is the "space-time of the system in the measurements of observables $\{\Omega\}$" in our theory.

The system which has the space-time can be used as a clock or a rule when we can observe the system with the other system simultaneously. Suppose that the system is given which is composed of independent systems 1, 2, $\cdots$ N, so that

$$(\mathrm{H}_1)\qquad \psi_M = \overset{N_1}{\Pi}\psi_i^{(1)}\overset{N_2}{\Pi}\psi_i^{(2)}\overset{N_3}{\Pi}\psi_i^{(3)}\overset{N_4}{\Pi}\psi_i^{(4)}.$$

We denote the total energy momentum of the system as $\mathfrak{H}_\lambda$. And we assume that we have the observables,

$$(\mathrm{H}_2)\qquad \Omega = \Omega^{(1)} \dot{+} \Omega^{(2)} \dot{+} \Omega^{(3)} \dot{+} \Omega^{(4)}$$

where

$$\Omega^{(\lambda)} = \Omega_1^{(\lambda)} \dot{+} \Omega_2^{(\lambda)} \dot{+} \cdots \dot{+} \Omega_{N_\lambda}^{(\lambda)}, \quad \lambda = 1,\ 2,\ 3,\ 4$$

and $\Omega^{(\lambda)}$ is defined to satisfy the following commutation relations.

$$(\mathrm{H}_3)\qquad \begin{array}{l} [\Omega^{(4)},\ \mathfrak{H}_4] \neq 0, \quad [\Omega^{(4)},\ \mathfrak{H}_l] = 0\ ; \\ [\Omega^{(l)},\ \mathfrak{H}_l] \neq 0, \quad [\Omega^{(l)},\ \mathfrak{H}_k] = 0 \quad \text{for} \quad l \neq k, \quad l,\ k = 1,\ 2,\ 3. \end{array}$$

If we measure $\Omega^{(4)}$ in the ordered observation $(O_1,\ O_2)$, we can compare the results of the measurements and the expectation values of the measurements in O_2

$$(\mathrm{H}_4)\qquad \langle \psi_i^{(1)} | \exp(-i\mathfrak{H}_4 C_4(O_1,\ O_2)\,\Omega^{(4)} \exp(i\mathfrak{H}_4 C_4(O_1,\ O_2))\,|\psi_i^{(1)}\rangle$$
$$i = 1,\ 2,\ \cdots N,$$

where $\psi_i^{(1)}$ is the state vector prepared by the measurement in O_1. When N_4 is large, we can give the most probable value to $C_4(O_1,\ O_2)$ by the law of probability.* We denote this statistical value as $\bar{C}_4(O_1,\ O_2)$. If we measure $\Omega^{(4)}$ in the ordered measurements of $O_1^4,\ O_2^4,\ O_3^4,\ \cdots$, we can get in a similar way

$$(\mathrm{H}_5)\qquad \bar{C}_4(O_1^4,\ O_2^4), \quad \bar{C}_4(O_2^4,\ O_3^4), \quad \bar{C}_4(O_3^4,\ O_4^4), \cdots$$

If the above mentioned $\Omega^{(1)}$, $\Omega^{(2)}$, $\Omega^{(3)}$ and $\Omega^{(4)}$ exist which allow simultaneous measurements in the ordered observation $(O_1,\ O_2,\ O_3 \cdots)$ we can give to every pair of $(O_i,\ O_{i+1})$ the four values $\bar{C}_\mu(O_i,\ O_{i+1})$, $\mu = 1,\ 2,\ 3,\ 4$. We call the assembly of $\bar{C}$ in all the possible ordered observations, as the space-time of the system in the measurements of $\Omega^{(\mu)}$'s, $\mu = 1,\ 2,\ 3,\ 4$.

When we consider the system ψ_M as a composition of two systems, i.e. $\psi_M = \psi_{M'} \cdot \psi_{M''}$, we can get two assemblies of $\bar{C}_\mu(O_i,\ O_{i+1})$ and $\bar{C}_{\mu'}(O_i,\ O_{i+1})$ in an ordered observation $O_1,\ O_2,\ O_3,\ \cdots$.

In general the equalities

$$(\mathrm{I}_1)\qquad \bar{C}_\mu(O_i,\ O_{i+1}) = \bar{C}_{\mu'}(O_i,\ O_{i+1}), \quad i = 1,\ 2,\ 3 \cdots$$

and

$$(\mathrm{I}_2)\qquad \bar{C}_\mu(O_i,\ O_{i+1}) + \bar{C}_\mu(O_{i+1},\ O_{i+2}) = \bar{C}_\mu(O_i,\ O_{i+2}),$$
$$i = 1,\ 2,\ 3,\ 4 \cdots$$

are not valid. We call this fact as the fluctuation of space-time. When (I_1), (I_2) are valid we call that the system has the exact space-time. The magnitudes of the fluctuations of the $\bar{C}_\mu(O_i, O_j)$ $\mu=1, 2, 3, 4$ give the measure of exactness of the space-time of the system determined by $\{\Omega\}$.

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* The law which gives $\bar{C}_\mu(O_1,\ O_2)$ is contained in the definition of the expectation value of (E_4).

3. *Ideal System*

We call, for convenience, the system of which the state vector and the observables have the form of (H_1) and (H_2) and N_μ is considered to be infinitely large, as "ideal" (in μ-direction). The system which contains an independent ideal system is an ideal system. The law of large numbers in the theory of probability infers that when we observe the two ideal systems simultaneously in an ordered measurements $O_1, O_2, O_3, \cdots$, the two ideal systems give the same assembly of values of $\bar{C}_\mu(O_i^\mu, O_j^\mu)$. Namely the equality (I_1) is valid for ideal system. We call that the ideal system is "local" when the equality (I_2) is valid and "non-local" when the equality (I_2) is not valid.

4. *Local ideal system*

For the local ideal system, $\bar{C}(O_i, O_j)$ is expressed as a difference of the form,

$$(I_2) \qquad \bar{C}_\mu(O_i, O_j)=\gamma(X_{\mu i}-X_{\mu j}), \quad \mu=1, 2, 3, 4.$$

where γ is a universal constant. Hence, if we have a local ideal system (in four directions), we can give to every O_i in the ordered measurement $O_1, O_2, O_3, \cdots$, four values $X_{\mu i}$ which is specified by a point in a four dimentional space spanned by four X_ν-axis. We call this space as X-space for the local ideal system.

Our theory is invariant for the rotation of $\sum^{(m)}$ as we mentioned in a preceding section. If we rotate the p_μ-axis in $\sum^{(m)}$, the energy-momentum four vector becomes

$$(J) \qquad \mathfrak{H}_\lambda'=\sum_\nu a_{\lambda\nu}\mathfrak{H}_\nu .$$

Since $C(O_i, O_j)$ is a four vector in $\sum^{(m)}$, $\bar{C}(O_1, O_2)$ is in the rotated representation

$$(J_1) \qquad \bar{C}_\lambda'(O_1, O_2)=\sum_\nu a_{\lambda\nu}\bar{C}_\nu(O_1, O_2),$$

hence

$$X_\lambda'=\sum_\nu a_{\lambda\nu}X_\nu .$$

The invariance property of the theory in $\sum^{(m)}$ implies the invariance in the Lorentz transformation in X-space.

The local ideal system can be used as an exact clock or an exact rule when we measure the system with the other system simultaneously. The local ideal system corresponds to the thermometer of ideal gas in themodynamics. The pi-meson clock mentioned in the previous paper is an illustrative example of the local ideal system (in 4-direction). However as we mention in § 6, the assumption of the existence of the local ideal system implies the local field theory, and hence the local ideal system has the divergence difficulty. In reality, therefore, we cannot have the local ideal system. And we cannot have the ideal system, because a particle in the system receives "the reaction of real particle "*. The space-time of any system fluctuates. The fluctuation of space-time is the original motive** of our theory.

We can say that the theory of relativity is only valid for the case in which the fluctuation of space-time of the system is neglected. This is the limit of applicability of the theory of relativity in view of our theory.

§ 5. Transformation theory

Our theory is based on the quantum mechanics and is described in the relation between operators and vectors in a Hilbert space so that the theory is also invariant in the unitary transformation of the Hilbert space. We mention here the transformations that are important for our discussions.

1. *Heisenberg representation*

The fundamental law (F) (G) is described in an equivalent way as follows.

(K) In an ordered observation (O_1, O_2), when the result of measurement of Ω_1 in O_1 is ω then the expectation value of measurement of Ω_2 in O_2 is $\langle\psi_\omega|\Omega_2'|\psi_\omega\rangle$, where $\Omega_1\psi_\omega=\omega\psi_\omega$ and

$$\text{(L)} \qquad \Omega_2'=\exp\Big(-i\sum_\lambda \mathfrak{H}_\lambda C_\lambda(O_1, O_2)\Big)\,\Omega_2\,\exp\Big(i\sum \mathfrak{H}_\lambda C_\lambda(O_1, O_2)\Big).$$

We may call (F) (G) the Schrödinger representation and (K) (L) the Heisenberg representation in our theory.

2. *Fock-representation*

It is often convenient to describe the theory in the representation as

$$\text{(M)} \qquad \psi=\begin{pmatrix} \psi_0 \\ \psi_1(s_1,\ p_1) \\ \psi_2(s_1,\ p_1;\ s_2,\ p_2) \\ \psi_3(s_1,\ p_1;\ s_2,\ p_2;\ s_3,\ p_3) \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix}$$

where $|\psi_n(s_1,\ p_1;\ \cdots;\ s_n,\ p_n)|^2$ is the probability finding the system of n particles having the co-ordinates $s_1, p_1;\ s_2, p_2;\ \cdots s_n, p_n$. (For simplicity, we considered one kind of particles.)

3. *Fourier transformation*

We can describe the theory in the x_μ-space transformed from the p-space by the Fourier transformation. The state vector in x-space is expressed in Fock-representation as

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* T. Tati, Soryushi-ron-Kenkyu, IV, p. 62 (1949).

** T. Tati, The fluctuation of space-time: Preliminary note of lectures for the annual meeting of the Physical Society of Japan (1952), p. 24.

$$\text{(N)} \qquad \psi = \begin{pmatrix} \varphi_0 \\ \varphi_1(s_1,\ x_1) \\ \varphi_2(s_1,\ x_1\ ;\ s_2,\ x_2) \\ \varphi_3(s_1,\ x_1\ ;\ s_2,\ x_2\ ;\ s_3,\ x_3) \\ \cdots\cdots\cdots\cdots \end{pmatrix}$$

where

$$\text{(N}_1\text{)} \qquad \varphi_n(s_1,\ x_1 :\ s_2,\ x_2\ ;\ \cdots\ ;\ s_n,\ x_n) = \int e^{i(p_1x_1+p_2x_2+\cdots+p_nx_n)} \psi(s_1,\ p_1 :\ \cdots;\ s_n,\ p_n)\, d^4p_1\cdots d^4p_n$$

$$\Big(px = \sum_{\mu=1}^{4} p_\mu x_\mu\Big).$$

x-space can be identified with X-space for the local ideal system as we mention in the following section. The real system is the non-local and non-ideal system and the space-time in our theory is not identical with x-space. Our theory insists that the x-space is not space-time but only Fourier transform of p-space and the fundamental theory cannot be based on the Schrödinger equation containing x.

The x-space will be useful also in our theory when we want to give the boundary condition of which the position of boundary is insensitive to the result of the calculation. In this case, on the boundary, we can identify x with the space-time.

§ 6. Correspondence between field theory and our theory

The space-time in the field theory is the x-space in (N). We show here that the X-space for the local ideal system is identified with x-space. This is the correspondence between the field theory and our theory.

For the local ideal system our fundamental equation (G) can be expressed as

$$\text{(O)} \qquad \psi(X) = \exp\Big(i\gamma \sum_{\lambda} \mathfrak{H}_\lambda X_\lambda\Big)\, \psi(0)$$

or in the form of differential equation

$$\text{(O}_1\text{)} \qquad \frac{i}{\gamma}\frac{\partial\psi(X)}{\partial X_\lambda} = -\mathfrak{H}_\lambda \psi(X) \qquad \lambda = 1,\ 2,\ 3,\ 4.$$

And (L) is expressed as

$$\text{(P)} \qquad \Omega(x) = \exp\Big(-i\gamma \sum_{\lambda} \mathfrak{H}_\lambda X_\lambda\Big)\, \Omega(0) \exp\Big(i\gamma \sum_{\lambda} \mathfrak{H}_\lambda X_\lambda\Big)$$

or in the differential equation

$$\text{(P}_1\text{)} \qquad \frac{i}{\gamma}\frac{\partial\Omega(X)}{\partial X_\lambda} = [\mathfrak{H}_\lambda, \Omega(X)].$$

(O_1) and (P_1) have the same form with the Schrödinger equation and the Heisenberg equation in the field theory when we put

$$(I_3) \qquad \gamma = (\hbar c)^{-1}.$$

We can say that the quantum mechanics based on the Schrödinger equation is only valid for the case in which the fluctuation of space-time is neglected. This is the limit of applicability of the quantum mechanics.

If we have the Schrödinger equation (O_1), our theory is shown to be identical with the field theory by the theory of second quantization starting from the particle aspect and we can get the quantized field equation in X-space. Hence we can say that the field theory is valid for the local ideal system. In other words, the field theory is valid for the local system which contains an independent local ideal system, i.e. an exact clock and a rule.

For the real system we cannot put the equation (O_1) or (P_1) in the basis of the theory. Our clock and rule fluctuate in principle. We can say that the field theory is valid only in the case where the fluctuation of space-time can be neglected. This is the limit of the field theory in view of our theory.

§ 7 Fluctuation of space-time and inconsistency in field theory

Various theories to avoid the divergence difficulty in the field theory based on the Schrödinger differential equation in x-space have the serious inconsistency in the following point. The fundamental differential equation (Tomonaga-Schwinger equation) in these theories do not satisfy the integrability condition. In other words, in these theories the state vector determined on a surface in x-space is to be given but can not be given in fact. This inconsistency disappears into the fluctuation of space-time in our theory, because our theory is based on the fundamental equation formulated without using the space-time co-ordinates.

In our theory, when a four-vector $\mathfrak{H}$ which leads to a finite theory is given, we have a consistent theory involving no divergence difficulty. The detailed form of $\mathfrak{H}$ can not be given at present and its determination depends on the results of future experiments. But we can presume at present the form of $\mathfrak{H}$ and make various theories each of which is characterized by the assumption of the form of $\mathfrak{H}$.

The system whose $\mathfrak{H}$ leads to a finite theory is the " non local system ". The space-time of the non local system has the fluctuation even when the system is an " ideal system " (§ 4). Different theories, characterized by their assumptions of the forms of $\mathfrak{H}$, give different space-time respectively, which are characterized by nature of their fluctuations in ideal systems.

The reason why the field theory has the above mentioned inconsistency and our theory has not is that the space-time is assumed in the field theory while it is derived in our theory, or in other words, our theory allows the existence of the fluctuation of space-time while the field theory does not.

Stanford experiments[2] of the electron-proton scattering have shown that the proton core has the spatial extension of the order of rms radius 0.7×10^{-13} cm. The interpretation[2] that the coulomb law breaks down at small dimension seems to be very attractive, because the interpretation suggests that the proton core extension is an observed effect of the non-localized interaction. The break-down of the coulomb law, however, will imply the break-down of the field theory because we have no consistent non local field theories.

In our theory, however, it is possible that the electromagnetic interaction part of the total energy-momentum operator $\mathfrak{H}$ contains or effectively contains the form factor (invariant in $\sum^{(m)}$) which cuts the high momentum part. In this case we may say that the effective radius of the proton core found in the electron scattering experiment is neither the radius of the proton nor the radius of the electron, but the radius of the fluctuation of space-time in the electron-proton scattering system.*

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The author dedicates this paper to Prof. S. Tomonaga.

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* We want to discuss about this problem elsewhere in more detail.

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Theory of Collective Oscillation of Electrons in Solids

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A general formulation of the collective oscillation of electrons in solids are developed to take into account fully effects of ion-core potential. Based on the second quantized scheme of the many-body system, the collective component of density fluctuation is examined carefully and it is shown that we can define a set of the collective normal co-ordinates under the specified conditions. In Section 3, the many-electron Hamiltonian is transformed into the Hamiltonian of the collective field and of the electrons with a modified electron-electron interaction and the interaction Hamiltonian between the collective field and the electrons. In Section 4, we derive the formula of the generalized "plasma" frequency and that of the dispersion relation.

§ 1. Introduction and summary

Recent developments of experimental studies of the scattering of fast electrons by thin solid films have made it possible to observe the energy spectrum of scattered electrons with extremely high energy-resolution. The spectrum consists of several lines, of which higher order lines appear to be multiples of a fundamental one. Since the line width is fairly narrow, it seems very hard to interpret the result in terms of single electron excitation through the inter-band transitions.[1] Thus, Bohm and Pines[2] have suggested a possible interpretation in terms of the excitation of a collective oscillation in solids by the interloper electrons.

The fundamental nature of the collective oscillation of electrons in metals has been studied theoretically by Pines.[3] As the Bohm-Pines theory dealt with the free electron system, many efforts[4),7)] have been forwarded for the extension of the Bohm-Pines theory to take into account the effects of the periodic potential of atomic cores.

The purpose of this paper is to present a general formulation of the collective motion of electrons in solids, based on the realistic physical picture that the collective motion in solids is due to the density fluctuation of electrons. So far all of the works on the present subject have used the canonical transformation method, which has restrictive limitation in the application, to formulate the present problem.

In the canonical transformation method, at first place one has to write down a Hamiltonian that describes the motion of the electrons and an auxiliary "plasma" field, then one tries to show this Hamiltonian is formally equivalent to the real Hamiltonian of the many-electron system under some subsidiary condition. This procedure can be carried out easily for the case of the Coulomb interaction, but not for other kinds of interaction such as the molecular interaction. Also in the canonical transformation method the statistical character of the particle can hardly be taken into account explicitly in the formulation of the problem.

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On the contrary, the density is a realistic physical quantity, and it can be defined uniquely for a many-body system with any kind of inreraction. By defining the density operator in the second quantized scheme, the statistics of the particles is also taken into account explicitly in the basic formulation of the problem. Furthermore, since this does not bring in any auxiliary dynamical variables, there is no need to introduce subsidiary conditions. Hence, the present approach is free from any ambiguity of the validity of the subsidiary condition.[8] Another possible approach to discuss the collective motion, which is free from such question of the subsidiary condition, has been developed by Nogami[9] and Ferrel,[10] based on the time dependent formulation of the Hartree-Fock self-consistent field. Tomonaga[11] has developed an elementary theory of the collective motion of many-body systems, which also enable us to discuss the collective motion without subsidiary condition.

In Section 2, we shall show that there exists a set of collective normal co-ordinates (q_k, p_k), defined by eqs. (18) and (39), under the conditions (I) and (II), which are presented in Section 2. It shall be shown also that the normal co-ordinate q_k executes a harmonic oscillation with the frequency ω. The frequency ω is determined by the dispersion relation (50).

In Section 3, we shall separate the original Hamiltonian of the many-electron system into the Hamiltonian of the collective field (65), the Hamiltonian (64) that describes the individual motion of the electrons with modified interaction and the interaction Hamiltonian between the collective field and the electron field (66). The Hamiltonian of the collective field is characterized by the "inertia" of the system $I(\boldsymbol{k}, \omega)$, which is an equivalent quantity introduced in the Tomonaga theory of the collective motion of the many-body system.

In Section 4, we shall develope a detailed discussion on the dispersion relation (50) and the proper frequency of the collective oscillation (51). The proper collective frequency is composed of two components, $\omega_s(\boldsymbol{k})$ is due to intra-band transitions, $\omega_a(\boldsymbol{k})$ due to inter-band transitions. They are given by eqs. (76) and (77), respectively. By using the fundamental properties of the Bloch wave function, $\omega_s(\boldsymbol{k})$ and $\omega_a(\boldsymbol{k})$ are reduced to the simpler forms (85) and (89). In the same way, the dispersion terms are also reduced to simpler expression (93). Finally, the collective frequency ω is summarized as eq. (97), which is refered to the case of the empty valence band electrons and the case of filled up valence band electrons through eqs. (95) and (96), respectively.

In the appendix, we shall give a comment with respect to Kanazawa's paper. In appendix III of his paper, Kanazawa has developed the calculation to derive the dispersion relation, according to the same approach used by the present author. At first sight, his dispersion relation (A·11) of reference 6) seems different from our dispersion relation (48). It will be, however, shown in the appendix that both of these relations are actually equivalent to each other.

The reason why he has reduced the form (48) to (A·2) by subtracting the quantity (A·1) might lie on the situation that the dispersion relation is derived in the form (A·2) by the canonical transformation of the Hamiltonian of the electron-plasma system (29) of reference 6). Here it must be noticed, however, that in his theory as well as in all other worker's theory, the electron field is described by the Bloch approximation, while the plasma field Hamiltonian introduced in eq. (29) of reference 6) is such one that is associated with the free electron system. Since the collective oscillation is the dynamical manifestation of a specified many-body system, the collective behavior of the Bloch electron system is different from that of the free electron system. Therefore, the introduction of the "plasmon" field as an universal physical entity hardly makes any sense.

The analysis of the experimental results based on the present theory will be illustrated in a separate paper.[12)]

§ 2. Normal co-ordinates of collective motion

To discuss the many-body system in general, it is appropriate to start with the second quantized representation of the many-body system, because it enables us to treat the statistical character of the system properly. We have a Hamiltonian for electrons in solids as follows,

$$H=H_0+H_1 \tag{1}$$

$$H_0=\int\psi^*(\boldsymbol{x})\left\{-\frac{\hbar^2}{2m}\Delta+U(x)\right\}\psi(\boldsymbol{x})\,d^3x \tag{2}$$

$$H_1=\frac{1}{2}\iint\psi^*(\boldsymbol{x})\,\psi^*(\boldsymbol{x}')\,V(\boldsymbol{x},\boldsymbol{x}')\,\psi(\boldsymbol{x}')\,\psi(\boldsymbol{x})\,d^3x\,d^3x' \tag{3}$$

where $U(x)$ is the periodic potential caused by the atomic nuclei in a lattice. The wave field $\psi(\boldsymbol{x})$ represents the electrons, in valence states as well as in ion-cores. $V(\boldsymbol{x},\boldsymbol{x}')$ is the Coulomb interaction between electrons. If the valence electrons are weakly bounded, we may neglect the polarization of the ion-core caused by the collective motion of the valence electrons. For such a case, we may regard $U(\boldsymbol{x})$ as a potential against the valence electrons due to the ion-cores, and discuss only the collective behavior of the valence electrons. In the strongly bounded case, the collective motion of the valence electrons may influence the motion of the core-electrons, thus it may induce the polarization of the ion-core. Based on eq. (1), the core electrons are treated on an equal basis with the valence electrons. Thus we can discuss the polarization effects of the ion-cores, which have been discussed in a semi-classical approach by Mott.[13)]

We expand the field quantities $\psi(\boldsymbol{x})$, $\psi^*(\boldsymbol{x})$ into

$$\psi(\boldsymbol{x})=\sum_{np\lambda}a_n(\boldsymbol{p},\lambda)\,\phi_n^{\boldsymbol{p}}(\boldsymbol{x},\lambda) \tag{4a}$$

$$\psi^*(\boldsymbol{x}) = \sum_{np\lambda} a_n^*(\boldsymbol{p}, \lambda)\phi_n^{p*}(\boldsymbol{x}, \lambda) \tag{4b}$$

where $a_n(\boldsymbol{p}, \lambda)$, $a_n^*(\boldsymbol{p}, \lambda)$ obey the commutation relations,

$$[a_n(\boldsymbol{p}, \lambda), a_m^*(\boldsymbol{q}, \mu)]_+ = \delta_{nm}\delta_{pq}\delta_{\lambda\mu} \tag{5a}$$

$$[a_n(\boldsymbol{p}, \lambda), a_m(\boldsymbol{q}, \mu)]_+ = [a_n^*(\boldsymbol{p}, \lambda), a_m^*(\boldsymbol{q}, \mu)]_+ = 0. \tag{5b}$$

Sufficies n, m stand for energy bands, $\boldsymbol{p}$, $\boldsymbol{q}$ for momentum parameters and λ, μ for spin states. $\phi_n^p(\boldsymbol{x}, \lambda)$ is the Bloch function multiplied by a spin function. The spatial part of $\phi_n^p(\boldsymbol{x}, \lambda)$, say $\phi_n^p(\boldsymbol{x})$, satisfies the Schroedinger equation,

$$\left\{-\frac{\hbar^2}{2m}\Delta + U(x) - E_n(\boldsymbol{p})\right\}\phi_n^p(\boldsymbol{x}) = 0. \tag{6}$$

As an eigen-function of electrons in a periodic potential, $\phi_n^p(\boldsymbol{x})$ must have the following structure,

$$\phi_n^p(\boldsymbol{x}) = \frac{1}{\sqrt{N}} e^{ipx} u_n^p(\boldsymbol{x}) \tag{7}$$

where N is the number of cells in the lattice. The momentum parameter $\boldsymbol{p}$ takes discrete values and $u_n^p(\boldsymbol{x})$ has the translational periodicity of the lattice. The factor $1/\sqrt{N}$ insures the ortho-normalization,

$$\int_V \phi^{np*}(\boldsymbol{x})\phi_m^q(\boldsymbol{x})\, d^3x = \int_v e^{i(q-p)x} u_n^{p*}(\boldsymbol{x}) u_m^q(\boldsymbol{x})\, d^3x = \delta_{p,q}\delta_{n,m} \tag{8}$$

where V is a total volume of the crystal lattice, and v is a unit cell volume.

Now, let us expand $\phi_n^p(\boldsymbol{x})$ and $V(\boldsymbol{x}, \boldsymbol{x}')$ into the Fourier series over the volume V,

$$\phi_n^p(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \phi_n^p(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{9}$$

$$V(\boldsymbol{x}\cdot\boldsymbol{x}') = \frac{1}{V} \sum_{\boldsymbol{k}} V(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{10}$$

The density operator $\rho(\boldsymbol{x})$ is expressed by

$$\rho(\boldsymbol{x}) = \sum_{\lambda} \psi^*(\boldsymbol{x}, \lambda)\psi(\boldsymbol{x}, \lambda) = \frac{1}{V}\sum_{\boldsymbol{k}} \rho(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{11}$$

with the definition of $\rho(\boldsymbol{k})$ as

$$\rho(\boldsymbol{k}) = \sum_{np}\sum_{mq}{}_{\lambda} F_{np}^{mq}(\boldsymbol{k})\, a_n^*(\boldsymbol{p}, \lambda)\, a_m(\boldsymbol{q}, \lambda) \tag{12}$$

where $F_{np}^{mq}(\boldsymbol{k})$ is the Fourier component of a correlation function of the electrons in the states $(n, \boldsymbol{p})$ and $(m, \boldsymbol{q})$, i. e.

$$F_{np}^{mq}(\boldsymbol{k}) = \int_V \phi_\lambda^{p*}(\boldsymbol{x})\phi_m^q(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d^3x \tag{13}$$

has a conjugate relation

$$F_{np}^{mq}(\boldsymbol{k})=F_{mq}^{np}(-\boldsymbol{k}). \tag{14}$$

For free electrons, since $u_n^p(\boldsymbol{x})$ is simply given as $v^{-1/2}$, $F_{np}^{mq}(\boldsymbol{k})$ is reduced to a following form,

$$F_{np}^{mq}(\boldsymbol{k})=\delta_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{q}}. \tag{15}$$

For the Bloch electrons, in the tightly bounded approximation, $u_n^p(\boldsymbol{x})$ is given by

$$u_n^p(\boldsymbol{x})=\sum_j e^{i\boldsymbol{p}(\boldsymbol{x}_j-\boldsymbol{x})}\varphi_n(\boldsymbol{x}-\boldsymbol{x}_j) \tag{16}$$

where $\varphi_n(\boldsymbol{x}-\boldsymbol{x}_j)$ is the atomic orbital wave function that is located about the point $\boldsymbol{x}_j$. Thus, $F_{np}^{mq}(\boldsymbol{k})$ is reduced as

$$F_{np}^{mq}(\boldsymbol{k})=\delta_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{q}}\int_v \varphi_n^*(\boldsymbol{x})\varphi_m(\boldsymbol{x})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d^3x \tag{17}$$

where overlapping integrals are neglected. It should be noticed here that the momentum conservation is very essential to derive the commutators (22) and (23) in the subsequent discussions.

First, we shall show that the co-ordinate of the collective density fluctuation can be defined by

$$q_k=\Omega\sum_{np}\sum_{\substack{mq\\ \lambda}}\frac{F_{np}^{mq}(\boldsymbol{k})}{\omega^2-\omega(m\boldsymbol{q}\,;n\boldsymbol{p})^2}a_n^*(\boldsymbol{p},\lambda)a_m(\boldsymbol{q},\lambda) \tag{18}$$

where $\hbar\omega(m\boldsymbol{q}\,;n\boldsymbol{p})=E_n(\boldsymbol{p})-E_m(\boldsymbol{q})$. Ω is a constant to be determined in such a way that the interaction effects are eliminated from the individual particle component of the density fluctuation. ω is the frequency of the collective density fluctuation, which shall be determined through the dispersion relation (50).

To show that the above defined q_k are the normal co-ordinates of the collective oscillation, let us introduce the quantities $\xi_{k,\omega}$ through the relation

$$q_k=(\xi_{k,\omega}-\xi_{k,-\omega})/2\omega \tag{19}$$

that is to say, $\xi_{k,\omega}$ is defined by

$$\xi_{k,\omega}=\Omega\sum_{np}\sum_{\substack{mq\\ \lambda}}\frac{F_{np}^{mq}(\boldsymbol{k})}{\omega-\omega(m\boldsymbol{q}\,;n\boldsymbol{p})}a_n^*(\boldsymbol{p},\lambda)a_m(\boldsymbol{q},\lambda). \tag{20}$$

Now, to prove the q_k are orthogonal with each other, we must show that it satisfies the canonical commutation relation,

$$[q_k,\ q_l]=0. \tag{21}$$

For this, it is sufficient to show that the following relations hold for the $\xi_{k,\omega}$ defined above,

$$[\xi_{k,\omega},\ \xi_{l,\omega}]=0 \tag{22}$$

$$[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},-\omega}]=-f(\boldsymbol{k},\ \omega)\,\delta(\boldsymbol{k}+\boldsymbol{l}) \tag{23}$$

where $f(\boldsymbol{k},\ \omega)$ is a factor to be determined through the following calculation.

Let us consider the first commutator. Using the commutators (5a) and (5b), it is reduced to

$$\begin{aligned}[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},\omega}]=\Omega^2\sum_{n\boldsymbol{p}}\underset{\lambda}{}\sum_{m\boldsymbol{q}}\frac{F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})}{\omega-\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})}\sum_{m',\boldsymbol{q}'}\\ \times\left\{\frac{F^{m'\boldsymbol{q}'}_{m\boldsymbol{q}}(\boldsymbol{l})}{\omega-\omega(m',\ \boldsymbol{q}'\ ;\ m,\ \boldsymbol{q})}a_n^*(\boldsymbol{p},\ \lambda)\,a_{m'}(\boldsymbol{q}',\ \mu)\right.\\ \left.-\frac{F^{n\boldsymbol{p}}_{m'\boldsymbol{q}'}(\boldsymbol{l})}{\omega-\omega(n\boldsymbol{p}\ ;\ m',\ \boldsymbol{q}')}a^*_{m'}(\boldsymbol{q}',\ \mu)\,a_m(\boldsymbol{q},\ \lambda)\right\}.\end{aligned} \tag{24}$$

Since we are interested in the collective motion of the electrons, which does not accompany one-particle real transitions of electrons between the different states $(n,\ \boldsymbol{p},\ \lambda)$ and $(m',\ \boldsymbol{q}',\ \mu)$, we retain only the diagonal terms of eq. (24). Clearly, when the excitation energy of the collective motion $\hbar\omega$ is nearly equal to the single electron excitation energy $E_m(\boldsymbol{q})-E_n(\boldsymbol{p})$, this procedure loses its validity, because for such case the non-diagonal term with $a_n^*(\boldsymbol{p},\ \lambda)\,a_m(\boldsymbol{q},\ \lambda)$ becomes predominant. In other words, for such case the collective motion of the electrons is no longer a stable state. Hence, under the condition described above, we get

$$[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},\omega}]=\Omega^2\sum_{n\boldsymbol{p}}\underset{\lambda}{}\sum_{m\boldsymbol{q}}\frac{|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2}{\omega^2-\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2}(N_n(\boldsymbol{p},\ \lambda)-N_m(\boldsymbol{q},\ \lambda))\,\delta(\boldsymbol{k}+\boldsymbol{l}). \tag{25}$$

In the derivation of eq. (25), we have used the momentum conservation expressed by eq. (17). In the second term of eq. (25), let us change the dummy indicies m, $\boldsymbol{q}$ to n, $-\boldsymbol{p}'$, and n, $\boldsymbol{p}$ to m, $-\boldsymbol{q}'$, respectively. Then it becomes

$$\begin{aligned}[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},\omega}]=\Omega^2\delta(\boldsymbol{k}+\boldsymbol{l})\left\{\sum_{n\boldsymbol{p}}\underset{\lambda}{}\sum_{m\boldsymbol{q}}\frac{|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2}{\omega^2-\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2}N_n(\boldsymbol{p},\ \lambda)\right.\\ \left.-\sum_{n,-\boldsymbol{p}'}\underset{\lambda}{}\sum_{m,-\boldsymbol{q}'}\frac{|F^{m,-\boldsymbol{q}'}_{n,-\boldsymbol{p}'}(\boldsymbol{k})|^2}{\omega^2-\omega(m,\ -\boldsymbol{q}'\ ;\ n,\ -\boldsymbol{p}')^2}N_n(-\boldsymbol{p}',\ \lambda)\right\}.\end{aligned} \tag{26}$$

For the case of no-external field, since the distribution of the electrons in the direction of $\boldsymbol{p}$ is symmetrical, we get required relation,

$$[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},\omega}]=0. \tag{22}$$

For the second commutator, we get similarly

$$\begin{aligned}[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},-\omega}]=-\Omega^2\delta(\boldsymbol{k}+\boldsymbol{l})\sum_{n\boldsymbol{p}}\underset{\lambda}{}\sum_{m\boldsymbol{q}}|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2\\ \times\left\{\frac{1}{[\omega-\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})]^2}-\frac{1}{[\omega+\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})]^2}\right\}N_n(\boldsymbol{p},\ \lambda).\end{aligned} \tag{27}$$

Introducing the following abbreviation,

$$f(\boldsymbol{k},\ \omega)=\omega 4\Omega^2\sum_{\substack{np\\ \lambda}}\sum_{mq}\frac{|F^{mq}_{np}(\boldsymbol{k})|^2}{[\omega^2-\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2]^2}\,\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})N_n(\boldsymbol{p},\ \lambda) \tag{28}$$

we can express it as follows,

$$[\xi_{\boldsymbol{k},\omega},\ \xi_{\boldsymbol{l},-\omega}]=-f(\boldsymbol{k},\ \omega)\,\delta(\boldsymbol{k}+\boldsymbol{l})\,. \tag{23}$$

Before going into further mathematical discussions, here we should summarize the physical conditions used to derive the commutators (22) and (23) :

Condition (I), we are only considering the collective motion of the many-electron system, which does not induce one-particle real transitions of electrons between the different single electron states.

Condition (II), the momentum of the electrons is a good quantum number.

As discussed above, if ω happens to be very close to $\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})$ the above derivation of the commutators loses its validity, and thus q_k loses the candidacy to be a normal co-ordinate of the system. That is to say, the state in which the collective motion is excited cannot be an eigen-state of the many-electron system, unless ω is greatly different from $\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})$.

Next, we must show the co-ordinates q_k are orthogonal to co-ordinates representing the motion of individual particles. Let us separate the density $\rho(\boldsymbol{k})$ into two components as

$$\rho(\boldsymbol{k})=\eta_k+q_k \tag{29}$$

where η_k is defined by

$$\eta_k=\sum_{\substack{np\\ \lambda}}\sum_{mq}F^{mq}_{np}(\boldsymbol{k})\left\{1-\frac{\Omega}{\omega^2-\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2}\right\}a_n{}^*(\boldsymbol{p},\ \lambda)\,a_m(\boldsymbol{q},\ \lambda), \tag{30}$$

and it will be shown later by the proper choice of Ω that η_k is purely of individual nature and it does not contain correlation effects due to the interaction between electrons. For the above defined η_k, it is straight-forward to show

$$[\eta_k,\ q_l]=0 \tag{31}$$

under the same condition which are used to establish the commutators (22) and (23).

Finally, it is quite easy to find the canonical conjugate momentum p_k to the q_k. If we define the operator A_k, $A_k{}^*$ by the relations

$$\xi_{k,+\omega}=i\{f(\boldsymbol{k},\ \omega)\}^{1/2}A_k \tag{32}$$

$$\xi_{k,-\omega}=i\{f(\boldsymbol{k},\ \omega)\}^{1/2}A_{-k}^* \tag{33}$$

the commutators of the A_k, $A_k{}^*$ are derived from eqs. (22) and (23) as follows,

$$[A_k,\ A_l{}^*]=\delta_{k,l} \tag{34}$$

$$[A_k,\ A_l]=[A_k{}^*,\ A_l{}^*]=0\,. \tag{35}$$

In terms of A_k and A_k^*, q_k are rewritten as

$$q_k = (i/2)\{f(\boldsymbol{k},\ \omega)/\omega^2\}^{1/2}(A_{+k} - A_{-k}^*) \tag{36}$$

or introducing quantity $I(\boldsymbol{k},\ \omega)$ by the relation

$$f(\boldsymbol{k},\ \omega) = 2\omega\hbar/I(\boldsymbol{k},\ \omega). \tag{37}$$

q_k can be expressed by the normal form

$$q_k = i\{\hbar/2\omega I\}^{1/2}(A_{+k} - A_{-k}^*). \tag{38}$$

Then, the canonical conjugate momentum p_k are found to be

$$p_k = \{\hbar\omega I/2\}^{1/2}(A_{+k} + A_{-k}^*) \tag{39}$$

from the requirement that p_k and q_k satisfy the canonical commutator

$$[q_k,\ p_l] = i\hbar\,\delta_{k,l}. \tag{40}$$

p_k can be expressed in terms of the particle operators as follows, corresponding to the expression eq. (18) of q_k,

$$p_k = -i2I(\boldsymbol{k},\ \omega)\,\Omega\sum_{np}\sum_{\substack{mq\\ \lambda}} F_{np}^{mq}(\boldsymbol{k})\,\frac{\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})}{\omega^2 - \omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2}\,a_n^*(\boldsymbol{p},\ \lambda)\,a_m(\boldsymbol{q},\ \lambda). \tag{41}$$

Similarly, $I(\boldsymbol{k},\ \omega)$ is expressed as

$$\frac{1}{I(\boldsymbol{k},\ \omega)} = \frac{2}{\hbar}\sum_{np}\sum_{\substack{mq\\ \lambda}}\frac{\Omega^2}{\{\omega^2 - \omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2\}^2}\,|F_{np}^{mq}(\boldsymbol{k})|^2\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})\,N_n(\boldsymbol{p},\ \lambda). \tag{42}$$

Now, we shall show that the q_k defined by eq. (18) executes a harmonic oscillations of the frequency ω, under the above discussed condition (I) and (II). Going back to eq. (19), we can see that if $\xi_{k,\omega}$ satisfies the equation of motion

$$\dot{\xi}_{k,\omega} + i\omega\xi_{k,\omega} = 0 \tag{43}$$

then, q_k obeys the equation of motion

$$\ddot{q}_k + \omega^2 q_k = 0. \tag{44}$$

The variation of $\xi_{k,\omega}$ in time can be calculated according to the relation

$$i\hbar\dot{\xi}_{k,\omega} = [\xi_{k,\omega},\ H]. \tag{45}$$

Using the Hamiltonian H, we find that

$$\begin{aligned}\dot{\xi}_{k,\omega} + i\omega\xi_{k,\omega} = {} & i\rho(\boldsymbol{k}) - \frac{i}{\hbar}\,\frac{1}{2V}\sum_{l}V(\boldsymbol{l})\sum_{np}\sum_{\substack{mq\\ \lambda}}\frac{F_{np}^{mq}(\boldsymbol{k})}{\omega - \omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})}\\ & \times\sum_{tr}\{[F_{mq}^{tr}(-\boldsymbol{l})\,a_n^*(\boldsymbol{p},\ \lambda)\,a_t(\boldsymbol{r},\ \lambda) - F_{tr}^{np}(-\boldsymbol{l})\,a_t^*(\boldsymbol{r},\ \lambda)\,a_m(\boldsymbol{q},\ \lambda)]\rho(+\boldsymbol{l})\\ & + \rho(-\boldsymbol{l})[F_{mq}^{tr}(+\boldsymbol{l})\,a_n^*(\boldsymbol{p},\ \lambda)\,a_t(\boldsymbol{r},\ \lambda) - F_{tr}^{np}(+\boldsymbol{l})\,a_t^*(\boldsymbol{r},\ \lambda)\,a_m(\boldsymbol{q},\ \lambda)]\}.\end{aligned} \tag{46}$$

However, we have shown $\xi_{\boldsymbol{k},\omega}$ is independent of all other $\xi_{\boldsymbol{l},\omega}$ except $\xi_{-\boldsymbol{k},-\omega}$. Therefore, in the second term of the right-hand side of eq. (46), it is consistent with the commutators (22) and (23) to omit all the terms except the term with $\boldsymbol{k}=\boldsymbol{l}$. Thus, we get

$$\dot{\xi}_{\boldsymbol{k},\omega}+i\omega\xi_{\boldsymbol{k},\omega}=i\rho(\boldsymbol{k})\left\{1-\frac{V(\boldsymbol{k})}{\hbar V}\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}\frac{|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2}{\omega-\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})}[N_n(\boldsymbol{p},\,\lambda)-N_m(\boldsymbol{q},\,\lambda)]\right\}. \tag{47}$$

Thus, we can see that $\xi_{\boldsymbol{k},\omega}$, hence $q_{\boldsymbol{k}}$, oscillates harmonically provided that ω satisfies the relation

$$1=\frac{V(\boldsymbol{k})}{\hbar V}\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}\frac{|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2}{\omega-\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})}[N_n(\boldsymbol{p},\,\lambda)-N_m(\boldsymbol{q},\,\lambda)] \tag{48}$$

$$=2\frac{V(\boldsymbol{k})}{\hbar V}\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}\frac{|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2}{\omega^2-\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})^2}\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})N_n(\boldsymbol{p},\,\lambda). \tag{49}$$

Rewriting this relation as

$$\omega^2=2\frac{V(\boldsymbol{k})}{\hbar V}\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})N_n(\boldsymbol{p},\,\lambda)$$

$$+2\frac{V(\boldsymbol{k})}{\hbar V}\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}\frac{|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2}{\omega^2-\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})^2}\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})N_n(\boldsymbol{p},\,\lambda), \tag{50}$$

we can see that the "plasma" frequency $\omega_p(\boldsymbol{k})^2$ of the electrons in a periodical potential is given as

$$\omega_p(\boldsymbol{k})^2=2\frac{V(\boldsymbol{k})}{\hbar V}\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}|F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})|^2\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})N_n(\boldsymbol{p},\,\lambda). \tag{51}$$

In fact, for the free electron system, the first term of eq. (50) is reduced to $4\pi e^2n/m$, while the second term is reduced to the dispersion term of the Bohm-Pines theory. We shall discuss in details the dispersion relation (50) and the generalized "plasma" frequency in the following section.

So far we are concerned with the collective component of the density fluctuation. In the following, we shall discuss the individual particle component $\eta_{\boldsymbol{k}}$ of the density fluctuation. Using the expression of $\eta_{\boldsymbol{k}}$ (30), the time variation of $\eta_{\boldsymbol{k}}$ is calculated as follows,

$$\ddot{\eta}_{\boldsymbol{k}}=-\sum_{\substack{n\boldsymbol{p}\\ \lambda}}\sum_{m\boldsymbol{q}}F^{m\boldsymbol{q}}_{n\boldsymbol{p}}(\boldsymbol{k})\,a_n{}^*(\boldsymbol{p},\,\lambda)\,a_m(\boldsymbol{q},\,\lambda)\left\{\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})^2\right.$$

$$\left.-\Omega\frac{\omega^2}{\omega^2-\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})^2}+\omega_p(\boldsymbol{k})^2\right\}. \tag{52}$$

Eq. (52) is valid under the condition (I). The time variation of $\eta_{\boldsymbol{k}}$ contains two components; the component $\eta_k^{(1)}$ is such one that is due to the random motion of electrons and that exists even in the absence of the interaction between electrons. We shall show that by choosing $\Omega=\omega_p(\boldsymbol{k})^2$ the latter component can be eliminated from the η_k. The time variation of $\eta_k^{(1)}$, in the absence of the interaction, can be calculated by using the Hamiltonian H_0 as follows,

$$\ddot{\eta}=-\sum_{np}\sum_{mq}{}_{\lambda} F_{np}^{mq}(\boldsymbol{k})\left\{1-\frac{\Omega}{\omega^2-\omega(m,\boldsymbol{q};n,\boldsymbol{p})^2}\right\}\omega(m,\boldsymbol{q};n,\boldsymbol{p})^2 a_n^*(\boldsymbol{p},\lambda)\,a_m(\boldsymbol{q},\lambda). \tag{53}$$

Now, eq. (52) can be rearranged in the following form, by using eq. (53),

$$\ddot{\eta}_k=\ddot{\eta}_k^{(1)}+(\Omega-\omega_p(\boldsymbol{k})^2)\rho_k. \tag{54}$$

Hence, by making the choice $\Omega=\omega_p(\boldsymbol{k})^2$, the individual particle component becomes purely of individual nature and does not contain the correlation effects of the interaction between electrons.

§ 3. Separation of the Hamiltonian into collective and individual particle part

In the preceding section, we have shown that there exists a set of normal co-ordinates (p_k, q_k) and that the co-ordinates q_k oscillate harmonically with the frequency $\omega(\boldsymbol{k})$. Therefore, we can introduce the Hamiltonian of the collective field as follows,

$$H_c=\frac{1}{2}\sum_k\left\{\frac{1}{I(\boldsymbol{k},\omega)}p_k^*p_k+I(\boldsymbol{k},\omega)\,\omega^2 q_k^*q_k\right\} \tag{55}$$

where $I(\boldsymbol{k},\omega)$ is defined by eq. (42) with $\Omega=\omega_p(\boldsymbol{k})^2$. $I(\boldsymbol{k},\omega)$ represents the inertia associated with the collective field. For free electrons, in the limit of $\omega^2\gg\omega(m,\boldsymbol{q};n,\boldsymbol{p})^2$, it reduces to

$$I(\boldsymbol{k},\omega)=m/(Nk^2) \tag{56}$$

where N is the total number of electrons in the volume V. Such a quantity, that has a dimension of a moment of inertia, has appeared also in the Tomonaga theory of the collective motions of particles.[11] The present theory can be regarded as a generalization of the Tomonaga theory in terms of the second quantization scheme.

Now, the original Hamiltonian can be expressed by the equivalent form,

$$H=H^*+H_c \tag{57}$$

where H^* is given by

$$H^*=H-H_c. \tag{58}$$

By substituting the representation of q_k, eq. (18), and that of p_k, eq. (39), H^* can be written down as follows,

$$
\begin{aligned}
H^* = {} & \sum_{np\lambda} E_n(\boldsymbol{p})\, N_n(\boldsymbol{p}, \lambda) + \frac{1}{2V} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) \sum_{\substack{np\\ \lambda}} \sum_{mq} \sum_{\substack{n'p'\\ \lambda'}} \sum_{m'q'} F^{mq}_{np}(-\boldsymbol{k})\, F^{m'q'}_{n'p'}(+\boldsymbol{k}) \\
& \cdot \left(1 - \Delta^{mq}_{np}(-\boldsymbol{k})\right) \left(1 - \Delta^{m'q'}_{n'p'}(+\boldsymbol{k})\right) a_n^*(\boldsymbol{p}, \lambda)\, a_m(\boldsymbol{q}, \lambda)\, a_{n'}^*(\boldsymbol{p}'\ \lambda')\, a_{m'}(\boldsymbol{q}', \lambda') \\
& + \frac{1}{2V} \sum_{\boldsymbol{k}} \{V(\boldsymbol{k}) - I(\boldsymbol{k}, \omega)\, \omega^2 V\} \sum_{\substack{np\\ \lambda}} \sum_{mq} \sum_{\substack{n'p'\\ \lambda'}} \sum_{m'q'} F^{mq}_{np}(-\boldsymbol{k})\, F^{m'q'}_{n'p'}(+\boldsymbol{k}) \\
& \cdot \Delta^{mq}_{np}(-\boldsymbol{k})\, \Delta^{m'q'}_{n'p'}(+\boldsymbol{k})\, a_n^*(\boldsymbol{p}, \lambda)\, a_m(\boldsymbol{q}, \lambda)\, a_{n'}^*(\boldsymbol{p}'\ \lambda')\, a_{m'}(\boldsymbol{q}', \lambda') \\
& + 2 \sum_{\boldsymbol{k}} I(\boldsymbol{k}, \omega) \sum_{\substack{np\\ \lambda}} \sum_{mq} \sum_{\substack{n'p'\\ \lambda'}} \sum_{m'q'} F^{mq}_{np}(-\boldsymbol{k})\, F^{m'q'}_{n'p'}(+\boldsymbol{k})\, \Delta^{mq}_{np}(-\boldsymbol{k})\, \Delta^{m'q'}_{n'p'}(+\boldsymbol{k}) \\
& \cdot \omega(m, \boldsymbol{q}\,;\, n, \boldsymbol{p})\, \omega(m', \boldsymbol{q}'\,;\, n', \boldsymbol{p}')\, a_n^*(\boldsymbol{p}, \lambda)\, a_m(\boldsymbol{q}, \lambda)\, a_{n'}^*(\boldsymbol{p}', \lambda')\, a_{m'}(\boldsymbol{q}', \lambda') \\
& + H_{\text{int}} \qquad (59)
\end{aligned}
$$

where $\Delta^{mq}_{np}(\boldsymbol{k})$ is defined by

$$\Delta^{mq}_{np}(\boldsymbol{k}) = \omega_p(\boldsymbol{k})^2 / [\omega^2 - \omega(m, \boldsymbol{q}\,;\, n, \boldsymbol{p})^2] \qquad (60)$$

and H_{int} is the interaction term of the collective field and electrons,

$$H_{\text{int}} = \frac{1}{2V} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) \sum_{\substack{np\\ \lambda}} \sum_{mq} \{ (1 - \Delta^{mq}_{np}(-\boldsymbol{k}))\, F^{mq}_{np}(-\boldsymbol{k})\, q_{\boldsymbol{k}}\, a_n^*(\boldsymbol{p}, \lambda)\, a_m(\boldsymbol{q}, \lambda) + [\boldsymbol{k} \rightarrow -\boldsymbol{k}] \}. \qquad (61)$$

Thus, using the relation

$$a_n^*\, a_m\, a_{n'}^*\, a_{m'} = N_n N_{m'} \delta_{nm} \delta_{n'm'} + (1 - N_{n'}) N_n \delta_{nm'} \delta_{n'm} + N.\,D.\,P. \qquad (62)$$

where $N.\,D.\,P.$ refers to the term with all different indices, the Hamiltonian H is transformed into the following form, which is composed of three parts, the modified Hamiltonian of the electron field, the Hamiltonian of the collective field and the interaction Hamiltonian of the collective field and the individual motion of electrons;

$$H = H_e + H_c + H_I \qquad (63)$$

where H_e, H_c and H_I are given as follows,

$$
\begin{aligned}
H_e = {} & \sum_{np\lambda} [E_n(\boldsymbol{p}) + 2 \sum_{\boldsymbol{k}} I(\boldsymbol{k}, \omega) \sum_{m, q} | F^{mq}_{np}(\boldsymbol{k})\, \Delta^{mq}_{np}(\boldsymbol{k}) |^2\, \omega(m, \boldsymbol{q}\,;\, n, \boldsymbol{p})^2 \\
& - \frac{1}{2V} \sum_{\boldsymbol{k}} \{V(\boldsymbol{k}) - I(\boldsymbol{k}, \omega)\, \omega^2 V\} \sum_{mq} | F^{mq}_{np}(\boldsymbol{k})\, \Delta^{mq}_{np}(\boldsymbol{k}) |^2 N_m(\boldsymbol{q}, \lambda) \\
& - \frac{1}{2V} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) \sum_{mq} | F^{mq}_{np}(\boldsymbol{k})\, (1 - \Delta^{mq}_{np}) |^2 N_m(\boldsymbol{q}, \lambda) \\
& - 2 \sum_{\boldsymbol{k}} I(\boldsymbol{k}, \omega) \sum_{mq} | F^{mq}_{np}(\boldsymbol{k})\, \Delta^{mq}_{np}(\boldsymbol{k}) |^2\, \omega(m, \boldsymbol{q}\,;\, n, \boldsymbol{p})^2 N_m(\boldsymbol{q}, \lambda) \\
& + \Delta E_n(\boldsymbol{p})]\, N_n(\boldsymbol{p}, \lambda) + N.\,D.\,P. \qquad (64)
\end{aligned}
$$

$$H_c = \sum_{\boldsymbol{k}} \hbar \omega \left(N_{\boldsymbol{k}} + \frac{1}{2} \right) \qquad (65)$$

$$H_I=\frac{i}{2V}\sum_{k}\sqrt{\frac{\hbar}{2I(\boldsymbol{k},\,\omega)\,\omega}}\,V(\boldsymbol{k})\sum_{\substack{n,\,p\\ m,\,q\\ \lambda}}\{F_{np}^{mq}(-\boldsymbol{k})\,(1-\varDelta_{np}^{mq})\,(A_{-k}-A_{k}^{*})+h.\,c.\}. \tag{66}$$

In eq. (64), $\varDelta E_n(\boldsymbol{p})$ is the abbreviation given as

$$\varDelta E_n(\boldsymbol{p})=\frac{1}{2V}\sum_{k}\{V(\boldsymbol{k})-I(\boldsymbol{k},\,\omega)\,\omega^2V\}\sum_{m,q}|F_{np}^{mq}(\boldsymbol{k})\,\varDelta_{np}^{mq}(\boldsymbol{k})\,|^2$$

$$+\frac{1}{2V}\sum_{k}V(\boldsymbol{k})\sum_{m,q}|F_{np}^{mq}(\boldsymbol{k})\,(1-\varDelta_{np}^{mq})\,|^2-\frac{1}{2V}\sum_{k}V(\boldsymbol{k}). \tag{67}$$

$\varDelta E_n(\boldsymbol{p})$ is the self energy due to the modified Coulomb interaction in the presence of the collective field, and it vanishes for the limit of $\omega\to 0$, i. e. in the absence of the collective field. The second term of the heavy bracket of H_e is the self energy of electron due to the modification of electron propagation by the collective field, the third and fourth terms are the potential energy of electrons in the Hartree field of the electron interaction modified by the collective field, the fifth and sixth terms are the exchange energy of the modified interaction. The effects of the third and fifth terms may be neglected, because for the limit of $\omega^2\gg\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})^2$, we have

$$I(\boldsymbol{k},\,\omega)^{-1}V(\boldsymbol{k})=\omega_p(\boldsymbol{k})^2V \tag{68}$$

and

$$\omega^2=\omega_p(\boldsymbol{k})^2 \tag{69}$$

thus they vanish for this limit. The *N. D. P.* in the H_e stands for the terms with all different suffices n, $\boldsymbol{p}$, for the operators $a_n(\boldsymbol{p},\,\lambda)$, and it gives rise to higher order correlation effects.

Since we have not introduced any auxiliary variables, we are not encountered with any subsidiary condition. In the Bohm-Pines theory the validity of the subsidiary condition is very doubtful for the state in which the collective oscillation is actually excited. The present formulation of the theory presents a complete scheme which is free from such defect.

§ 4. Frequency of collective motion

In Section 2, we have obtained the collective frequency ω in the form of dispersion relation (50). There the state $(n,\,\boldsymbol{p})$ is refered to a valence state as well as to a core electron state. If, however, the state $(n,\,\boldsymbol{p})$ stands for the core state, the excitation energy of electron from the state $(n,\,\boldsymbol{p})$ to the upper band state $(m,\,\boldsymbol{q})$ may be much larger than $\hbar\omega$. Then, by expanding the second term of eq. (50) into power series of $\omega/\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})$, it can be concluded that the virtual transitions of electrons from the core state to the band state do not participate directly to the collective oscillation, but it may do indirectly through higher order effects of $\omega/\omega(m,\,\boldsymbol{q}\,;\,n,\,\boldsymbol{p})$. Hence, in the following discussions we shall divide the states $(n,\,\boldsymbol{p})$ and $(m,\,\boldsymbol{q})$ into two groups ;

$$(n,\ \boldsymbol{p}\ ;\ m,\ \boldsymbol{q})\ ;\ \omega>|\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})| \tag{70}$$

$$(n',\ \boldsymbol{p}'\ ;\ m',\ \boldsymbol{q}')\ ;\ \omega<|\omega(m',\ \boldsymbol{q}'\ ;\ n',\ \boldsymbol{p}')| \tag{71}$$

where the state $(n,\ \boldsymbol{p})$ stands for the valence band, the $(m,\ \boldsymbol{q})$ stands for the lower excited band, while the $(n',\ \boldsymbol{p}')$ stands for the core states as well as the valence band, correspondingly the $(m',\ \boldsymbol{q}')$ stands for all of the excited band, or the higher excited band which cannot be grouped in (70) when the $(n',\ \boldsymbol{p}')$ stands for the valence band.

Now, by expanding eq. (50) in powers of $\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})/\omega$ and $\omega/\omega(m',\ \boldsymbol{q}'\ ;\ n',\ \boldsymbol{p}')$ up to second order terms, we get the equation

$$\omega^2=\omega_p(\boldsymbol{k})^2+2\frac{V(\boldsymbol{k})}{\hbar V}\left\{\sum_{np}\underset{\lambda}{}\sum_{mq}|F_{np}^{mq}(\boldsymbol{k})|^2\frac{\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^3}{\omega^2}N_n(\boldsymbol{p},\ \lambda)\right.$$

$$\left.-\sum_{n'p'}\underset{\lambda'}{}\sum_{m'q'}|F_{n'p'}^{m'q'}(\boldsymbol{k})|^2\frac{\omega^2}{\omega(m',\ \boldsymbol{q}'\ ;\ n',\ \boldsymbol{p}')}N_{n'}(\boldsymbol{p}',\ \lambda')\right\} \tag{72}$$

which gives one positive solution for ω^2, i. e. a real solution for ω,

$$\omega^2=\frac{{\omega_p}^2}{\varepsilon_c}+2\frac{V(\boldsymbol{k})}{\hbar V}\sum_{np}\underset{\lambda}{}\sum_{mq}\frac{\omega(m,\ \boldsymbol{q}\ ;\ n,\ \boldsymbol{p})^2}{{\omega_p}^2}N_n(\boldsymbol{p},\ \lambda) \tag{73}$$

where

$$\varepsilon_c=1+2\frac{V(\boldsymbol{k})}{\hbar V}\sum_{n'p'}\underset{\lambda'}{}\sum_{m'q'}\frac{|F_{n'p'}^{m'q'}(\boldsymbol{k})|^2}{\omega(m',\ \boldsymbol{q}'\ ;\ n',\ \boldsymbol{p}')}N_n(\boldsymbol{p}',\ \lambda'). \tag{74}$$

The other solution is thrown away, since it gives imaginary values for ω. The result derived above corresponds to Pines' result, eqs. (44) and (45) in reference 7).

Let us examine the proper collective frequency in more detail. The summation over the states $(m,\ \boldsymbol{q})$ in eq. (51) can be divided into two parts due to intra-band transitions and to inter-band transitions, respectively. The $\omega_p(\boldsymbol{k})^2$ can be written as

$$\omega_p(\boldsymbol{k})^2=\omega_s(\boldsymbol{k})^2+\omega_d(\boldsymbol{k})^2 \tag{75}$$

where

$$\omega_s(\boldsymbol{k})^2=2\frac{V(\boldsymbol{k})}{\hbar^2 V}\sum_{n}\underset{\lambda}{}\sum_{pq}|F_{np}^{nq}(\boldsymbol{k})|^2(E_n(\boldsymbol{q})-E_n(\boldsymbol{p}))N_n(\boldsymbol{p},\ \lambda) \tag{76}$$

and

$$\omega_d(\boldsymbol{k})^2=2\frac{V(\boldsymbol{k})}{\hbar^2 V}\sum_{np}\underset{\lambda}{}\sum_{mq}|F_{np}^{mq}(\boldsymbol{k})|^2(E_m(\boldsymbol{q})-E_n(\boldsymbol{p}))N_n(\boldsymbol{p},\ \lambda) \tag{77}$$

$\omega_s(\boldsymbol{k})^2$ is due to the intra-band transitions, $\omega_d(\boldsymbol{k})^2$ is due to inter-band transitions. The former vanishes for the completely filled-up band, because of the symmetry of $\boldsymbol{q}$ and $\boldsymbol{p}$. On the other hand, the latter does contribute to the collective frequency, even if the state $(n,\ \boldsymbol{p})$ is completely filled up.

Now, we shall reduce the above derived general formulae, eqs. (76) and (77), to a simpler form which can be used to calculate explicitly the collective frequency for specified examples. In the following discussions, we shall consider the case of limit $\boldsymbol{k}\to 0$ for the sake of simplicity. Firstly, let us consider the $\omega_d(\boldsymbol{k})^2$. By rewriting the $\omega_d(\boldsymbol{k})^2$ as follows,

$$\omega_d(\boldsymbol{k})^2=2\frac{4\pi e^2}{\hbar^2 k^2}\frac{1}{V}\sum_{\substack{np\\ \lambda}}\sum_{mq}\frac{|F_{np}^{mq}(\boldsymbol{k})|^2}{E_m(\boldsymbol{q})-E_n(\boldsymbol{p})}(E_m(\boldsymbol{q})-E_n(\boldsymbol{p}))^2 N_n(\boldsymbol{p},\lambda) \tag{78}$$

we can fully use the relation

$$\{E_m(\boldsymbol{q})-E_n(\boldsymbol{p})\}F_{np}^{mq}(\boldsymbol{k})=-i\frac{\hbar^2}{2m}k\int d^3x e^{-ikx}[\phi_n^{p*}(\boldsymbol{x})\vec{e}\vec{\nabla}\phi_m^q(\boldsymbol{x})-\vec{e}\vec{\nabla}\phi_n^{p*}(\boldsymbol{x})\cdot\phi_m^q(\boldsymbol{x})] \tag{79}$$

which is derived from eqs. (6) and (13), and $\vec{e}$ is the direction vector of $\boldsymbol{k}$. Using the general formula of $\phi_n^p(\boldsymbol{x})$, eq. (7), we get for the limit of $\boldsymbol{k}\to 0$,

$$\begin{aligned}\int d^3x[\phi_n^{p*}(\boldsymbol{x})\vec{e}\vec{\nabla}\phi_m^q(\boldsymbol{x})-\vec{e}\vec{\nabla}\phi_n^{p*}(\boldsymbol{x})\phi_m^q(\boldsymbol{x})]\\ =\frac{i}{N}(\boldsymbol{q}-\boldsymbol{p})\int_v d^3xe^{-i(q-p)x}u_n^{p*}(\boldsymbol{x})u_m^q(\boldsymbol{x})+\frac{1}{N}\int_v d^3xe^{-i(p-q)x}\\ \cdot[u_n^{p*}(\boldsymbol{x})\vec{e}\vec{\nabla}u_m^q(\boldsymbol{x})-\vec{e}\vec{\nabla}u_n^{p*}(\boldsymbol{x})\cdot u_m^q((\boldsymbol{x})].\end{aligned} \tag{80}$$

Since u_n^p, u_m^q have the periodicity of the unit cell, this can be written as

$$=i\delta_{p,q}(\boldsymbol{p}-\boldsymbol{q})\int_v d^3xu_n^{p*}(\boldsymbol{x})u_m^q(\boldsymbol{x})+\delta_{p,q}\int_v d^3x[u_n^{p*}(\boldsymbol{x})\vec{e}\vec{\nabla}u_m^q(\boldsymbol{x})-(\vec{e}\vec{\nabla}u_n^{p*}(\boldsymbol{x}))\cdot u_m^q(\boldsymbol{x})]. \tag{81}$$

The first term vanishes identically, and thus we obtain

$$\{E_m(\boldsymbol{q})-E_n(\boldsymbol{p})\}F_{np}^{mq}(\boldsymbol{k})=\frac{\hbar}{m}k\int_v d^3xu_n^{p*}(\boldsymbol{x})\vec{e}\vec{p}u_m^q(\boldsymbol{x})\delta_{p,q}. \tag{82}$$

Therefore, we get the following expression for the $\omega_d(\boldsymbol{k})^2$,

$$\omega_d^2=\frac{4\pi e^2}{m}\frac{1}{V}\sum_{\substack{np\\ \lambda}}\frac{2}{m}\sum_m\frac{|\int_v d^3xu_n^{p*}(\boldsymbol{x})\vec{e}\vec{p}u_m^p(\boldsymbol{x})|^2}{E_m(\boldsymbol{p})-E_n(\boldsymbol{p})}N_n(\boldsymbol{p},\lambda). \tag{83}$$

Finally, using the f-sum rule for Bloch's one electron wave function[15)], this is simplified as

$$\omega_d^2=\frac{4\pi e^2}{m}\frac{1}{V}\sum_{\substack{np\\ \lambda}}\left\{1-\frac{m}{\hbar^2}(\vec{e}\vec{\nabla}_p)^2E_n(\boldsymbol{p})\right\}N_n(\boldsymbol{p},\lambda). \tag{84}$$

Since in the course of the calculation of the f-sum rule one uses the completeness relation, the summation over $(n, \boldsymbol{p})$ extends to the states, some of which violate the condition (70). The error introduced by this, however, is expected to be unappreciable, since $F_{np}^{mq}(\boldsymbol{k})$ is very small for such states that contradict the condition (70). For the case of no-external field, the direction of the momentum transfer is isotropic, and we may take an average of eq. (84) over the direction $\boldsymbol{k}$. Thus, the ω_d^2 is given as

$$\omega_d{}^2 = \frac{4\pi e^2}{m}\left\{n - \frac{m}{3\hbar^2}\,\frac{1}{V}\sum_{\substack{np\\ \lambda}} \nabla_p{}^2 E_n(\boldsymbol{p})\, N_n(\boldsymbol{p}, \lambda)\right\} \tag{85}$$

where n is the density of electrons in the valence state.

Next, we shall consider the $\omega_s(\boldsymbol{k})^2$. As already mentioned, the $\omega_s(\boldsymbol{k})^2$ contributes only for the unfilled valence band $(n, \boldsymbol{p})$. For the unfilled band we can calculate it as follows; for the limit of $\boldsymbol{k} \to 0$, we have

$$F^{nq}_{np}(\boldsymbol{k}) = \delta_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{q}} \int d^3x\, e^{-ikx}\, |\varphi_n(\boldsymbol{x})|^2 = 1 + O(\boldsymbol{k}^2) \tag{86}$$

and

$$E_n(\boldsymbol{q}) = E_n(\boldsymbol{p}) + \boldsymbol{k}\cdot\vec{\nabla}_p E_n(\boldsymbol{p}) + \frac{1}{2}k^2 (\vec{e}\vec{\nabla}_p)^2 E_n(\boldsymbol{p}). \tag{87}$$

In general the current of valence electrons is odd function of $\boldsymbol{p}$, therefore the second term of eq. (87) does not contribute to the $\omega_s(\boldsymbol{k})^2$. Thus, we get

$$\omega_s{}^2 = \frac{4\pi e^2}{m}\,\frac{1}{V}\sum_{\substack{np\\ \lambda}} \frac{m}{\hbar^2} (\vec{e}\vec{\nabla}_p)^2 E_n(\boldsymbol{p})\, N_n(\boldsymbol{p}, \lambda) \tag{88}$$

or taking an average over the direction of $\boldsymbol{k}$, we get

$$\omega_s{}^2 = \frac{4\pi e^2}{m}\,\frac{m}{3\hbar^2}\,\frac{1}{V}\sum_{\substack{np\\ \lambda}} \nabla_p{}^2 E_n(\boldsymbol{p})\, N_n(\boldsymbol{p}, \lambda). \tag{89}$$

Lastly, we shall discuss the dispersion term

$$2\,\frac{V(\boldsymbol{k})}{\hbar^4 \omega_p{}^2}\,\frac{1}{V}\sum_{\substack{np\\ \lambda}} J_p^n(\boldsymbol{k})\, N_n(\boldsymbol{p}, \lambda) \tag{90}$$

of eq. (73), where $J_p^n(\boldsymbol{k})$ is defined by

$$J_p^n(\boldsymbol{k}) = \sum_{mq} |F^{mq}_{np}(k)|^2 (E_m(\boldsymbol{q}) - E_n(\boldsymbol{p}))^3. \tag{91}$$

Using eq. (79), this can be written as

$$\begin{aligned} J_p^n(\boldsymbol{k}) = \left(\frac{\hbar^2}{2m}\right)^2 k^2 \cdot \frac{1}{2}\sum_{mq}\Bigg[\int d^3x\, e^{-ikx}\,[\phi_n^{p*}(\boldsymbol{x})\,\vec{e}\vec{\nabla}\phi_m^q(\boldsymbol{x}) - \vec{e}\vec{\nabla}\phi_n^{p*}(\boldsymbol{x})\cdot\phi_m^q(\boldsymbol{x})] \\ \cdot\left\{\int d^3y\, e^{+iky}\,[\phi_n^p(\boldsymbol{y})\,(\vec{e}\vec{\nabla})\phi_m^{q*}(\boldsymbol{y}) - \vec{e}\vec{\nabla}\phi_n^p(\boldsymbol{y})\cdot\phi_m^{q*}(\boldsymbol{y})](E_m(\boldsymbol{q}) - E_n(\boldsymbol{p})) + \text{c. c.}\right]. \end{aligned} \tag{92}$$

After lengthy but straight-forward calculation, we obtain

$$\begin{aligned} J_p^n(\boldsymbol{k}) = \left(\frac{\hbar^2}{2m}\right)^2 k^2 \cdot 2\Bigg[\int_v d^3x\, u_n^{p*}(\boldsymbol{x})\cdot(\vec{e}\vec{\nabla})^2 U(\boldsymbol{x})\cdot u_n^p(\boldsymbol{x}) \\ -3\,\frac{\hbar^2}{m}k^2 \int_v d^3x\, u_n^{p*}(\boldsymbol{x})\,(\vec{e}\vec{\nabla})^2\, u_n^p(\boldsymbol{x}) + \frac{\hbar^2}{4m}k^4\Bigg]. \end{aligned} \tag{93}$$

Thus, after taking an average of eq. (93) over the direction of $\boldsymbol{k}$, we get

$$(90)=\frac{4\pi e^2}{m^2\omega_p{}^2}\frac{1}{V}\sum_{\substack{np\\ \lambda}}\left[\frac{1}{3}\int_v d^3x\,u_n^{p*}(\boldsymbol{x})\nabla^2U(x)\,u_n^p(\boldsymbol{x})+\frac{1}{m}\int_v d^3x\,u_n^{p*}(\boldsymbol{x})\,p^2\,u_n^p(\boldsymbol{x})\,k^2\right.$$

$$\left.+\frac{\hbar^2}{4m}k^4\right]N_n(\boldsymbol{p},\lambda). \tag{94}$$

In the derivation of eq. (93), as well as in the derivation of eq. (84), we have extended the summation over $(m, \boldsymbol{q})$ to the states that violate the condition (70). For such states that are designated as $(m', \boldsymbol{q}')$ in eq. (71), $F_{np}^{m'q'}(\boldsymbol{k})$ is expected to be very small and $\omega(m', \boldsymbol{q}'; n', \boldsymbol{p}')$ is very large. Therefore, it is consistent with eqs. (84) and (94) to take $\mathcal{E}_c=1$.

Thus, we may summarize the results obtained here as follows:

(I) If the valence band is partly occupied, combining eqs. (85) and (89), we have

$$\omega_p{}^2=\omega_s{}^2+\omega_d{}^2=\frac{4\pi e^2}{m}n. \tag{95}$$

Though this is derived under the limit $\boldsymbol{k}\to 0$, one can show easily that this is the exact result which is valid for $\boldsymbol{k}\neq 0$.

(II) If the valence band is completely filled up, we have

$$\omega_p{}^2=\omega_d{}^2=\frac{4\pi e^2}{m}\left\{n-\frac{m}{3\hbar^2}\frac{1}{V}\sum_{p\lambda}\nabla_p{}^2E_n(\boldsymbol{p})\cdot N_n(\boldsymbol{p},\lambda)\right\} \tag{96}$$

which is exact only for the case of $\boldsymbol{k}\to 0$. However, the higher order terms depending on $\boldsymbol{k}$ contain higher derivatives of $E_n(\boldsymbol{p})$ and higher multipole moments of the valence electron distribution. The correction due to these higher order term can be neglected in comparison with the $\boldsymbol{k}$ dependence terms of the dispersion term, since the involved momentum transfer $\boldsymbol{k}$ in the collective excitation of electrons is far smaller than the momentum transfer for which the higher multipole moments become relevant and in the effective mass approximation the higher derivatives of $E_n(\boldsymbol{p})$ do not come in the problem. Therefore, we shall take eq. (96) as a general expression for the "plasma" frequency of the filled up valence band electrons.

(III) Thus, the frequency of collective oscillation $\omega(\boldsymbol{k})^2$ is given by the following relation,

$$\omega(\boldsymbol{k})^2=\omega_p{}^2+\frac{4}{3}\frac{\pi e^2}{m^2\omega_p{}^2}\frac{1}{V}\sum_{np\lambda}\int_v d^3x\,u_n^{p*}(\boldsymbol{x})\cdot\nabla^2U(\boldsymbol{x})\cdot u_n^p(\boldsymbol{x})\,N_n(\boldsymbol{p},\lambda)$$

$$+\frac{8\pi e^2}{m^2\omega_p{}^2}\frac{1}{V}\sum_{np\lambda}\int_v d^3x\,u_n^{p*}(\boldsymbol{x})\frac{P^2}{2m}u_n^p(\boldsymbol{x})\,N_n(\boldsymbol{p},\lambda)\,k^2+\frac{4\pi e^2}{m\omega_p{}^2}n\cdot\frac{\hbar^2}{4m^2}k^4 \tag{97}$$

where $\omega_p{}^2$ is given by eq. (95), or by eq. (96), for the concerned case, respectively.

The application of the above derived formulae to the analysis of the energy loss of electron will be discussed in the separate paper.[12] We shall be particularly interested in the case of the material with the filled up valence band.

In conclusion, the author would like to express his deep gratitude to Professor S. Hayakawa for many illuminating discussions on this problem. He also would like to thank the members of the Laboratory of Nuclear Studies for their hospitality.

Appendix

Here we shall discuss Kanazawa's work in comparison with the present theory. We shall show that eq. (A·11) of reference 6) is equivalent to the dispersion relation (48) of Section 2. We have

$$1=\frac{V(\boldsymbol{k})}{V}\sum_{np}\sum_{\substack{mq\\ \lambda}}\frac{|F_{np}^{mq}(\boldsymbol{k})|^2}{\hbar\omega-E_m(\boldsymbol{q})+E_n(\boldsymbol{p})}(N_n(\boldsymbol{p},\lambda)-N_m(\boldsymbol{q},\lambda)). \tag{48}$$

Subtract the quantity,

$$\frac{V(\boldsymbol{k})}{V}\sum_{np}\sum_{\substack{mq\\ \lambda}}\frac{|F_{np}^{mq}(\boldsymbol{k})|^2}{\hbar\omega}(N_n(\boldsymbol{p},\lambda)-N_m(\boldsymbol{q},\lambda)) \tag{A·1}$$

which is identically equal to zero, because of the relation

$$F_{np}^{mq}(\boldsymbol{k})=F_{m,-q}^{n,-p}(\boldsymbol{k}),\quad \sum_{p}\sum_{q}\cdots N_n(\boldsymbol{p},\lambda)=\sum_{-p}\sum_{-q}\cdots N_n(-\boldsymbol{p},\lambda)$$

from the right-hand side of eq. (48). Then, we obtain

$$1=\frac{V(\boldsymbol{k})}{\hbar\omega}\frac{1}{V}\sum_{np}\sum_{\substack{mq\\ \lambda}}|F_{np}^{mq}(\boldsymbol{k})|^2\frac{E_m(\boldsymbol{q})-E_n(\boldsymbol{p})}{\hbar\omega-E_m(\boldsymbol{q})+E_n(\boldsymbol{p})}(N_n(\boldsymbol{p},\lambda)-N_m(\boldsymbol{q},\lambda)) \tag{A·2}$$

which is eq. (A·11) of reference 6).

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π^-–P Interaction in High Energy Region

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It is shown that the elastic and total cross sections for π^-–p collision above 1.0 Bev can almost be explained as the shadow effect of the inelastic collision. Based on this result, we discuss whether or not the second maximum of the total cross section for this reaction can be interpreted in terms of inelastic scattering.

§ 1. Introduction and summary

The existence of the seond maximum for π^-–p scattering cross section at 0.9 Bev has been confirmed by recent experiments at Brookhaven. It is, however, not quite clear whether the maximum is due to the resonance with a certain state of a nucleon, or due to other mechanisms.

Recently, Yang[1)] has pointed out that the angular momentum J of the resonating state should be considerably large, if one tries to interpret the maximum as due to the resonance with a single state. Takeda[2)] and Dyson[3)] independently have tried to give an explanation of this second maximum by assuming the strong pion-pion interaction.

On the other hand, we have to pay our attention to the multiple pion production in the high energy π^-–p scattering. The cross section for elastic scattering is nearly equal to that for inelastic scattering at about 0.8 Bev, so the former ought to be affected by the latter. We have shown before that π^-–p scattering data at 1.4 Bev[7)] can fairly well be explained as due to the shadow effect of the multiple production. Recently, by using the dispersion relations, Sternheimer[4)] has also shown that the absorptive part of the forward scattering amplitude is predominantly larger than the dispersive part. These situations suggest us that the large shadow effects of the inelastic collision might play an important role in appearance of the second maximum.

In this paper, first of all, we intend to show the relation between the constants contained in our empirical formula[5)] which has previously been adopted in the analysis of π^-–p interaction at 1.4 Bev, and parameters of the optical model for this reaction, that is, nucleon radius and transparency. In § 3, we show that the elastic and total cross sections above 1.0 Bev can almost be explained by the shadow effect of the inelastic collision. Therefore, in such a high energy region, our theoretical interest may be concentrated to examining the inelastic collision. On the other hand, the phenomena in the energy region lower than 1.0 Bev can hardly be explained by such a simple consideration. However, it will be shown that the existence of, at least, one maximum value for σ_{tot} in the energy range between 1.0 Bev and the threshold energy for $\pi + N \rightarrow 2\pi + N$ reaction can readily be predicted by the theory which describes correctly the inelastic cross sections above 1.0 Bev.

§ 2. Angular distribution of elastic scattering

Measurements of angular distributions for elastic $\pi^- - p$ scattering at 1.0 Bev[6] as well as 1.5 Bev[7] have shown the large scattering at very small angles. This may be interpreted in terms of the diffraction scatttering. Walker[8] has obtained a histogram (Fig. 1) of $d\sigma_{\text{el}}/d\Omega$ as a function of $k \sin\theta$, where θ and k are the scattering angle and the momentum of pion in the center-of-mass system, respectively.

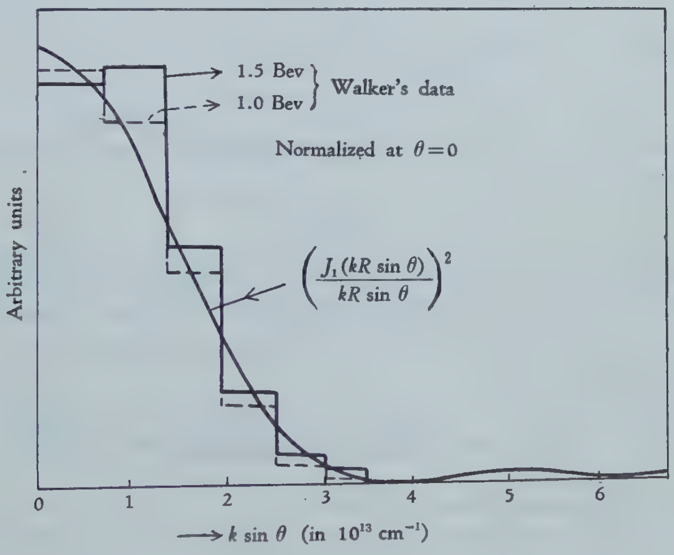

Fig. 1

As is well known, the angular distribution of diffraction scattering by a sphere of radius R is expressed as follows:

$$d\sigma_{\text{el}}/d\Omega \sim |J_1(kR\sin\theta)/(kR\sin\theta)|^2. \qquad (2\cdot1)$$

The solid line in Fig. 1, which was calculated by Walker for

$$R = 1 \times 10^{-13}\ \text{cm}, \qquad (2\cdot2)$$

agrees fairly well with the experimental data at 1.0 Bev and 1.5 Bev (see Fig. 1).

On the other hand, we have previously introduced the following empirical formula for the cross section of the elastic scattering

$$d\sigma_{\mathrm{el}}/d\Omega=(1/4)[b/(a-\cos\theta)]^2, \tag{2·3}$$

which explained the experimental data at 1.4 Bev when $a=1.13$. We now want to point out that the expressions (2·1) and (2·3) are essentially the same with regard to the important points, although they seem to be entirely different expressions each other at first sight. In the region of the small angles where $d\sigma_{\mathrm{el}}/d\Omega$ has a large value,

$$\frac{J_1(kR\sin\theta)}{kR\sin\theta}\approx\left(1-\frac{(kR)^2}{8}\theta^2+\cdots\right)\approx\frac{1}{1+(kR)^2\theta^2/8}.$$

Considering quantities up to the order of θ^2, we may set as follows :

$$\theta^2\approx2(1-\cos\theta),$$

then

$$\frac{J_1(kR\sin\theta)}{kR\sin\theta}\sim\frac{1}{\{1+4/(kR)^2\}-\cos\theta}.$$

This expression for the scattering amplitude has the same form with ours

$$(b/2)/(a-\cos\theta).$$

Thus it is found that a and b in (2·3) are closely connected with the parameters of the optical model, that is,

$$a=1+(2/kR)^2. \tag{2·4}$$

Putting the value of $a=1.13$ at 1.4 Bev ($k=3.8\times10^{13}\,\mathrm{cm}^{-1}$) into Eq. (2.4), we obtain $R=1.44\times10^{-13}$ cm. It is worthwhile to notice that this value is nearly equal to the value mentioned in (2·2). For the sake of convenience to practial analysis, we shall at first employ the formula (2·3).

§ 3. Diffraction scattering

According to the general theory of scattering

$$\sigma_{\mathrm{el}}=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)|1-\eta_l e^{2i\delta_l}|^2, \tag{3·1}$$

$$\sigma_{\mathrm{inel}}=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)(1-\eta_l^2). \tag{3·2}$$

If we make a crude assumption that the elastic scattering is entirely due to the diffraction,*

$$\sigma_{\mathrm{el}}=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)(1-\eta_l)^2, \tag{3·3}$$

$$\sigma_{\mathrm{tot}}=\sigma_{\mathrm{el}}+\sigma_{\mathrm{inel}}=2\pi\lambda\!\!\!^{-2}\sum_l(2l+1)(1-\eta_l). \tag{3·4}$$

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* The word " diffraction " means here such an approximation that all the δ_l's can be pute qual to zero.

When the angular distribution for elastic scattering is given by (2·3), η_l can be calculated by

$$\lambdabar(1-\eta_l)=\frac{1}{2}\int_{-1}^{1}\frac{b}{a-x}P_l(x)\,dx, \tag{3·5}$$

as was shown in the previous paper.[5)]

Making use of the orthogonality of Legendre function, the cross sections can be expressed in terms of a and b as follows:

$$\sigma_{\mathrm{el}}=(\pi b^2)/(a^2-1), \tag{3·6}$$

$$\sigma_{\mathrm{tot}}=(2\pi\lambdabar b)/(a-1). \tag{3·7}$$

Although the parameters a and b can be determined by the experimental values σ_{el} and σ_{tot}, we think it more convenient to discuss R and γ which are connected with a and b by the following relations

$$a=1+(2/kR)^2, \tag{3·8}$$

$$b=2\lambdabar\gamma. \tag{3·9}$$

Then

$$\sigma_{\mathrm{tot}}=\gamma\pi R^2, \tag{3·10}$$

$$\sigma_{\mathrm{el}}=\gamma^2\pi R^2/\{2+(2/kR)^2\}, \tag{3·11}$$

$$\sigma_{\mathrm{inel}}=\gamma\pi R^2[1-\gamma/\{2+(2/kR)^2\}]. \tag{3·12}$$

From these, γ and R are expressed by σ_{el}, σ_{inel} and σ_{tot}.

$$2/\gamma=1+(\sigma_{\mathrm{inel}}/\sigma_{\mathrm{el}})-4\pi\lambdabar^2/\sigma_{\mathrm{tot}}, \tag{3·13}$$

$$\pi R^2=\sigma_{\mathrm{tot}}/\gamma. \tag{3·14}$$

Employing the experimental values for the cross sections which have been reported by Shutt[9)] at the Sixth Rochester Conference, we obtain values of γ and R (see Table 1).

Table 1

E_π (Bev)	σ_{tot} (mb)	$\sigma_{\mathrm{el}}/\sigma_{\mathrm{inel}}$	γ	πR^2 (mb)	$R(10^{-13}\mathrm{cm})$
0.7	43	1.6	1.78	24.6	0.89
1.0	46	1.0	1.18	39.4	1.14
1.5	30	0.5	0.74	40.8	1.18
4.5	20	0.25	0.5	40.0	1.16

It is remarkable that R is nearly constant in the energy range 1.0～4.5 Bev. And this fact will probably mean that our assumption is fairly correct. Moreover, we plot in Fig. 2 values of σ_{tot} and σ_{el} which are calculated under the assumption of $R=1.16\times10^{-13}$ cm by employing the experimental values of σ_{inel}.

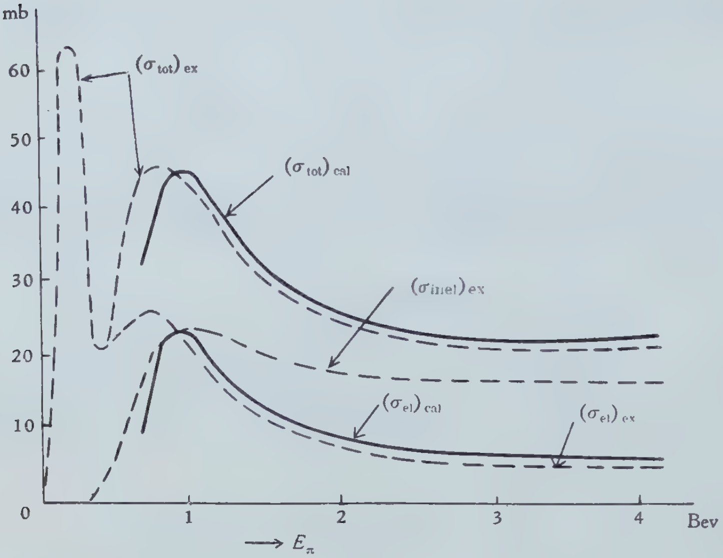

Fig. 2

Here we must pay our attention to the following important points.

(1) Calculated values of both σ_{tot} and of σ_{el} in the neighborhood of 1.0 Bev have a tendency to decrease with increasing energy.

(2) Moreover the calculated σ_{tot} at 1.0 Bev is nearly twice as large as that at 400 Mev.

From these facts we conculde that the existence of the second maximum for the cross section may indirectly be predicted from a theory if only the theory can correctly describe the inelastic collision at or above 1.0 Bev.

The above study is a simple sketch of our explanation for the second maximum of $\pi^- - p$ scattering and the effects of neither the charge independence nor the spin dependence of the interaction are taken into consideration. These effects will be examined in our next work.

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On the Structure of Extensive Air Showers*

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Assuming a simple model of extremely high energy collisions we investigate the relation between the one-dimensional development of extensive air showers (EAS) and the mechanism of multiple meson production. The gross structure of EAS based on the model is given graphically or numerically, in a form useful, as a first step, for further investigations. In particular, varying the value of inelasticity of high energy collisions we discuss its effect on various quantities. As a preliminary conclusion it is shown that it seems difficult to explain the experimental result of the constancy of the attenuation length of the electron component over a wide range of primary energies, $10^{14}\sim10^{17}$eV, without introducing any energy dependence of the inelasticity or a stronger energy dependence of multiplicity with increasing energy, though it is desirable that further experiments furnish us with more abundant and reliable data. Attention will also be paid to the problem of what quantities must be measured to determine more directly and less ambiguously the angular and energy distributions of secondary particles produced by high energy collisions.

§ 1. Introduction

The purpose of this paper is to analyse the structure of extensive air showers (EAS) with a view of investigating the mechanism of multiple meson production in extremely high energy collisions. The skeleton of EAS is the nucleonic cascade process in the atomosphere developed due to high energy collisions, through which particles of other components, namely electrons, photons, muons, etc., are produced secondarily. There are not very many analyses of EAS which treat this process directly, because the mathematical treatments are very difficult and the process of high energy collisions involves many unknown factors. The works which have been published may be roughly classified into two types: (a) Existing theories of multiple meson production are applied to analyse EAS and they are examined according to whether or not they can explain experimental results well. (b) A phenomenological model is assumed and parameters introduced in it are fitted by comparison with experimental data. The works of Amaldi et al.[1)] and Oda[2)] belong to the former and those of the Soviet group,[3 4.5)] Messel et al.[6,7)] and Budini and Molière[8)] to the latter.

However, their works have the following defects: Some of their calculations were made with a fixed value of the primary energy, in which case it is not clear how the calculated quantities vary with primary energy. Therefore one cannot draw definite conclusions by comparing theoretical values with experimental ones which are derived by averaging or integrating over the energy spectrum of primary particles. In particular the effect of the inelasticity of high energy collisions on the developement of EAS has not been sufficiently investigated.* In the treatment (a) the value of inelasticity is inherent in the theory applied, for example, it has the value one in Fermi's theory[9] in its original form. In the treatment (b) it must be regarded as a parameter. But their results are not conclusive for the above mentioned reason. On the other hand the intentions of some of these works seem to be to obtain the over-all aspect of EAS and not to investigate high energy collisions themselves.

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* The main contents are distributed in Japanese, Uchusen Kenkyu (mimeographed circular) 1 (1956), 163.

Considering these defects, we shall analyse the structure of EAS from the following standpoint which is similar to (b). In the mechanism of high energy collisions, the multiplicity and the inelasticity contribute mainly to the one-dimensional development of EAS, the angular distribution of secondary particles to the lateral distribution and the energy distribution to both. However to take into account all these factors at the same time makes the analysis intricate. In this paper we shall therefore confine ourselves to treat the development of EAS one-dimensionally and pay special attention to the effect of inelasticity on EAS. On the relation between multiplicity and energy of a colliding particle with an air nucleus we shall assume a phenomenological model, as comprehensive as possible considering all existing theories. Though parameters which will be introduced in our model must be determined by comparison with experimental results, it is yet too early to do it owing to insufficiency and lack of experimental information. Thus we shall be concerned with the problem of which parameters strongly affect which quantities measured. That is, which quantities must be measured to determine the values of the parameters.

Secondly we shall always take into account the following situation. Our model is physically based in the case that a particle having an energy larger than 10^{12}eV collides with an air nucleus, because in this case the collision process is well approximated by the so-called jet model,[15] and the character of an elementary process, a nucleon (or pion)-nucleon collision, seems to be well conserved in spite of many-body processes.[16,17] However, in the lower energy region, the process becomes complicated and the elementary process may be obscured due to the nucleonic cascade in a nucleus, so that our model, especially the multiylicity expressed by (1), may lose its physical meaning. Therefore we shall try to pick out processes in EAS which are initiated by a particle of energy larger than 10^{12}eV. In the previously mentioned works the authors have not focused their attention on this aspect of the problem. We shall also try to derive conclusions which do not depend strongly on the model introduced.

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* Recently Koba[10] has also pointed out that it is necessary to take into account the effect of inelasticity to discriminate between three models for multiple meson production, those of Landau,[11] Takagi[12] and Heisenberg,[13] which are compatible with the experimental information, concerning the composition and the distribution of transverse momenta[14] of secondary particles.

The assumptions of our model which are similar to those introduced by Rozental'[4] are as follows:

1) The collision cross section is independent of the energy of a colliding particle and also independent of the kind of colliding particle, i.e., nucleon or charged pion.

2) The number of secondary particles produced by a particle of energy* E in the Lab system is given by

$$n=a(\eta E)^{\gamma}. \tag{1}$$

Here a, η, γ are parameters. In particular η represents the inelasticity.** An energy of amount ηE is transferred to secondary particles and $(1-\eta)E$ carried away by the colliding particle.

3) The energy transferred to secondary particles, ηE, is equipartitioned among them.***

4) Two of the secondary particles are nucleons.**** Of the $(n-2)$ particles, $\varDelta(n-2)$ particles are charged pions and $(1-\varDelta)(n-2)$ particles are neutral pions. The fraction, $\varDelta$, is constant. The production of nucleon pairs and heavy mesons are disregarded.

5) Nuclear active particles of energy lower than the critical value, E_c, do not contribute to the further development of the nucleonic cascade process in the atmosphere.

The parameters introduced in the assumptions are a, η, γ, $\varDelta$, λ_c (collision mean free path) and E_c. Some of the parameters are roughly determined from theoretical and experimental analyses, i.e., $\gamma\simeq 1/4$, $\varDelta\cong 2/3$ and $\lambda_c\cong$ geometrical m.f.p. The critical energy, E_c, may be considered to be several to ten Mc^2. In our calculations we shall take

$$a=2, \quad \gamma=1/4, \quad \varDelta=0.7, \quad E_c=10Mc^2, \quad \lambda_c=90\ \mathrm{gcm}^{-2},$$

and investigate the η-dependence of all quantities. The change of the value of E_c by a factor of about $0.5\sim 2$ does not affect the behaviour of the electron component. This fact was confirmed by our actual calculation. However this change leads to the serious changes of the total numbers of the nucleonic and muon components. λ_c depends seriously on the nuclear radius, $R=r_0A^{1/3}$. Our value corresponds to $r_0=1.15\cdot 10^{-13}$cm which is obtained from the electron scattering experiment and is different from that used by the Soviet group, 70 gcm^{-2}, which corresponds to $r_0=1.3\cdot 10^{-13}$cm. However, as will be seen in the last section, this difference is not very essential in our arguments.

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* In this paper energy is measured by the nucleon rest energy, Mc^2.

** The relation between the multiplicity and the inelasticity is not uniquely defined even in nucleon-nucleon collisions.[10] Furthermore our definition is concerned with nucleon (or pion) -nucleus collisions, not with nucleon (or pion) -nucleon collisions. However the parameter, η, will fill the role of inelasticity and thus we shall call η "inelasticity" hereafter.

*** This assumption may be sufficient to evaluate the number of electrons because this number is mainly governed by the amount of energy supply from nuclear active particles to neutral pions at each altitude and is not much affected by the manner of energy supply, i.e., the energy distribution of secondary particles.

**** This value may be considered as the average number of nucleons in the "tunnel" which the colliding particle burrows in the air nucleus.

In the next section we shall describe the method of calculation of each component. For the electron component, calculations will be made independently of models of high energy collisions as long as the nucleonic cascade process is solved in terms of the generation method. The results for each component will be given numerically, and will show the gross structure of EAS. The reliability of our model is apparently different from component to component. These facts will be shown quantitatively in several tables. Using the results in § 3, we shall note the method of determining the inelasticity and in § 4 shall pick out an appropriate quantity to determine the value of the inelasticity. The comparison of such a quantity with experimental results now available will be made. The criticism of the work done by the Soviet group will be given. Further remarks on investigating angular and energy distributions of secondary particles will be given.

§ 2. The calculation method for each component

Calculations are made according to the generation method. This method makes it easy not only to see the contribution from each generation, i.e., the altitudes at which nuclear active particles are mainly produced, but also to pick out the high energy particles in each component. The notations x and t are used respectively to represent the atmospheric depth when measured by collision mean free path and radiation length.

2.1 *Nucleonic component*

The diffusion equations of nucleons and charged pions are given by

$$\begin{aligned}\frac{\partial N(E,x)}{\partial x}&=-N(E,x)+\int S_{NN}(E,E')N(E',x)\,dE'+\int S_{N\pi}(E,E')\pi(E',x)\,dE',\\ \frac{\partial \pi(E,x)}{\partial x}&=-\left(1+\frac{B}{Ex}\right)\pi(E,x)+\int S_{\pi N}(E,E')N(E',x)\,dE'\\ &\qquad+\int S_{\pi\pi}(E,E')\pi(E',x)\,dE'.\end{aligned}\tag{2}$$

Here $N(E,x)dE$ and $\pi(E,x)dE$ are the differential numbers of nucleons and charged pions having energy between E and $E+dE$, respectively, and B is the decay constant of charged pions. $S_{\pi N}(E,E')$ is the production spectrum of charged pions when a nucleon of energy E' collides, and so on.

From the assumptions 2), 3) the production spectrums are represented by

$$\begin{aligned}S_{NN}(E,E')&=\delta(E-(1-\eta)E')+S_N(E,E'),\\ S_{N\pi}(E,E')&=S_N(E,E'),\\ S_{\pi N}(E,E')&=\Delta S_\pi(E,E'),\\ S_{\pi\pi}(E,E')&=\delta(E-(1-\eta)E')+\Delta S_\pi(E,E'),\end{aligned}\tag{3}$$

where

$$S_N(E,E')=2\delta(E-\eta E'/n'),$$

$$S_\pi(E, E') = (n'-2)\,\delta(E-\eta E'/n'),$$
$$n' = a(\eta E')^{\Upsilon}. \tag{4}$$

Here $\delta(E-(1-\eta)E')$ represents a survival of a colliding particle and $S_N(E, E')$ and $S_\pi(E, E')$ are the production spectrums respectively of nucleons and pions (including neutral pions) by a particle of energy E'.

To solve the equations, (2), expanding $N(E, x)$ and $\pi(E, x)$ as

$$N(E, x) = \sum_{l=0}^{\infty} \frac{e^{-x}x^l}{l!} N_l(E),$$
$$\pi(E, x) = \sum_{l=0}^{\infty} \frac{e^{-x}x^l}{l!} \pi_l(E), \tag{5}$$

and inserting them into (2), we obtain the recursion formulae for each generation :

$$N_l(E) = \int \delta(E-(1-\eta)E')N_{l-1}(E')\,dE' + \int S_N(E, E')\,\{N_{l-1}(E') + \pi_{l-1}(E')\}\,dE',$$
$$\pi_l(E) = \frac{lE}{lE+B}\Big[\int \delta(E-(1-\eta)E')\pi_{l-1}(E')\,dE' \tag{6}$$
$$+\int \Delta S_\pi(E, E')\,\{N_{l-1}(E') + \pi_{l-1}(E')\}\,dE'\Big].$$

Assuming that a primary particle is a proton of energy E_0, the initial condition for (6) is

$$N_0(E) = \delta(E-E_0), \qquad \pi_0(E) = 0. \tag{7}$$

$N_l(E)$ and $\pi_l(E)$ can be obtained successively from (6) using (3), (4) and (7). Over the course of calculations all particles whose energies become smaller than $10\,Mc^2$ are cut off under assumption 5).

2.2 *Electron component*

The electron component is given by the superposition of cascade showers originated by two photons, the decay products of the neutral pions of each generation.

When a cascade shower is initiated by one neutral pion having energy E, the number of shower electrons of energies larger than $\mathcal{E}$, at depth t, is given by

$$\Pi(E, \mathcal{E}, t) = \int_0^E \frac{2dW}{E} \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} ds M(s) \left(\frac{W}{\mathcal{E}}\right)^s \exp[\lambda_1(s)t - \tfrac{1}{2}\log s]$$
$$= \frac{2}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} ds \exp[ys + \lambda_1(s)t - \tfrac{1}{2}\log s - \log(s+1) + \log M(s)], \tag{8}$$

where $y = \log(E/\mathcal{E})$ and the other notations are the same as defined by Rossi and Greisen.[18] From (5) and (6) the number of neutral pions having energy E which are produced by nuclear active particles of the $(l-1)$th generation is given by

$$\pi_l^{\,0}(E,\,x)\,dE=dE\int(1-\Delta)\,S_\pi(E,\,E')\,\{N_{l-1}(E')+\pi_{l-1}(E')\}\,dE'\cdot\frac{e^{-x}x^{l-1}}{(l-1)!}$$

$$\equiv\pi_l^{\,0}(E)\,dE\cdot\frac{e^{-x}x^{l-1}}{(l-1)!}\,. \tag{9}$$

Combining (8) with (9), the number of shower electrons of energies larger than $\mathcal{E}$ at the depth t, initiated by neutral pions which are produced by the nuclear active particles of the $(l-1)$th generation and which have the energy spectrum $\pi_l^{\,0}(E)\,dE$ is given by

$$\Pi_l(E,\,\mathcal{E},\,t)\,dE=dE\int_0^t\pi_l^{\,0}\Big(E,\,\frac{t'}{\beta}\Big)\frac{dt'}{\beta}\,\Pi(E,\,\mathcal{E},\,t-t')$$

$$=\pi_l^{\,0}(E)\,dE\cdot\frac{1}{\pi i\beta^l}\int_{\delta-i\infty}^{\delta+i\infty}ds\,\exp\big[ys+\lambda_1(s)\,t-\tfrac{1}{2}\log s-\log(s+1)+\log M(s)$$

$$+\log\{f_{l-1}(\xi t)/\xi^l\}\big]\,, \tag{10}$$

where β is the ratio of radiation length to collision mean free path and

$$f_l(\xi t)=\int_0^{\xi t}\frac{e^{-y}y^l}{l!}dy\,,\qquad \xi\equiv\xi(s)=\frac{1}{\beta}+\lambda_1(s)\,. \tag{11}$$

The integration of (10) is carried out by the saddle-point method, i.e.,

$$\Pi_l(E,\,\mathcal{E},\,t)\,dE=\pi_l^0(E)\,dE\Big(\frac{2}{\pi}\Big)^{1/2}\frac{M(\bar{s})}{(\bar{s}+1)[\bar{s}g_l''(\bar{s})]^{1/2}}\Big(\frac{E}{\mathcal{E}}\Big)^{\bar{s}}\exp[\lambda_1(\bar{s})\,t]$$

$$\cdot\,(\beta\bar{\xi})^{-l}f_{l-1}(\bar{\xi}t)\,, \tag{12}$$

where

$$g_l(s)=ys+\lambda_1(s)\,t-\tfrac{1}{2}\log s-\log(s+1)+\log M(s)+\log\{f_{l-1}(\xi t)/\xi^l\}\,,$$

and $\bar{s}$ is defined by $g_l'(\bar{s})=0$ and $\bar{\xi}=\xi(\bar{s})$. The total number of electrons is also derived by using the approximation B and is expressed as

$$\Pi_l(E,\,0,\,t)\,dE=\pi_l^{\,0}(E)\,dE\Big(\frac{2}{\pi}\Big)^{1/2}\frac{M(\bar{s})K_1(\bar{s},\,-\bar{s})}{(\bar{s}+1)[\bar{s}g_l''(\bar{s})]^{1/2}}\Big(\frac{E}{\mathcal{E}_c}\Big)^{\bar{s}}\exp[\lambda_1(\bar{s})\,t]$$

$$\cdot\,(\beta\bar{\xi})^{-l}f_{l-1}(\bar{\xi}t)\,, \tag{13}$$

where $\mathcal{E}_c$ is the critical energy in air and $K_1(\bar{s},\,-\bar{s})$ is also defined by Rossi and Greisen.[18]

It must be emphasized that in (12) and (13) the factor depending on the mechanism of high energy collisions is only $\pi_l^0(E)\,dE$ as long as we solve the diffusion equation (2) by the generation method. The calculation of the factors remaining in (12) and (13) was numerically carried out by putting into them all combinations of generations $l=1\sim15$ and the ratio $E/\mathcal{E}=1\sim10^6$, where $\mathcal{E}$ was replaced by $\mathcal{E}_c$ in the approximation B.

2.3 *Muon component*

Muons are taken into account only when their parental charged pions have energies larger than $10\,Mc^2$, in which case their decays and ionization losses are neglected. The number of muons at depth x which are the decay products of charged pions of the l-th generation is given by

$$\mu_l(E_\mu, x)\,dE_\mu = \pi_l(E)\,dE\int_0^x \frac{B}{Ey}\,\frac{e^{-y}y^l}{l!}\,dy \tag{14}$$

$$\equiv \pi_l(E)\,dE\frac{B}{lE}\,f_{l-1}(x)\,.$$

Table 1. Values of Constants.

Collision Mean Free Path, λ_c	90 gcm^{-2}
Decay Constant, B	139 Bev$=$148 Mc^2
Radiation Unit	38 gcm^{-2}
Critical Energy, ε_c	85 Mev$=9.0\cdot10^{-2}\,Mc^2$

The values of constants used in the numerical calculations are tabulated in Table 1. The numerically calculated results concerned with all components are shown graphically, or tabulated with three values of the inelasticity, η, (1, 0.5 and 0.3) and primary energies 10^4Mc^2 and 10^6Mc^2. In the graphs representing altitude dependences of all components, the letters S_0 and S_{10} mean the numbers of electrons having energies larger than zero and $10\,Mc^2$, respectively, and N and μ respectively are the numbers of nucleonic and muon components.

§ 3. Calculated results

In this section we shall tabulate the results calculated for each component. These results will be useful as an orientation for further investigation. A few results have irregularities of numerical values. This arose from the fact that nuclear active particles having energies less than $10\,Mc^2$ are cut off artificially.

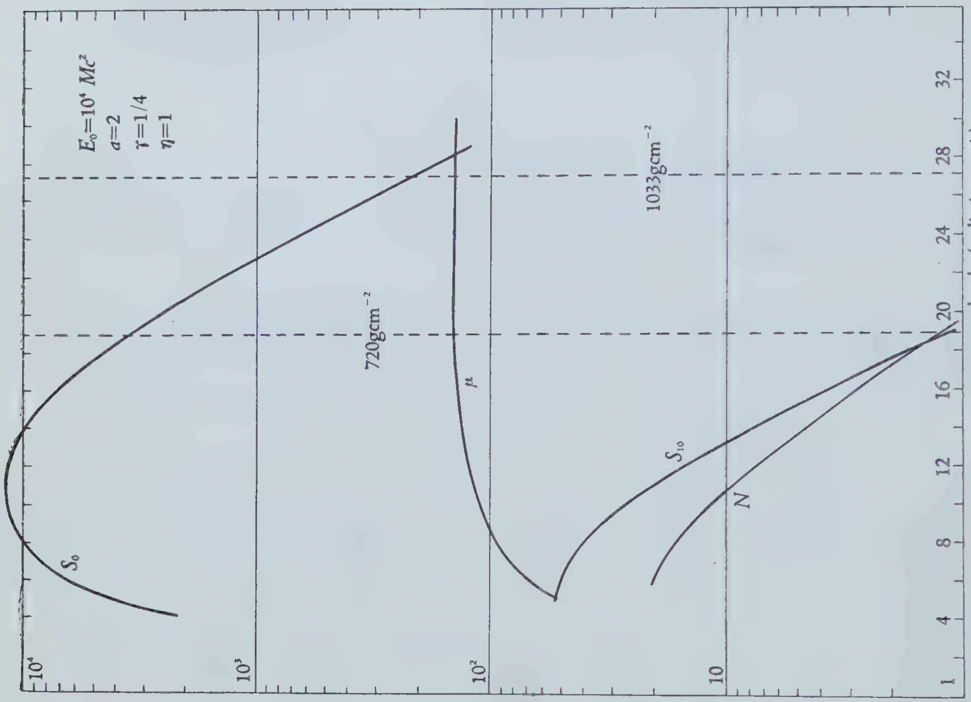
3.1 *Nucleonic component*

The results for the nucleonic component are shown in Tables 2 and 3. S in Table 2 is defined by assuming that the relation between the primary energy and the number of nuclear active particles has a form

$$\text{number} \propto E_0^s. \tag{15}$$

Table 2. The numbers of nucleonic components at mountain altitude (720 gcm^{-2}) and sea level (1033 gcm^{-2}). S is defined by the relation (15). Attenuation lengths are obtained from the numbers at both altitudes. Values in the brackets are obtained from the derivatives of altitude-dependent curves at sea level.

Depth	720 gcm^{-2}		S	1033 gcm^{-2}		S	Attenuation Length	
E_0 / η	10^4	10^6		10^4	10^6		10^4	10^6
1	1.2	$6.5\cdot10^2$	1.4	$9.0\cdot10^{-2}$	$7.9\cdot10$	1.5	120	150
0.5	8.4	$1.4\cdot10^3$	1.1	1.7	$4.5\cdot10^2$	1.2	190	270
0.3	$2.0\cdot10$	$2.3\cdot10^3$	1.0	$1.4\cdot10$	$1.8\cdot10^3$	1.1	510 (360)	1380 (540)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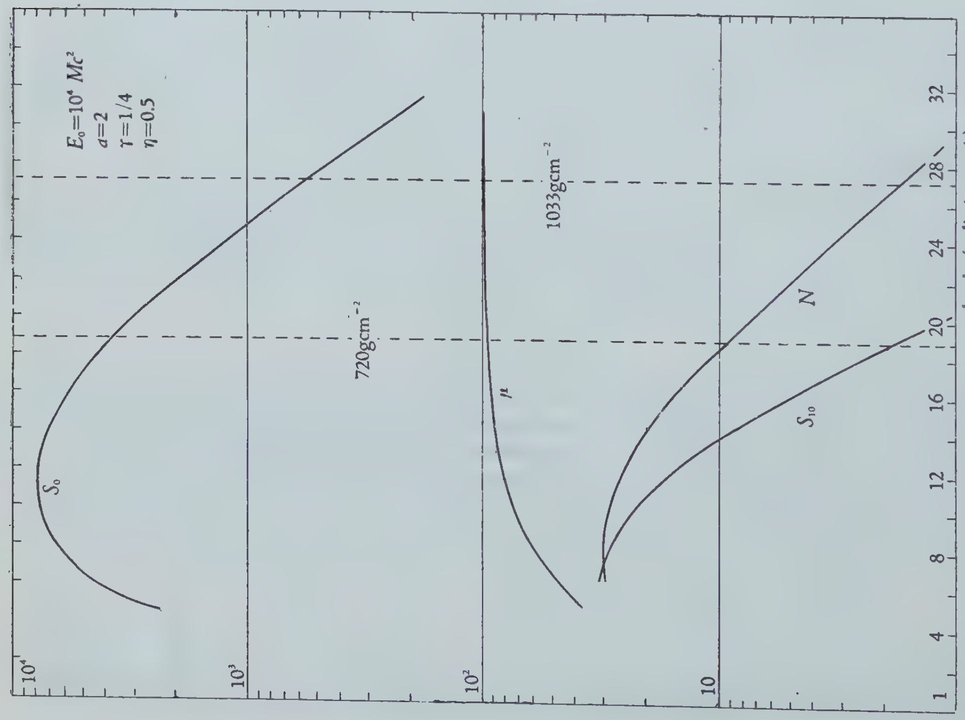
$a=2$
$\gamma=1/4$
$\eta=1$
S_0
μ
S_{10}
N
720gcm^{-2}
1033gcm^{-2}
10^4
10^3
10^2
10
1
4
8
12
16
20
24
28
32
depth (radiation unit)

Fig. 1.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$a=2$
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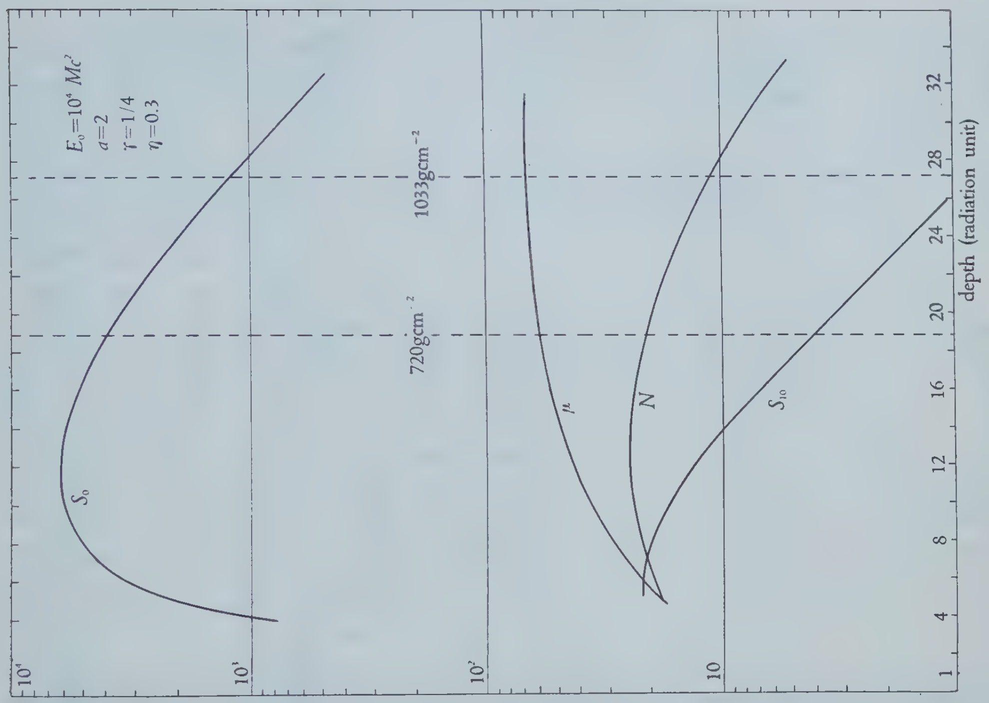
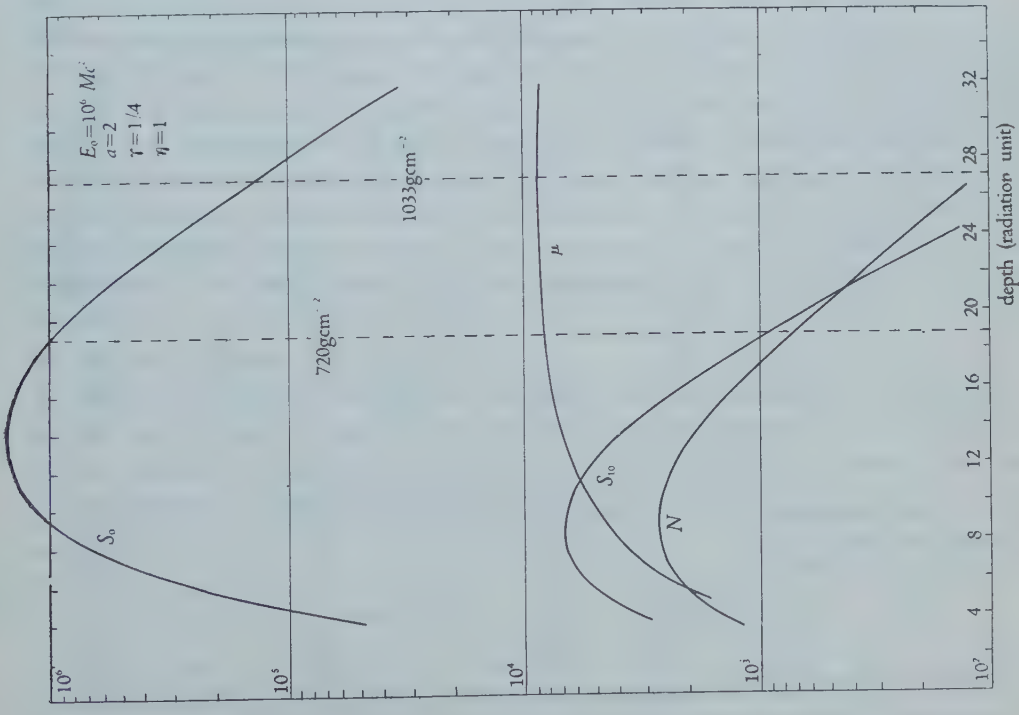
S_0
μ
S_{10}
N
720gcm^{-2}
1033gcm^{-2}
10^4
10^3
10^2
10
1
4
8
12
16
20
24
28
32
depth (radiation unit)

Fig. 2.



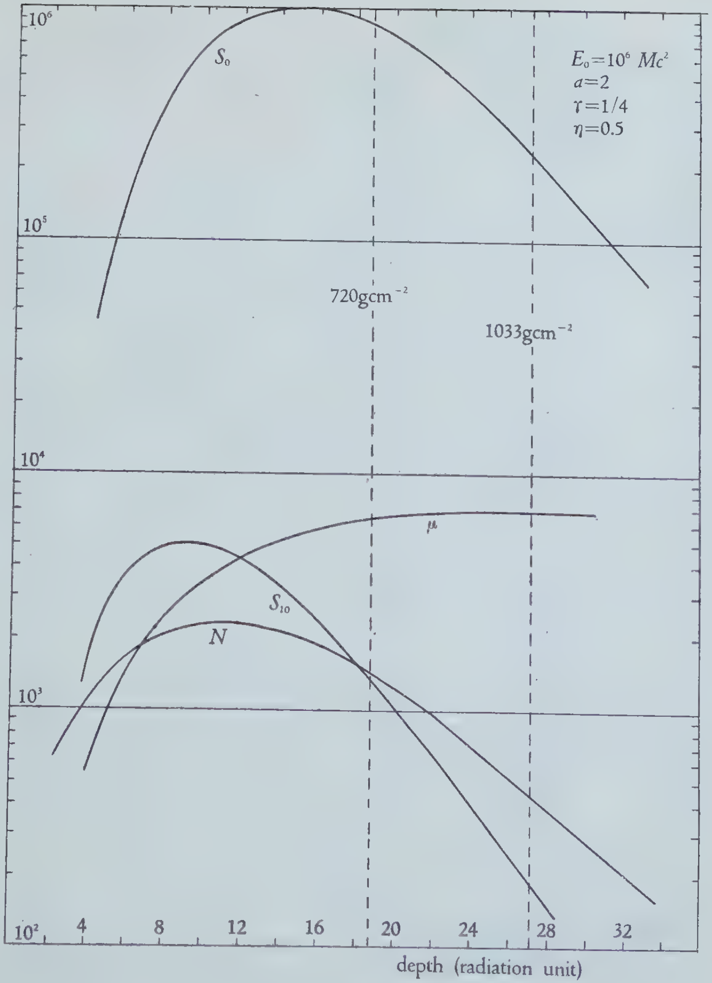
$E_0=10^4\ Mc^2$
$a=2$
$\gamma=1/4$
$\eta=0.3$
10^4
10^3
10^2
10
S_0
μ
N
S_{10}
720gcm^{-2}
1033gcm^{-2}
1
4
8
12
16
20
24
28
32
depth (radiation unit)

Fig. 3.



$E_0=10^6\ Mc^2$
$a=2$
$\gamma=1/4$
$\eta=1$
10^6
10^5
10^4
10^3
10^2
S_0
S_{10}
N
μ
720gcm^{-2}
1033gcm^{-2}
4
8
12
16
20
24
28
32
depth (radiation unit)

Fig. 4.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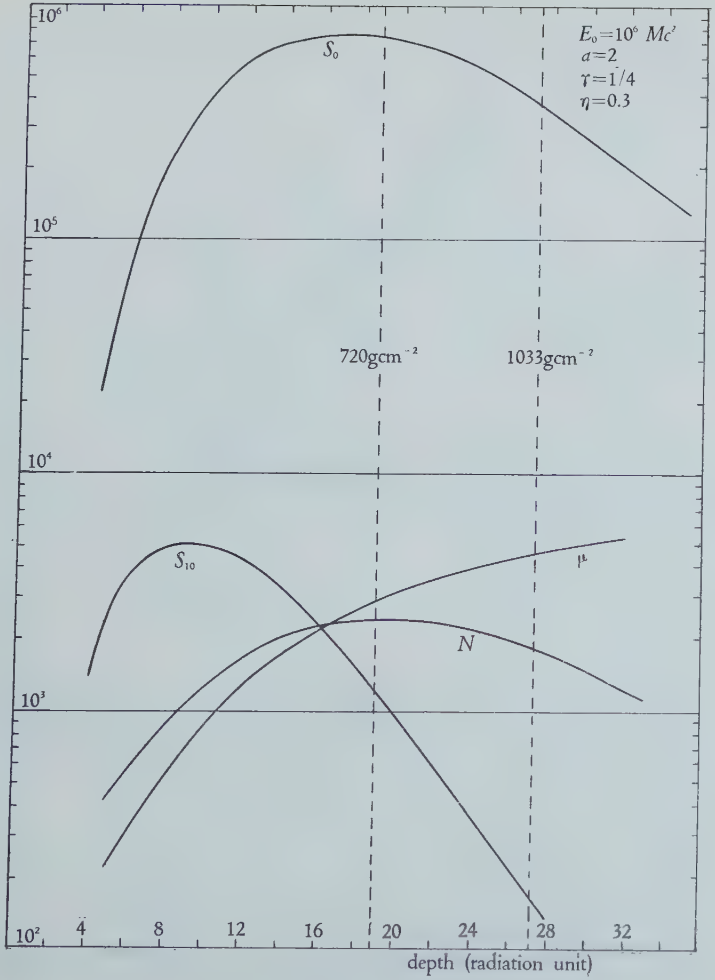
$a=2$
$\gamma=1/4$
$\eta=0.5$
S_0
S_{10}
N
μ
720gcm^{-2}
1033gcm^{-2}
10^6
10^5
10^4
10^3
10^2
4
8
12
16
20
24
28
32
depth (radiation unit)

Fig. 5.


$E_0=10^6\ Mc^2$
$a=2$
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S_0
S_{10}
N
μ
720gcm^{-2}
1033gcm^{-2}
10^6
10^5
10^4
10^3
10^2
4
8
12
16
20
24
28
32
depth (radiation unit)

Fig. 6.

Table 3. The fractions of the numbers of the nucleonic component of each generation at both altitudes.

Depth	720 gcm^{-2}						1033 gcm^{-2}					
E_0	10^4			10^6			10^4			10^6		
η / l	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3
1	7	1					4					
2	32	5	2	1			26	2		1		
3	61	26	3	16	4	1	70	11	1	11	1	
4		13	6	83	20	3		8	2	88	8	
5		19	11		20	7		18	4		11	2
6		22	15		17	11		29	7		14	4
7		11	17		20	16		21	12		24	8
8		2	18		14	18		4	18		24	12
9		1	14		4	16		6	19		11	16
10			9		1	14			19		4	20
11			4			9			13		2	18
12			1			4			4		1	12
13						1			1			5
14												2
15												1

Attenuation lengths are evaluated from the numbers at mountain altitude (720 gcm^{-2}) and at sea level. For the case $\eta=0.3$ the values evaluated from the derivatives at sea level of altitude-dependent curves are also shown in the brackets. The quantities shown in Table 2 have a fairly strong η-dependence. For $\eta=0.3$, the position of the maximum is very deep and nearly at mountain altitude. The depth variation is very slow (see fig. 6) and the attenuation length is very large. Table 3 shows the fractions of the number of particles of each generation to the total number. Naturally with decreasing η, the generations which contribute strongly appear later, that is, the altitude where particles are mainly produced becomes lower. The same situation also holds for the other components. The energy spectrums at both mountain altitude and sea level are almost independent of η, and thus are not given here.

3.2 *Electron component*

The total number of electrons was calculated with the approximation B and that of electrons having energies larger than $10\,Mc^2$ with the approximation A.

Table 4. The numbers of electron components at mountain altitude and sea level. S is defined by the relation (15). Attenuation lengths are obtained from the numbers at both altitudes.

Depth	720 gcm^{-2}		S	1033 gcm^{-2}		S	Attenuation length	
E_0 / η	10^4	10^6		10^4	10^6		10^4	10^6
1	$3.3\cdot10^3$	$9.6\cdot10^5$	1.2	$2.8\cdot10^2$	$1.4\cdot10^5$	1.3	130	160
0.5	$3.4\cdot10^3$	$8.7\cdot10^5$	1.2	$5.5\cdot10^2$	$2.5\cdot10^5$	1.3	170	250
0.3	$3.8\cdot10^3$	$7.3\cdot10^5$	1.1	$1.2\cdot10^3$	$3.9\cdot10^5$	1.3	270	490

Table 5 The fractions of the numbers of total electrons of each generation at both altitudes.

Depth	720 gcm^{-2}						1033 gcm^{-2}					
E_0	10^4			10^6			10^4			10^6		
l \ η	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3
1	52	21	11	43	24	15	46	9	2	42	11	3
2	33	26	14	30	21	16	36	15	5	26	15	6
3	15	22	15	18	20	16	18	20	9	17	16	8
4		16	21	9	15	15		20	14	11	17	11
5		9	14		10	13		16	15	4	15	15
6		4	11		6	10		10	14		11	13
7		2	7		3	6		6	13		8	12
8			4		1	4		3	11		4	10
9			2			3		1	8		2	8
10			1			1			6		1	6
11						1			3			4
12												2
13												1
14												1

The results for the total number is shown in Tables 4 and 5. S is again defined by (15). The η-dependence of the total number is weak, and hence the shower size is a useful datum for estimating the primary energy. The attenuation length, however, has a strong η-dependence. Comparing Tables 3 and 5, early generations contribute to the electron component much more than they do to the nucleonic component. This fact shows that high energy neutral pions contribute to the electron number much more strongly than high energy nuclear active particles contribute to the nucleonic component, so that the electron component reflects high energy collisions more directly than does the nucleonic component. Actually the fraction of the electron number initiated from neutral pions having energies larger than 100 Mc^2 was estimated to be 40～50 percent at both altitudes in our model regardless of the value of η. On the contrary, in the nucleonic component, particles predominate which are produced by particles having energies in the range from several ten to about one hundred Mc^2.

Table 6. The numbers of electrons having energies larger than 10 Mc^2 at both altitudes. The values in the brackets are ratios of the numbers of such electrons to the numbers of total electrons.

Depth	720 gcm^{-2}		1033 gcm^{-2}	
η \ E_0	10^4	10^6	10^4	10^6
1	1.2 $(3.6\cdot10^{-4})$	$9.1\cdot10^2$ $(9.5\cdot10^{-4})$	$4.0\cdot10^{-2}$ $(1.4\cdot10^{-4})$	$5.3\cdot10^2$ $(3.8\cdot10^{-4})$
0.5	2.2 $(6.5\cdot10^{-4})$	$1.2\cdot10^3$ $(1.4\cdot10^{-3})$	$2.0\cdot10^{-1}$ $(3.6\cdot10^{-4})$	$1.7\cdot10^2$ $(6.8\cdot10^{-4})$
0.3	4.6 $(1.2\cdot10^{-3})$	$1.7\cdot10^3$ $(2.3\cdot10^{-3})$	1.1 $(9.2\cdot10^{-4})$	$5.2\cdot10^2$ $(1.3\cdot10^{-3})$

Table 7. The fractions of the numbers of electrons having energies larger than 10 Mc^2, of each generation at both altitudes.

Depth	720 gcm^{-2}						1033 gcm^{-2}					
E_0	10^4			10^6			10^4			10^6		
l \ η	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3
1	62	12	3	46	15	5	60	5	2	45	5	1
2	38	21	8	31	20	10	40	11	3	27	10	2
3		30	15	17	23	14		17	7	18	15	5
4		16	15	6	16	18		19	11	10	17	8
5		9	14		12	14		19	14		17	10
6		7	14		6	12		16	15		15	12
7		5	12		5	9		13	14		12	14
8			8		2	7			11		3	13
9			5		1	5			9		3	12
10			3			3			7		1	9
11			2			2			5			7
12			1			1			2			5
13												2

In order to see how the generations giving the main contribution occur earlier when the lower limit of the energy is raised, we have selected electrons having energies larger than 10 Mc^2; the results are listed in Tables 6 and 7. Comparing Tables 5 and 7 we see that even for electrons of energy larger than 10 Mc^2 the positions of the generations giving the bulk of the contribution to the electron number remain fairly constant. This fact means that the parental neutral pions have not necessarily large energies even if the electrons produced by them have large energies. It is noted that the number of high energy electrons at sea level depends fairly heavily on η.

3.3 *Muon component*

The results for the muon component whose parental pions have energies larger than 10 Mc^2 are shown in Tables 8 and 9. Collisions in the lower energy region affect the muon component as well as the nucleonic component, though earlier generations contribute slightly more to the former than to the latter.

Table 8 The numbers of muons at both altitudes.

Depth	720 gcm^{-2}		1033 gcm^{-2}	
η \ E_0	10^4	10^6	10^4	10^6
1	$1.3\cdot10^2$	$8.4\cdot10^3$	$1.4\cdot10^2$	$8.7\cdot10^3$
0.5	$9.6\cdot10$	$6.2\cdot10^3$	$9.9\cdot10$	$7.1\cdot10^3$
0.3	$5.7\cdot10$	$3.0\cdot10^3$	$6.8\cdot10$	$4.8\cdot10^3$

Table 9 The fractions of the numbers of muons of each generation at both altitudes.

Depth	720 gcm^{-2}						1033 gcm^{-2}					
E_0	10^4			10^6			10^4			10^6		
η / l	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3
1	2	4	6				2	4	6			
2	27	30	36	1	1	3	26	29	31	1	1	2
3	71	45	9	19	18	6	72	44	8	18	16	4
4		7	10	80	38	10		7	9	81	35	6
5		7	10		19	13		7	9		19	9
6		5	9		9	15		6	9		9	11
7		2	8		9	15		3	8		10	13
8			6		5	13			8		7	13
9			3		1	10			5		2	13
10			2			8			4		1	12
11			1			4			3			9
12						2						5
13						1						2
14												1

Table 10 The numbers of muons at both altitudes whose parental pions have energies larger than 100 Mc^2. Values in the brackets are ratios of the numbers of such muons to the numbers of total muons.

Depth	720 gcm^{-2}		1033 gcm^{-2}	
E_0 / η	10^4	10^6	10^4	10^6
1	2.9 $(2.2\cdot10^{-2})$	$6.1\cdot10$ $(7.2\cdot10^{-3})$	2.9 $(2.1\cdot10^{-2})$	$6.1\cdot10$ $(7.0\cdot10^{-3})$
0.5	$1.0\cdot10$ $(1.1\cdot10^{-2})$	$4.7\cdot10^2$ $(7.6\cdot10^{-2})$	$1.1\cdot10$ $(1.1\cdot10^{-1})$	$4.9\cdot10^2$ $(6.9\cdot10^{-2})$
0.3	$1.3\cdot10$ $(2.2\cdot10^{-1})$	$2.9\cdot10^2$ $(9.9\cdot10^{-2})$	$1.3\cdot10$ $(1.9\cdot10^{-1})$	$3.2\cdot10^2$ $(6.7\cdot10^{-2})$

However, as shown in Table 11, when we pick out muons whose parental pions have energies larger than 100 Mc^2, we see that such muons are produced in a relatively few and relatively early generations independently of the value of η. Since a nuclear active particle which produces such muons secondarily has an energy larger than 10^{12}eV, the behaviours of such high energy muons, especially the lateral distribution, are directly affected by high energy collisions.

The ratios of various combinations of the numbers of the three components are listed in Table 12. These ratios may be influenced by fluctuations, and hence the values in the table must be interpreted statistically.

Table 11 The fractions of the numbers of muons of each generation whose parental pions have energies larger than 100 Mc^2 at both altitudes.

Depth	720 gcm^{-2}						1033 gcm^{-2}					
E_0	10^4			10^6			10^4			10^6		
l \\ η	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3	1	0.5	0.3
1	100	33	30	1			100	33	30	1		
2		46	40	99	18	29		45	40	99	16	27
3		21	30		44	64		22	30		42	60
4					35	5					34	1
5					3	1					2	1
6						1					2	1
7											2	1
8											2	1
9												1
10												2
11												2
12												2
13												1

Table 12. The ratios of various combinations of the numbers of the three components. M.L. and S.L. mean mountain level (720 gcm^{-2}) and sea level (1033 gcm^{-2}).

E_0		10^4			10^6		
η		1	0.5	0.3	1	0.5	0.3
$\frac{N+\mu}{\text{tot.no.}}$	M.L.	$3.8\cdot10^{-2}$	$2.9\cdot10^{-2}$	$2.0\cdot10^{-2}$	$9.7\cdot10^{-3}$	$8.6\cdot10^{-3}$	$7.2\cdot10^{-3}$
	S.L.	$3.3\cdot10^{-1}$	$1.5\cdot10^{-1}$	$6.1\cdot10^{-2}$	$5.9\cdot10^{-2}$	$2.9\cdot10^{-2}$	$1.6\cdot10^{-2}$
$\frac{\mu}{\text{tot.no.}}$	M.L.	$3.8\cdot10^{-2}$	$2.7\cdot10^{-2}$	$1.5\cdot10^{-2}$	$8.9\cdot10^{-3}$	$7.0\cdot10^{-3}$	$4.1\cdot10^{-3}$
	S.L.	$3.3\cdot10^{-1}$	$1.5\cdot10^{-1}$	$5.2\cdot10^{-2}$	$5.8\cdot10^{-2}$	$2.7\cdot10^{-2}$	$1.2\cdot10^{-2}$
$\frac{N}{N+\mu}$	M.L.	$9.2\cdot10^{-3}$	$8.4\cdot10^{-2}$	$2.6\cdot10^{-1}$	$7.1\cdot10^{-2}$	$1.8\cdot10^{-1}$	$4.3\cdot10^{-1}$
	S.L.	$6.4\cdot10^{-4}$	$1.7\cdot10^{-2}$	$1.4\cdot10^{-1}$	$9.0\cdot10^{-4}$	$5.9\cdot10^{-2}$	$2.7\cdot10^{-1}$

3.4 *A remark on combinations of parameters*

We have so far confined our attention to the η-dependence of various quantities with fixed values of a and γ. If an arbitrary combination of the parameters a, γ and η are adopted, there may be a possibility of obtaining results not essentially different from those given above. Here we shall give one remark on this point.

We have calculated the altitude dependence of the electron component taking two combinations

$$(a, \gamma, \eta)=(6, 1/4, 1),$$

$$(a', \gamma', \eta')=(2, 1/3, 1),$$

for a primary energy $E_0=10^6 Mc^2$. The results were quite similar to each other. This degeneracy comes from the fact that, when $\eta=1$ and the values of the parameters are adjusted to satisfy $a(\eta E_0)\simeq a'(\eta' E_0)^{\gamma'}=200$, the development of the electron component is governed mainly by the first collision of a primary particle. As η becomes smaller, however, many generations come to contribute appreciably and the possibility of obtaining similar results for two different combinations of the parameters may be decreased. Moreover, even if similar results for one quantity are obtained for such combinations with a fixed primary energy, this degeneracy may be resolved by comparing the other quantities and also by varying the primary energy. A similar risk arises in connection with the value of λ_c; this point we shall discuss in the next section. Therefore it is desirable to perform experiments in such a way that as many measurements as possible of the relevant quantities of one EAS can be made at the same time. From the theoretical point of view it seems to be more desirable to find general tendencies of parameters with varying E_0 in connection with experimental information than to determine the values of parameters by comparison with present experiments.

§ 4. Discussions and remarks

In the preceding section we have investigated the η-dependence of various quantities concerned with each component of EAS, without considering the possibility of observations and the experimental conditions. The important fact given there is that most of the quantities based on our model depend strongly on the primary energy. To determine the value of η, we must select the quantities which satisfy the two conditions that they have a strong η-dependence and they are directly affected by high energy collisions, whence the value of η could be determined by investigating experimentally the dependences of such quantities on the primary energy. Furthermore it is to be noted that the primary energy of each EAS must be estimated from quantities not depending strongly on η, but rather on models, for it cannot be measured directly.

The attenuation length of the electron component is effective to test the η-dependence of high energy collisions because it has a strong η-dependence and is affected by high energy collisions as stated in § 3.2 (see also Table 4). On the other hand, though the behaviour of the nucleonic and muon components are obscured by low energy collisions, some of quantities concerned with these components depend strongly on η. Therefore it may be worthwhile to consider these quantities as data for fixing a rough standard.

Let us compare the attenuation length of the electron component, λ_a, with the experimental data now available. The experimental value is nearly constant over a wide range of primary energies, $10^{14}\sim10^{17}$eV, and nearly equal to 200 gcm^{-2}.[19] This is a very striking feature. Our theoretical values are nearly equal to the experimental ones if we take $\eta=0.3\sim0.5$ for $E_0=10^4Mc^2$ and $\eta=0.5\sim1$ for $E_0=10^6Mc^2$.

A work similar to ours was done by the Soviet group. The differences between their model and ours are, a) $\lambda_c=70$ gcm^{-2} instead of 90 gcm^{-2}, b) $n=A^{1/4}E^{1/4}$, A being the average mass number of air nuclei, and c) secondary nucleons are not emitted. The calculated value of λ_a in the case $\eta=0.3$ agreed with its experimental value, 200 gcm^{-2}.

Thus they concluded $\eta \simeq 0.3$.[5] Even in their model, however, λ_a and other quantities should vary with primary energy, thus such a conclusion cannot be drawn from their results with only one value of primary energy.

Moreover, even if we restrict our discussion to the case of only one fixed primary energy, λ_a is altered by the value of λ_c adopted. Therefore the value of inelasticity cannot at the same time be determined uniquely from such results as they derived. In fact the attenuation length of the nucleonic component based on our model changes its value from 1350 gcm^{-2} for $\lambda_c = 90\ \mathrm{gcm}^{-2}$ to 380 gcm^{-2} for $\lambda_c = 70\ \mathrm{gcm}^{-2}$ in the case of $E_0 = 10^6 Mc^2$ and $\eta = 0.3$.* Correspondingly it is expected that λ_a may be reduced from 510 gcm^{-2} to 200∼300 gcm^{-2}.

Summarizing these discussions, the constancy of λ_a cannot be explained by our model nor by that of the Soviet group. To explain this difficulty it seems to be necessary to introduce an energy dependence of the inelasticity or a stronger energy dependence of the multiplicity with increasing energy,** even if some fixed value be taken for λ_c, though the possibility that λ_c would become small with increasing energy is not excluded. Thus the conclusion derived by Rosental' and Cernavskij[3], namely $\gamma = 1/4$, is not yet decisive.

Finally we shall comment on how to deduce a detailed knowledges of angular and energy distributions of secondary particles produced by high energy collisions.*** First, as stated in § 3.3, high energy muons give important data for investigating the angular distribution. For this purpose it would be desirable to perform an experiment as was done by the Cornell group.[21] Secondly the electron component is also important. We stated in § 3.2 that 40∼50 percent of the total electrons are produced at both altitudes by neutral pions having energies larger than 100 Mc^2 irrespective of the values of η. Considering this result and the fact that the number of electrons is roughly proportional to the energy of their parental neutral pion, the behaviour of electrons will give useful data for investigating the energy distribution of secondary particles. In this sense the behaviour of electrons and of high energy muons are complements of each other as data for investigating the energy and angular distributions of secondary particles in high energy collisions. It must be noticed here that it is not necessarily apparent from the result described in § 3.2 that the behaviour of high energy electrons near shower axes is more directly affected by high energy collisions than is the behaviour of total electrons, though some analyses[22]–[25] were made assuming this fact to be true. More detailed analyses on these problems are now being undertaken.

We should like to express our sincere gratitude to Prof. T. Inoue for his continual

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* The difference b) makes λ_a small as compared with our value, even if the same value of λ_c is adopted in both models. But the difference between the λ_a's appears to be almost all due to the difference a).

** Recently the research group of EAS in Osaka City University[20] carried out an experiment at Mt. Norikura (720gcm^{-2}) and obtained a different result, namely that the attenuation length of the total number varies from 225 gcm^{-2} for shower size $3.10^4 \sim 2.10^5$ to 386 gcm^{-2} for shower size $\gtrsim 6.10^5$. This result indicates the constancy of the inelasticity with increasing energy.

*** The necessity for introducing the sharpness of the angular or the energy distribution of secondary particles has been pointed out qualitatively by several authors.[2][3][14][10] See also the footnote on p. 270)

encouragement. We should also like to express our sincere thanks to Professors J. Nishimura, M. Oda, S. Takagi and the researching group of extremely high energy phenomena in Japan for their valuable advices and stimulating discussions, especially during the "Norikura Symposium on Extremely High Energy Phenomena" held at the Research Institute for Fundamental Physics, Kyoto University, Jan. 1957 and also held at the Institute for Nuclear Studies, University of Tokyo, Mar. 1957. Thanks are also due to Miss K. Uemura for her assistance in doing numerical calculations.

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On the Subsidiary Conditions in the Bohm-Pines Theory of Electron Plasma

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The subsidiary conditions in the Bohm-Pines theory are too stringent. There is no normalizable wave function which satisfies the subsidiary conditions. We modify the subsidiary conditions and show that the mathematical inconsistency in the Bohm-Pines theory may be removed.

§ 1. Introduction

Bohm and Pines[1] (BP) have developed a collective description of the behavior of the electrons in an electron gas. They introduce a set of new field variables, representing extra degrees of freedom of the system. These new variables, after a series of canonical transformations, appear as coordinates and momenta of the plasma oscillators. In order that the total number of degrees of freedom of the system remains unchanged, it is necessary to require that a set of subsidiary conditions, equal in number to the extra degrees of freedom introduced for the plasma field, be satisfied by the wave functions.

The Hamiltonian for the system is given by

$$H=\sum_i p_i^2/2m+\sum_k 2\pi e^2/k^2\cdot(\rho_k\rho_{-k}-n), \tag{1·1}$$

where n is the number of electrons and ρ_k their density fluctuation

$$\rho_k=\sum_i e^{-ikx_i}\,. \tag{1·2}$$

BP introduce the plasma field by adding certain terms to the original Hamiltonian (1·1). The resultant extended Hamiltonian is given by

$$H_{\text{ext}}=H-1/2\cdot\sum_{k<k_c} p_{-k}p_k-i\sum_{k<k_c} M_k p_k\rho_k\,, \tag{1·3}$$

where $M_k=(4\pi e^2/k^2)^{1/2}$. The subsidiary conditions are

$$p_k\psi=0\quad(k<k_c). \tag{1·4}$$

BP relate the p_k to the density fluctuation ρ_k by applying the unitary transformation $\psi_{\text{old}}=S\psi_{\text{new}}$, where

$$S=\exp\Big(\sum_{k<k_c}\hbar^{-1}M_k q_k\rho_{-k}\Big). \tag{1·5}$$

The new subsidiary conditions become

$$(p_k - iM_k\rho_{-k})\psi = 0 \quad (k < k_c). \tag{1·6}$$

Adams[2)] has pointed out that the wave function

$$\psi = \exp\left(-\sum_{k<k_c} \hbar^{-1} M_k q_k \rho_{-k}\right) \cdot \chi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) \tag{1·7}$$

which satisfies the subsidiary conditions (1·6) is incompatible with the oscillators being in their ground states and in fact the expectation value of the oscillator energy diverges. On the ground of this divergence he has objected to the neglect of the nonlinear plasma-electron interaction. Kuper[3)] has criticized the neglect of the subsidiary conditions. He has made the selection of the set of plasma oscillators variationally. He has shown that when the subsidiary conditions are neglected—a trial wave function is introduced whose form is a product of oscillator ground state functions and a Slater determinant—, and the set of oscillators is chosen so as to minimize the expectation value of the Hamiltonian, the expectation value is allowed to become infinitely negative.

It should be noted that the wave function (1·7) which satisfies the subsidiary conditions (1·6) is not normalizable. In order to normalize the wave function ψ we must integrate $|\psi|^2$ over $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ and over the oscillator variables. Since $|\psi|^2 = |\chi|^2$ because of $q_k^* = -q_{-k}$ and $\rho_k{}^* = \rho_{-k}$, the norm (ψ, ψ) diverges. This situation is analogous to the corresponding situation in ordinary wave mechanics where the eigenfunctions of the canonically conjugate variables q and p are not normalizable.

In quantum electrodynamics it was recognized that the subsidiary condition $\partial A_\alpha / \partial x_\alpha \cdot \psi = 0$ (the subsidiary condition (1·4) is proportional to the k-th component of div E $\psi = 0$) is too stringent as a subsidiary condition. In fact, if we imposed this subsidiary condition as such, there would be no states, ψ, which satisfy it. Various methods have been proposed to circumvent this difficulty. One procedure[4)] is to define a family of normalizable state vectors depending on a parameter which is allowed to approach a limit in such a manner that the subsidiary condition is satisfied in the limit only (method ***a***). Another method has been proposed by Gupta[5)] and Bleuler[5)]. They modify the subsidiary condition so that it holds only for the destruction part of the operator, i.e. $\dfrac{\partial A_\alpha^{(-)}}{\partial x_\alpha}\psi = 0$ (method ***b***). One more method is to modify the subsidiary condition so that the expectation values of $\dfrac{\partial A_\alpha}{\partial x_\alpha}$ and $\left(\dfrac{\partial A_\alpha}{\partial x_\alpha}\right)^2$ vanish[6)] (method ***c***).

In the next section we shall investigate the three methods and try to remove the defects in the mathematical basis of BP work.

§ 2. Modification of the subsidiary conditions

The Hamiltonian (1·3) of the extended system may be rewritten as

$$H_{\text{ext}} = H + 1/4 \cdot \sum_{k<k_c} \hbar\omega(a_k{}^* a_k + a_k a_k{}^* + a_k a_{-k} + a_k{}^* a_{-k}{}^*) \tag{2·1}$$

$$+ \sum_{k<k_c} f_k \rho_k (a_k^* + a_{-k}),$$

where $f_k=(2\pi e^2/\hbar\omega k^2)^{1/2}$ and the creation and destruction operators of plasma field, a_k^* and a_k, are defined by

$$\begin{aligned} q_k&=(\hbar/2\omega)^{1/2}(a_k-a_{-k}{}^*),\\ p_k&=i(\hbar\omega/2)^{1/2}(a_k{}^*+a_{-k}). \end{aligned} \tag{2·2}$$

The subsidiary conditions (1·4) can be written in the form

$$\Omega_k\psi=0 \quad (k<k_c), \tag{2·3}$$

where

$$\Omega_k=a_k{}^*+a_{-k}. \tag{2·4}$$

The state vector of the extended system is in general given by

$$\psi=\prod_{\substack{k<k_c\\k_z>0}} \Phi_k\chi(x_1,\cdots,x_n), \tag{2·5}$$

where

$$\Phi_k=\sum_{\substack{m=0\\n=0}}^{\infty} C_k(m,n)/\sqrt{m!n!}\cdot(a_k{}^*)^m(a_{-k}{}^*)^n\Phi_0. \tag{2·6}$$

Φ_0 is the eigenvector of $a_k{}^*a_k\,(k<k_c)$ with eigenvalue zero. Let us consider the equation $\Omega_k\Phi_k=0$, which may be written as[7]

$$\sum_{\substack{m=0\\n=0}}^{\infty}(n+1/(m+1)!(n+1)!)^{1/2}\{\sqrt{m+1}\,C_k(m,n)+\sqrt{n+1}\,C_k(m+1,n+1)\}$$

$$\cdot(a_k{}^*)^{m-1}(a_{-k}{}^*)\,\Phi_0+\sum_{n=0}^{\infty}(n+1/n!)^{1/2}\,C_k(0,n+1)\,(a_k{}^*)^n\Phi_0=0.$$

Therefore

$$C_k(0,n+1)=0 \tag{2·7a}$$

and

$$C_k(m,n)=-(n+1/m+1)^{1/2}\,C_\kappa(m+1,n+1). \tag{2·7b}$$

Because of the symmetry of $C_k(m,n)$ with respect to m and n, we obtain

$$C_k(m,n)=0 \quad \text{for} \quad m\neq n \tag{2·8a}$$

and

$$C_k(n,n)=-C_k(n+1,n+1)=(-1)^nC_k(0,0). \tag{2·8b}$$

Hence the state vector which satisfies the subsidiary conditions (2·3) must be of the form

$$\psi=\prod_{\substack{k<k_c\\k_z>0}}\sum_{n=0}^{\infty}C_k(0,0)(-1)^n/n!\cdot(a_k{}^*)^n(a_{-k}{}^*)^n\Phi_0\chi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n). \tag{2·9}$$

The norm of ψ is given by

$$(\psi,\psi)=\Pi|C_k(0,0)|^2(1+1+\cdots), \quad \text{if} \quad (\chi,\chi)=1.$$

Therefore $\psi=0$ if $C_k(0,0)=0$ or $(\psi,\psi)=\infty$ if $C_k(0,0)\neq 0$. Thus there is no normalizable state vector which satisfies the subsidiary conditions $(2\cdot3)$.*

In the following we shall investigate the three methods mentioned in § I.

(*a*) We define a family of statevectors by setting[4)]

$$\psi^\lambda=\prod_{\substack{k<k_c\\ k_z>0}}(1-\lambda^2)^{1/2}\exp(-\lambda a_k{}^*a_{-k}{}^*)\cdot\Phi_0\chi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n), \tag{2·10}$$

where $(\chi,\chi)=1$. The norm of these vectors is given by

$$(\psi^\lambda,\psi^\lambda)=\Pi(1-\lambda^2)\sum_{n=0}^{\infty}\lambda^{2n}=1,\quad\text{for}\quad|\lambda|<1. \tag{2·11}$$

For the left-hand side of the subsidiary conditions $(2\cdot3)$, we obtain

$$\alpha_k^\lambda\equiv(a_k{}^*+a_{-k})\psi^\lambda=(1-\lambda)a_k{}^*\psi^\lambda. \tag{2·12}$$

The norm of α_k^λ can easily be calculated. We get

$$(\alpha_k^\lambda,\alpha_k^\lambda)=(1-\lambda)/(1+\lambda). \tag{2·13}$$

It is seen from the last expression that this norm approaches zero with $\lambda\to1$. Thus the subsidiary conditions are satisfied for the limit $\lambda=1$.

The extended system $(2\cdot1)$ is equivalent to the original system $(1\cdot1)$ in the sense that if χ_n is the eigenfunction of the original Hamiltonian $(1\cdot1)$, i.e.

$$H\chi_n=E_n\chi_n, \tag{2·14}$$

then the equation holds

$$\lim_{\lambda\to1}(\psi_n^\lambda,H_{\text{ext}}\psi_n^\lambda)=E_n, \tag{2·15}$$

where

$$\psi_n^\lambda=\prod_{\substack{k<k_c\\ k_z>0}}(1-\lambda^2)^{1/2}\exp(-\lambda a_k{}^*a_{-k}{}^*)\,\Phi_0\chi_n.$$

Since this procedure is somewhat awkward, we do not enter into further details.

(*b*) We impose the following subsidiary conditions instead of $(2\cdot3)$

$$\Omega_k^{(-)}\psi=0\,(k<k_c), \tag{2·16}$$

where

$$\Omega_k^{(-)}=a_k. \tag{2·17}$$

Our extended Hamiltonian is given by

$$H_{\text{ext}}=H+1/4\cdot\sum_{k<k_c}\hbar\omega(a_k{}^*a_k+a_ka_k{}^*+a_ka_{-k}+a_k{}^*a_{-k}{}^*)$$
$$-1/4\cdot\sum_{k<k_c}\hbar\omega+\sum_{k<k_c}f_k\rho_k(a_k{}^*+a_{-k}). \tag{2·18}$$

The state vector which satisfies the subsidiary conditions $(2\cdot16)$ is given by

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* Galasiewicz[7)] did not refer to this difficulty.

$$\psi=\Phi_0\chi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) \tag{2·19}$$

where $(\chi, \chi)=1$. If χ_n is the eigenfunction of the original Hamiltonian (1·1), i.e.

$$H\chi_n=E_n\chi_n,$$

then the equation holds*

$$(\psi_n, H_{\text{ext}}\psi_n)=E_n,$$

where

$$\psi_n=\Phi_0\chi_n.$$

Applying the unitary transformation (1·5), we get

$$(\Omega_k^{(-)})_{\text{new}}=a_k-f_k\rho_k. \tag{2·20}$$

The new Hamiltonian becomes

$$H_{\text{ext}}=H_0+H_{\text{coll}}+H_1+U+H_{\text{s.r.}}+H'_{\text{coll}}-\sum_{k<k_c}\hbar\omega/4-2\pi e^2 n\sum_{k<k_c}1/k^2, \tag{2·21}$$

where

$$H_0=\sum_i p_i^2/2m, \tag{2·22a}$$

$$H_{\text{coll}}=\sum_{k<k_c}\hbar\omega(a_k{}^*a_k+1/2), \tag{2·22b}$$

$$H_1=(e/m)\sum_{i,\,k<k_c}(2\pi\hbar/\omega)^{1/2}\{\boldsymbol{\varepsilon}_k\cdot(\boldsymbol{p}_i-\hbar\boldsymbol{k}/2)\,a_k e^{ik\cdot x_i}+e^{-ik\cdot x_i}a_k{}^*\boldsymbol{\varepsilon}_k\cdot(\boldsymbol{p}_i-\hbar\boldsymbol{k}/2)\}, \tag{2·22c}$$

$$H_{\text{s.r.}}=2\pi e^2\sum_{\substack{k>k_c\\ i\neq j}}e^{ik\cdot(x_i-x_j)}/k^2, \tag{2·22d}$$

$$U=(\pi\hbar e^2/m)\sum_{\substack{i,\,k\,l<k_c\\ k\neq -l}}(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{\varepsilon}_l)(a_k-a_{-k}{}^*)(a_l-a_{-l}{}^*)e^{i(k+l)\cdot x_i}, \tag{2·22e}$$

$$H'_{\text{coll}}=\sum_{k<k_c}\hbar(\omega_p{}^2-\omega^2)/4\omega\cdot(a_k{}^*a_k+a_k a_k{}^*-a_k a_{-k}-a_k{}^*a_{-k}{}^*), \tag{2·22f}$$

where $\boldsymbol{\varepsilon}_k=\boldsymbol{k}/|\boldsymbol{k}|$ and $\omega_p=(4\pi e^2 n/m)^{1/2}$.

The state vector which satisfies the new subsidiary conditions (2·20) is

$$\psi=\exp\Big(-1/2\cdot\sum_{k<k_c}f_k^2\rho_k\rho_{-k}+\sum_{k<k_c}f_k\rho_k a_k{}^*\Big)\cdot\Phi_0\chi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n). \tag{2·23}$$

It is easily seen that $(\psi, \psi)=1$ if $(\chi, \chi)=1$. Using the above expression for the state vector, we obtain the following expressions for the expectation values of (2·22)

$$(\psi, H_0\psi)=\sum_i\langle p_i^2\rangle/2m+\sum_{k<k_c}\hbar\omega/4, \tag{2·24a}$$

$$(\psi, H_{\text{coll}}\psi)=\sum_{k<k_c}\hbar\omega/2+2\pi e^2\sum_{k<k_c}\langle\rho_k\rho_{-k}\rangle/k^2, \tag{2·24b}$$

---

* If we require that the extended system is equivalent to the original system in the sense that $H_{\text{ext}}\psi_n=E_n\psi_n$ holds, we would have to impose the BP's subsidiary conditions (2.3).

$$(\psi,\ H_1\,\psi) = -\sum_{k<k_c} \hbar\omega/2, \tag{2·24c}$$

$$(\psi,\ H_{\text{s.r.}}\psi) = 2\pi e^2 \sum_{k>k_c} \{\langle\rho_k\rho_{-k}\rangle - n\}/k^2, \tag{2·24d}$$

$$(\psi,\ U\psi) = 0, \tag{2·24e}$$

$$(\psi,\ H'_{\text{coll}}\psi) = \sum_{k<k_c} \hbar(\omega_p{}^2 - \omega^2)/4\omega, \tag{2·24f}$$

where $\langle g(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)\rangle \equiv (\chi,\ g\chi)$. Putting $\omega = \omega_p$, we find

$$(\psi,\ H_{\text{ext}}\psi) = (\chi,\ H\chi) \tag{2·25}$$

as is expected. If we adopt Kuper's procedure, we obtain the same result as (2·25). Hence there is no divergence difficulty. So far our subsidiary conditions (2·16) seem to be satisfactory. However we meet with a difficulty, when we go over to the BP representation, in which H_1 is eliminated by a canonical transformation, as we shall show below.

The BP transformation, which is introduced to eliminate the plasma-electron interaction H_1, is generated by[1)]

$$S = -(ie/m\hbar) \sum_{\substack{i \\ k<k_c}} (2\pi\hbar/\omega)^{1/2} \left\{ \frac{\boldsymbol{\varepsilon}_k \cdot (\boldsymbol{p}_i - \hbar\boldsymbol{k}/2)}{\omega - \boldsymbol{k}\cdot\boldsymbol{p}_i/m + \hbar k^2/2m}\, e^{i\boldsymbol{k}\boldsymbol{x}_i} a_k \right.$$

$$\left. - a_k{}^* e^{-i\boldsymbol{k}\boldsymbol{x}_i} \frac{\boldsymbol{\varepsilon}_k \cdot (\boldsymbol{p}_i - \hbar\boldsymbol{k}/2)}{\omega - \boldsymbol{k}\cdot\boldsymbol{p}_i/m + \hbar k^2/2m} \right\}. \tag{2·26}$$

The Hamiltonian in the BP representation is given by[1)]

$$e^{-iS} H_{\text{ext}} e^{iS} = H_{\text{electron}} + H_{\text{s.r.}} + H_{\text{r.p.}} + H_{\text{coll}} - \sum_{k<k_c} \hbar\omega/4, \tag{2·27}$$

where higher order terms are neglected. Since we need not the explicit expression of this Hamiltonian in our discussion, we do not write down H_{electron}, $H_{\text{s.r}}$ and $H_{\text{r.p.}}$ explicitly, but we only refer to the fact that these three terms depend on only electron variables. The subsidiary conditions, to the same order of approximation as the Hamiltonian, are given by

$$\Omega_k^{(-)}\psi = 0\ (k < k_c),$$

where

$$\Omega_k^{(-)} = 1/2\cdot(a_k - a_{-k}{}^*) - f_k \sum_i e^{-i\boldsymbol{k}\boldsymbol{x}_i} \frac{\omega}{\omega - \boldsymbol{k}\cdot\boldsymbol{p}_i/m + \hbar k^2/2m}. \tag{2·28}$$

There is no normalizable state vector which satisfies the above conditions.* This is most easily seen by rewriting (2·28) as follows

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* This is due to the approximation used in deriving the Hamiltonian (2.27) and the subsidiary conditions (2.28). If all the higher order terms were retained, the state vector which satisfies the subsidiary conditions would be normalizable.

$$\{q_k-(4\pi e^2/\omega^2k^2)^{1/2}\sum_i e^{-ikx_i}\}\psi=0 \qquad (k<k_c),$$

where q_k is defined by $q_k=(\hbar/2\omega)^{1/2}(a_k-a_{-k}^*)$ and $\omega/(\omega-\boldsymbol{k}\cdot\boldsymbol{p}_i/m+\hbar k^2/2m)$ is approximated by unity.

(***c***) We consider the following subsidiary conditions:

$$\left.\begin{aligned}&(\psi,\ \Omega_k\psi)=0,\\&(\Omega_k\psi,\ \Omega_k\psi)=(\psi,\ \Omega_{-k}\Omega_k\psi)=1,^*\end{aligned}\right\}\quad (k<k_c) \tag{2·29}$$

where $\Omega_k=a_k^*+a_{-k}$. The Hamiltonian of the extended system is given by (2·18). The state vector which satisfies these subsidiary conditions may be obtained by replacing the infinite sum in (2·9) by a finite sum, i.e.

$$\psi=\prod_{\substack{k<k_c\\k_z>0}}\Phi_k\chi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n) \tag{2·30}$$

where

$$\Phi_k=(1/N_k+1)^{1/2}\sum_{n=0}^{N_k}(-1)^n/n!\cdot(a_k^*)^n(a_{-k}^*)^n\Phi_0. \tag{2·31}$$

In fact, we find

$$\Omega_k\Phi_k=(1/N_k+1)^{1/2}(-1)^{N_k}/N_k!\cdot(a_k^*)^{N_k+1}(a_{-k}^*)^{N_k}\Phi_0.$$

Therefore

$$\begin{aligned}(\Phi_\kappa,\ \Omega_k\Phi_k)&=(1/N_k+1)\sum_{n=0}^{N_k}(-1)^{n+N_k}/n!N_k!\cdot(\Phi_0,\ (a_k)^n(a_{-k})^n(a_k^*)^{N_k+1}(a_{-k}^*)^{N_k}\Phi_0)\\&=0\end{aligned}$$

and

$$\begin{aligned}(\Omega_k\Phi_k,\ \Omega_k\Phi_k)&=(1/N_k+1)\,(1/N_k!)^2(\Phi_0,(a_k)^{N_k+1}(a_k^*)^{N_k+1}(a_{-k})^{N_k}(a_{-k}^*)^{N_k}\Phi_0)\\&=1.\end{aligned}$$

Also we get $(\psi,\ \psi)=1$ if $(\chi,\ \chi)=1$. The extended system is equivalent to the original system in the same sense as in (***b***).

Applying the unitary transformation (1·5), we get

$$\begin{aligned}&(\psi_{\text{new}},\ (\Omega_k)_{\text{new}}\psi_{\text{new}})=0,\\&(\psi_{\text{new}},\ (\Omega_{-k})_{\text{new}}(\Omega_k)_{\text{new}}\psi_{\text{new}})=1,\end{aligned} \tag{2·32}$$

where

$$(\Omega_k)_{\text{new}}=a_k^*+a_{-k}-2f_k\rho_{-k}.$$

The new Hamiltonian is given by (2·21). The new state vector, which satisfies (2·32), is given by**

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* We cannot adopt $(\Omega_k\psi,\ \Omega_k\psi)=0$ as subsidiary conditions, unless we apply a limiting procedure as in (***a***).

** If $N_k=0$ for all $\boldsymbol{k}(<k_c)$, this state vector reduces to (2.23).

$$\psi=\exp(-1/2\sum_{k<kc} f_k^2\rho_k\rho_{-k}+\sum_{k<k_c} f_k\rho_k a_k^*)\prod_{\substack{k<k_c\\ k_z>0}}(1/N_k+1)^{1/2}\sum_{n=0}^{N_k}(-1)^n/n!$$

$$\cdot(a_k^*-f_k\rho_{-k})^n(a_{-k}^*-f_k\rho_k)^n\Phi_0\chi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n).$$

In the BP representation the Hamiltonian is given by (2·27) and the subsidiary conditions become

$$(\psi,\ \Omega_k\psi)=0,$$

$$(\psi,\ \Omega_{-k}\Omega_k\psi)=1, \tag{2·33}$$

where

$$\Omega_k=-2f_k\sum_i\frac{\omega^2}{\omega^2-(\boldsymbol{k}\cdot\boldsymbol{p}_i/m-\hbar k^2/2m)^2}e^{ikx_i}.$$

It is to be noted that the field variables occur only in H_{coll} and the subsidiary conditions no longer relate field and electron variables. The state vector which satisfies (2·33) may be written in the form

$$\psi=\prod_{k<k_c}(1/N!)^{1/2}(a_k^*)^{N_k}\Phi_0\cdot\varphi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n), \tag{2·34}$$

where φ must satisfy the subsidiary conditions (2·33). $N=\sum_{k<k_c}N_k$, which is the total number of plasmons that are excited, is an arbitrary integer and might have some connection with that which appeared in (2·31), though we could not show this definitely.

In our opinion the method (c) is the most satisfactory one, being free from any inconsistency.

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The Parity Conservation and the Strength of the Interaction of Elementary Particles

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In addition to the usual space and time, the two degrees of freedom which may be concerned in the mass of elementary particles are introduced. Based on the fundamental postulate with respect to the symmetry in the above new degrees of freedom, as well as the invariance under the proper Lorentz-transformation, we shall try to formulate a theory and approach the violation law of the parity conservation and of the invariance under the particle conjugation in connection with the strength of interactions. It is shown that our theory has a close relation to the two-component neutrino theory proposed by Lee and Yang and also to Landau's theory.

§ 1. Introduction

Recently, Lee and Yang¹⁾ proposed an important question of the parity conservation in the weak interactions. Being motivated by their proposal, Wu et al.²⁾ made an experiment of β-decay of the oriented nuclei and demonstrated the right and left asymmetry by which Lee and Yang's anticipation, the violation of the parity conservation and that of the invariance with respect to the particle conjugation, was confirmed. Thereupon, Lee and Yang,³⁾ and Landau⁴⁾ have proposed a theory of the two component neutrino, expecting that such unfamiliar results of experiments may be attributed exclusively to the special property of the neutrino which has the vanishing mass. Further, Salam⁵⁾ also has proposed independently a similar theory from a slightly different point of view. Although their theories seem to be successful in giving the violation law of the parity conservation in the processes in which the neutrino takes part in, they do not solve the well-known $\tau-\theta$ puzzle. Furthermore, the above authors⁴⁾·⁶⁾ have pointed out the possibility that the weak interactions are invariant under the particular combination among the space reflection, the time reversal and the particle conjugation, for example, the combination of the space reflection and the particle conjugation as in Landau's theory. These attempts, however, cannot explain why the weak interactions are invariant under a special combination of reflections, but not under the others. In other words they have not given any answer in what way the violation of reflection invariances takes place in each process.

Now, we believe that the violation law of the conservation of the parity and the strangeness or the isotopic spin in the weak interactions⁷⁾ is so closely related to the fundamental aspect of the structure of elementary particles that it may give a hopeful clue to the possibility for future theory. In order to turn the present negative situation into a positive direction, we should try to investigate the violation law of the invariances from a broad view-point.

As a result of the discovery of many new particles, we became aware of the distinctions of elementary particles, for instance, about the strength of the couplings. However, there seems to exist a slightly profound aspect concerning the property of the strong and the weak couplings themselves. For example, in the hypothesis of the composite theory of new particles proposed by Sakata,[8] interactions of such as

$$\bar{\Lambda}\Lambda\bar{\Lambda}\Lambda, \quad \bar{N}N\bar{N}N, \quad \bar{\Lambda}\Lambda\bar{N}N \tag{1·1}$$

are conceived to be strong, and

$$\text{i)} \quad \bar{N}N\binom{\bar{\mu}}{e}\nu, \quad \bar{\mu}e\bar{\nu}\nu, \quad \bar{\Lambda}N\binom{\bar{\mu}}{e}\nu,$$

$$\text{ii)} \quad \bar{\Lambda}N\bar{N}N \tag{1·2}$$

to be weak.[9] It is noteworthy that, qualitatively speaking, in the strong interactions the same kind of particles appear necessarily in pair, but in the weak interactions the combination of particles is not always so. That is, the strong coupling may be characterized by the conservation of the type of elementary particles. Therefore, we can speculate that each elementary particle possesses some internal mass structure concerned with the parity, and that the condition of the overlapping of these internal mass wave functions of elementary particles interacting on each other is responsible for the distinction of the strong and the weak interactions and for the violation of the conservation of the parity. However, we like to point out that there seems to exist some distinction among the weak interactions themselves. In fact, the weak interaction of (ii)−(1·2), which is considered to be responsible for *K*-meson decay,[9] differs from that of (i)−(1·2) in the following point. In the former interaction only baryons, each of which has strong couplings with other particles, take part in, while in the latter interactions leptons, each of which has only weak couplings with other particles, take part in. We shall hereafter call the interactions (i) and (ii) in (1·2) the weak interaction of the first kind and that of the second kind, respectively. In order to explain theoretically the distinction between the strong interactions (1·1) and the weak interaction of the second kind (ii)−(1·2), both of which are concerned only in baryon fields, it seems necessary to consider a kind of structure of elementary particle reflecting also some entity by which the nucleon and the Λ-particle may be distinguished, in addition to the parity mentioned above. In the theories[7] of Nishijima and Gell-Mann, the term "strangeness" or "η-charge" seems to represent such an entity and just the violation of its conservation characterizes the weak interaction of the second kind.

In the present paper, we shall confine ourselves only to the violation of the parity conservation as the first step and tentatively try to explain mainly the distinction between the strong interaction and the weak interaction of the first kind and to obtain the violation law of the parity conservation, introducing the new degrees of freedom concerning the mass structure of elementary particles.

§ 2. The fundamental equation

As discussed in the preceding section, the mass structure and the parity seem to be intimately related. Therefore, it is quite desirable that the basic equation reflects the situation of the above relation between mass and parity. Let us consider a basic equation of the spinor field, where 1 and γ_5 of the Dirac matrix play rather an important role for its parity determination as is well known. On the other hand, the main features of the usual Dirac equation should be kept in our basic equation.

Taking into consideration the above requirement, we propose the six dimensional basic equation,

$$(\gamma_\mu\cdot\partial/\partial x_\mu+\gamma_5\cdot\partial/\partial x_5+\gamma_6\cdot\partial/\partial x_6)\,\Psi(x_\mu,\ x_a)=0\,, \qquad (2\cdot 1)$$

in place of the usual four dimensional Dirac equation. In the above equation, $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ and $\gamma_6=i$. x_a's ($a=5,6$) denote the new degrees of freedom and are real variables with the dimension of length. From $(2\cdot 1)$, the following equation is derived,

$$(\partial^2/\partial x_\mu{}^2+\partial^2/\partial x_5{}^2+\partial^2/\partial x_6{}^2)\,\Psi(x_\mu,\ x_a)=0\,. \qquad (2\cdot 2)$$

Comparing $(2\cdot 2)$ with the Klein-Gordon equation, we find that $-(\partial^2/\partial x_5{}^2+\partial^2/\partial x_6{}^2)$ can be interpreted as a square of mass operator and consequently the new degrees of freedom relate to the mass.

Now, let us postulate the fundamental principle of our theory: *The theory should be invariant under the proper Lorentz transformation* (R_L) *and the proper rotation in the* (x_5, x_6) *space* (R_M), *while it need not be invariant with respect to all inversions such as the space reflection, the time reversal or the particle conjugation.*

From the requirement for the basic equation $(2\cdot 1)$ to be invariant, the transformation property of $\Psi(x_\mu,\ x_a)$ under R_L and R_M can be determined as follows: Under the rotation with α-angle in $(\sigma,\ \rho)$-plane

$$\Psi(x_\mu,\ x_a)\rightarrow\Psi'(x_\mu{}',\ x_a)=\exp(-\gamma_\sigma\gamma_\rho\alpha/2)\,\Psi(x_\mu,\ x_a)\,. \quad (R_L) \qquad (2\cdot 3)^*$$

Under the rotation with α-angle in $(5\cdot 6)$-plane

$$\Psi(x_\mu,\ x_a)\rightarrow\Psi'(x_\mu,\ x_a{}')=\exp(-\gamma_5\gamma_6\alpha/2)\,\Psi(x_\mu,\ x_a)\,. \quad (R_M) \qquad (2\cdot 4)^{**}$$

Let us consider the solution of the basic equation $(2\cdot 1)$. It can be shown that the general solution of $(2\cdot 1)$ is expressed as the linear superposition of the special solutions which are determined by the suitable boundary condition in the $(x_5,\ x_6)$-space and characterized by some quantum numbers. From our fundamental postulate given above, these quantum numbers must be invariant under the transformations, R_L and R_M. We can take as such quantities the mass operator and the total angular momentum j in the

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* As usual, α means the imaginary angle for the rotation which relates to the time coordinate x_4.

** The basic equation $(2\cdot 1)$ is invariant not only under R_L and R_M, but also under the arbitrary rotation in the six dimensional space $(x_\mu,\ x_a)$. The transformation of $\Psi(x_\mu,\ x_a)$ under the rotation of α-angle in the $(\mu,\ a)$-plane is given by

$$\Psi(x_\mu,\ x_a)\rightarrow\Psi'(x_\mu{}',\ x_a{}')=\exp(-\gamma_\mu\gamma_a\alpha/2)\,\Psi(x_\mu,\ x_a).$$

new degrees of freedom (x_5, x_6). That is, the nomal modes of the solution are specified by

$$\left.\begin{aligned}&(\partial^2/\partial x_5^2+\partial^2/\partial x_6^2)\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a)=-\kappa^2\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a),\\&j\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a)=\mp(l+\tfrac{1}{2})\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a),\end{aligned}\right\}\quad(2\cdot5)$$

$$(l=0,\ 1,\ 2,\ 3,\ 4,\ \cdots)$$

where

$$j=-i(x_5\cdot\partial/\partial x_6-x_6\cdot\partial/\partial x_5)+\gamma_5/2.\quad(2\cdot6)$$

Henceforth, we shall regard $\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a)$ as *the new representation of the elementary particle which has the internal mass structure.*

The explicit form of $\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a)$'s can be given as follows,

$$\begin{aligned}\Psi^{+}_{\kappa l}(x_\mu,\ x_a)&=1/2\pi\int_0^{2\pi}d\theta\, e^{i(l+1/2)\theta+i\gamma_5\theta/2}\psi^{+}(x_\mu,\ \kappa l)\\&\qquad\times\exp[-i\kappa(x_5\sin\theta+x_6\cos\theta)]\\&=\tfrac{1}{2}(-i)^{l+1}e^{i(l+1/2)\varphi+i\gamma_5\varphi/2}\\&\qquad\times\{J_{l+1}(\kappa r)(1+\gamma_5)+iJ_l(\kappa r)(1-\gamma_5)\}\psi^{+}(x_\mu,\ \kappa l),\end{aligned}\quad(2\cdot7)$$

$$\begin{aligned}\Psi^{-}_{\kappa l}(x_\mu,\ x_a)&=1/2\pi\int_0^{2\pi}d\theta\, e^{-i(l+1/2)\theta+i\gamma_5\theta/2}\psi^{-}(x_\mu,\ \kappa l)\\&\qquad\times\exp[-i\kappa(x_5\sin\theta+x_6\cos\theta)]\\&=\tfrac{1}{2}(-i)^{l+1}e^{-i(l+1/2)\varphi+i\gamma_5\varphi/2}\\&\qquad\times\{iJ_l(\kappa r)(1+\gamma_5)+J_{l+1}(\kappa r)(1-\gamma_5)\}\psi^{-}(x_\mu,\ \kappa l),\end{aligned}\quad(2\cdot8)$$

where r and φ are the polar coordinates of $(x_5,\ x_6)$, $x_5=r\sin\varphi$, $x_6=r\cos\varphi$ and $\psi^{\pm}(x_\mu,\ \kappa l)$'s are the usual Dirac spinors satisfying the equation

$$(\gamma_\mu\cdot\partial/\partial x_\mu+\kappa)\psi^{\pm}(x_\mu,\ \kappa l)=0.\quad(2\cdot9)$$

The confirmation of the above solution will be given in the appendix III. Now, we can define the particle conjugate fields of $\Psi^{\pm}_{\kappa l}(x_\mu,\ x_a)$'s by

$$\Psi^{\pm A}_{\kappa l}(x_\mu,\ x_a)=C\bar{\Psi}^{\pm}_{\kappa l}(x_\mu,\ -x_a),\quad(2\cdot10)^*$$

so that they also satisfy the basic equation (2·1)

$$(\gamma_\mu\cdot\partial/\partial x_\mu+\gamma_5\cdot\partial/\partial x_5+\gamma_6\cdot\partial/\partial x_6)\Psi^{\pm A}_{\kappa l}(x_\mu,\ x_a)=0,\quad(2\cdot11)$$

and

$$j\Psi^{\pm A}_{\kappa l}(x_\mu,\ x_a)=\pm(l+\tfrac{1}{2})\Psi^{\pm A}_{\kappa l}(x_\mu,\ x_a).\quad(2\cdot12)$$

Therefore, one sees that the angular momentum of the antiparticle defined above is equal and opposite to that of the particle. The explicit form of $\Psi^{\pm A}_{\kappa l}(x_\mu, x_a)$'s are given by

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* $C\gamma_\mu^T C^{-1}=-\gamma_\mu$, $\bar{\Psi}\equiv\Psi^*\gamma_4$.

$$\Psi^{+A}_{\kappa l}(x_\mu, x_a)=\tfrac{1}{2}(-i)^{l+1}e^{-i(l+1/2)\varphi+i\gamma_5\varphi/2}$$
$$\times\{iJ_l(\kappa r)(1+\gamma_5)+J_{l+1}(\kappa r)(1-\gamma_5)\}\psi^{+A}(x_\mu, \kappa l), \tag{2·13}$$
$$\Psi^{-A}_{\kappa l}(x_\mu, x_a)=\tfrac{1}{2}(-i)^{l+1}e^{i(l+1/2)\varphi+i\gamma_5\varphi/2}$$
$$\times\{J_{l+1}(\kappa r)(1+\gamma_5)+iJ_l(\kappa r)(1-\gamma_5)\}\psi^{-A}(x_\mu, \kappa l), \tag{2·14}$$

where $\psi^{\pm A}(x_\mu, \kappa l)=C\overline{\psi}^{\pm}(x_\mu, \kappa l)$ and J denotes the Bessel function.

By making use of the spinor representation,

$$\gamma_5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad \alpha=\begin{pmatrix}\sigma & 0\\ 0 & -\sigma\end{pmatrix},\quad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{2·15}$$

we can rewrite $\Psi^{\pm}_{\kappa l}(x_\mu, x_a)$'s as follows,

$$\Psi^{\pm}_{\kappa l}(x_\mu, x_a)=(-i)^{l+1}\begin{pmatrix}ie^{il\varphi}J_l(\kappa r)u^+(x_\mu, \kappa l)\\ e^{i(l+1)\varphi}J_{l+1}(\kappa r)v^+(x_\mu, \kappa l)\end{pmatrix}, \tag{2·16}$$

$$\Psi^{-}_{\kappa l}(x_\mu, x_a)=(-i)^{l+1}\begin{pmatrix}e^{-i(l+1)\varphi}J_{l+1}(\kappa r)u^-(x_\mu, \kappa l)\\ ie^{-il\varphi}J_l(\kappa r)v^-(x_\mu, \kappa l)\end{pmatrix}, \tag{2·17}$$

where
$$\psi^{\pm}(x_\mu, \kappa l)=\begin{pmatrix}u^{\pm}(x_\mu, \kappa l)\\ v^{\pm}(x_\mu, \kappa l)\end{pmatrix}, \tag{2·18}$$

and $u^{\pm}$'s, v^{+}'s are the usual two-component spinors. Thus, in general, two kinds of solutions characterized by symbols ±, are obtained. Then, imposing the suitable boundary condition for $\Psi^{+}_{\kappa l}(x_\mu, x_a)$ and $\Psi^{-}_{\kappa l}(x_\mu, x_a)$ in the (x_5, x_6)-space, we get the corresponding two discrete mass spectra κ^{+}_{lm} and κ^{-}_{lm}. However, although we shall here not specify the detailed form of the boundary condition, it should be satisfied that this boundary condition must be invariant under the transformation R_M, that is, it must be dependent only on the radial variable r.

The general solution $\Psi_1(x_\mu, x_a)$ of (2·1) are given as follows,

$$\Psi_1(x_\mu, x_a)=\sum_{lm}\Psi^{+}_{lm}(x_\mu, x_a)+\sum_{lm}\Psi^{-}_{lm}(x_\mu, x_a), \tag{2·19}$$

where

$$\Psi^{+}_{lm}(x_\mu, x_a)=(-i)^{l+1}\begin{pmatrix}ie^{il\varphi}J_l(\kappa^{+}_{lm}r)u^+(x_\mu, \kappa^{+}_{lm})\\ e^{i(l+1)\varphi}J_{l+1}(\kappa^{+}_{lm}r)v^+(x_\mu, \kappa^{+}_{lm})\end{pmatrix}, \tag{2·20}$$

$$\Psi^{-}_{lm}(x_\mu, x_a)=(-i)^{l+1}\begin{pmatrix}e^{-i(l+1)\varphi}J_{l+1}(\kappa^{-}_{lm}r)u^-(x_\mu, \kappa^{-}_{lm})\\ ie^{-il\varphi}J_l(\kappa^{-}_{lm}r)v^-(x_\mu, \kappa^{-}_{lm})\end{pmatrix}, \tag{2·21}$$

and

$$\psi^{\pm}(x_\mu, \kappa^{\pm}_{lm})=\begin{pmatrix}u^{\pm}(x_\mu, \kappa^{\pm}_{lm})\\ v^{\pm}(x_\mu, \kappa^{\pm}_{lm})\end{pmatrix}. \tag{2·22}$$

Comparing the expressions (2·20) with (2·21), one finds that $\Psi^{+}_{lm}(x_\mu, x_a)$ can be changed into $\Psi^{-}_{lm}(x_\mu, x_a)$ only by replacing l, κ^{+}_{lm}, and $\psi^{+}(x_\mu, \kappa^{+}_{lm})$ with $-(l+1)$, κ^{-}_{lm}, and $\psi^{-}(x_\mu, \kappa^{-}_{lm})$ in the expression of $\Psi^{+}_{lm}(x_\mu, x_a)$ (2·20), and vice versa.

Now, as seen from (2·19), $\Psi_{\mathrm{I}}(x_\mu, x_a)$ is regarded as the unified field which involve two kinds of elementary particles with mass spectra κ^{+}_{lm} and κ^{-}_{lm}. However, it should be noticed that we can so relate these two kinds of fields, Ψ^{+} and Ψ^{-}, as to be particle-conjugate to each other. This is done by assuming that the Dirac fields $\psi^{+}(x_\mu, \kappa l)$ and $\psi^{-}(x_\mu, \kappa l)$ in (2·7) and (2·8) are in the relation of a particle conjugation to each other,

$$\psi^{+}(x_\mu, \kappa l)=\psi^{-A}(x_\mu, \kappa l)=C\bar{\psi}^{-}(x_\mu, \kappa l). \tag{2·23}$$

Then, by comparing (2·7) and (2·8) with (2·14), we easily find the following relations,

$$\left.\begin{aligned} &\Psi^{+}_{\kappa l}(x_\mu, x_a)=\Psi^{-A}_{\kappa l}(x_\mu, x_a), \\ \text{or}\quad &\Psi^{-}_{\kappa l}(x_\mu, x_a)=\Psi^{+A}_{\kappa l}(x_\mu, x_a). \end{aligned}\right\} \tag{2·24}$$

Further, in this case, only one discrete mass spectrum κ_{lm} may be obtained, since the behavior of one of $\Psi^{+}_{\kappa l}(x_\mu, x_a)$ and $\Psi^{-}_{\kappa l}(x_\mu, x_a)$ in the (x_5, x_6)-space must be determined from the boundary condition of the other one owing to the above relation (2·24). The corresponding unified field $\Psi_{\mathrm{II}}(x_\mu, x_a)$ becomes

$$\Psi_{\mathrm{II}}(x_\mu, x_a)=\sum_{lm}\Psi^{+}_{lm}(x_\mu, x_a)\ \Big(=\sum_{lm}\Psi^{-A}_{lm}(x_\mu, x_a)\Big). \tag{2·25}$$

Thus, we obtained two possible choices of the unified fields Ψ_{I} and Ψ_{II} from two alternative standpoints.* However it seems of course premature to say, at the present stage, which one is superior to the other.

Now, it should be noticed that our theory has a close relation to the theory of the two component neutrino proposed by Lee and Yang,[3)] and also Landau.[4)] In fact, if in our theory the mass spectrum can involve the vanishing mass, which may correspond to the neutrino, the angular momentum j of its mode must be $\pm\frac{1}{2}$, e. g., $l=0$, as easily seen from (2·7) or (2·8) and the corresponding field becomes

$$\left.\begin{aligned} \Psi^{+}(x_\mu, x_a)&=\tfrac{1}{2}(1-\gamma_5)\psi^{+}(x_\mu, \kappa=0)=\begin{pmatrix} u^{+}(x_\mu, \kappa=0) \\ 0 \end{pmatrix}, \\ \Psi^{+A}(x_\mu, x_a)&=\tfrac{1}{2}(1+\gamma_5)\psi^{+A}(x_\mu, \kappa=0)=\begin{pmatrix} 0 \\ \sigma_2 u^{+*}(x_\mu, \kappa=0) \end{pmatrix}, \end{aligned}\right\} \tag{2·26}$$

or

$$\left.\begin{aligned} \Psi^{-}(x_\mu, x_a)&=\tfrac{1}{2}(1+\gamma_5)\psi^{-}(x_\mu, \kappa=0)=\begin{pmatrix} 0 \\ u^{-}(x_\mu, \kappa=0) \end{pmatrix}, \\ \Psi^{-A}(x_\mu, x_a)&=\tfrac{1}{2}(1-\gamma_5)\psi^{-A}(x_\mu, \kappa=0)=\begin{pmatrix} -\sigma_2 u^{-*}(x_\mu, \kappa=0) \\ 0 \end{pmatrix}. \end{aligned}\right\} \tag{2·27}$$

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* We postulate the second quantization of the unified field Ψ_{I} or Ψ_{II} is performed by taking the usual commutation relation for the Dirac fields $\psi^{\pm}(x_\mu, \kappa^{\pm}_{lm})$, which are involved in the expression of the unified fields, that is, $[\psi^{\varepsilon}(x_\mu, \kappa^{\varepsilon}_{lm}), \bar{\psi}^{\varepsilon'}(x_\mu', \kappa^{\varepsilon'}_{l'm'})]_{+}=1/i\cdot S(x-x')\times\delta_{\varepsilon\varepsilon'}\delta_{lm,l'm'}$, where ε (or ε') denotes the symbols $\pm$.

This is just the result of the theory of the above authors.

At the end of this section, we shall show that the rotation R_M $(2\cdot 4)$ can be represented by a unitary transformation U_α as follows,

$$U_\alpha \Psi(x_\mu,\ x_a')U_\alpha^{-1}=\exp(-\gamma_5\gamma_6\,\alpha/2)\,\Psi(x_\mu,\ x_a), \qquad (2\cdot 28)$$

or

$$U_\alpha \Psi_{lm}^{\pm}(x_\mu,\ x_a')U_\alpha^{-1}=\exp(-\gamma_5\gamma_6\,\alpha/2)\,\Psi_{lm}^{\pm}(x_\mu,\ x_a). \qquad (2\cdot 29)$$

It is easily verified that such a unitary operator U_α can be determined by

$$U_\alpha \psi^{\pm}(x_\mu,\ \kappa_{lm}^{\pm})U_\alpha^{-1}=e^{\pm i(l+1/2)\varphi}\,\psi^{\pm}(x_\mu,\ \kappa_{lm}^{\pm}). \qquad (2\cdot 30)$$

Therefore, from $(2\cdot 7)$ and $(2\cdot 8)$, we obtain

$$U_\alpha \Psi_{lm}^{\pm}(x_\mu,\ x_a)U_\alpha^{-1}=e^{\pm i(l+1/2)\alpha}\,\Psi_{lm}^{\pm}(x_\mu,\ x_a). \qquad (2\cdot 31)$$

Thus, one sees that the rotation R_M is equivalent to the gauge transformation of the first kind.

In this section, we have discussed the general solution of the basic equation, based on our fundamental postulate in § 1 and consequently representing the elementary particles by the eigensolution of the mass operator and the angular momentum in the new degrees of freedom, both of which are invariant under the rotation R_M. In the appendix I, modifying our fundamental postulate slightly, we shall investigate the other possible picture of the elementary particles, where the elementary particle is specified by the plane wave solution with the square boundary in the new degrees of freedom.

§ 3. Interactions of elementary particles

In this section, we shall investigate the interaction of elementary particles, taking the non-linear interaction of the Fermi type for the unified field. In order to set up the interaction Hamiltonian of the Fermi type, which must be invariant under R_L and R_M from our fundamental principle, at first we have to form the bilinear quantities of the unified fields, which are irreducible representation of R_L and R_M. These quantities are given in Table I. It may be easily confirmed, by making use of the transformation property of the unified field $(2\cdot 3)$ and $(2\cdot 4)$, that each quantity has the transformation property shown in the Table.

Table I

type		R_L	R_M	type		R_L	R_M
I	$\overline{\Psi}\gamma_a\Psi$	scalar	vector	I′	$\overline{\Psi}\gamma_a\partial_a\Psi$	scalar	scalar
II	$\overline{\Psi}\gamma_\mu\Psi$	vector	scalar	II′	$\overline{\Psi}\gamma_\mu\partial_a\Psi$	vector	vector
III	$i\overline{\Psi}\gamma_5\gamma_\mu\Psi$			III′	$i\overline{\Psi}\gamma_5\gamma_\mu\partial_a\Psi$		
IV	$\overline{\Psi}\gamma_{[\mu}\gamma_{\nu]}\gamma_a\Psi$	tensor	vector	IV′	$\overline{\Psi}\gamma_{[\mu}\gamma_{\nu]}\gamma_a\partial_a\Psi$	tensor	scalar

$$\gamma_{[\mu}\gamma_{\nu]}=\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu),\quad \mu=1,\ 2,\ 3,\ 4,\ a=5,\ 6.$$

The interaction Hamiltonians of the Fermi type can be obtained as the product of each quantity of this Table, that is,

$$H^{(A)}(x_\mu) = g_A \int \bar{\Psi}\Omega_A \Psi \bar{\Psi}\Omega_A \Psi \, dx_5 dx_6, \tag{3·1}$$

and

$$\begin{aligned} H^{(A)\prime}(x_\mu) &= f_A \int \bar{\Psi}\Omega_A(\vec{\partial})\Psi \bar{\Psi}\Omega_A(\vec{\partial})\Psi \, dx_5 dx_6 \\ &\quad + f_A^* \int \bar{\Psi}\Omega_A(\overleftarrow{\partial})\Psi \bar{\Psi}\Omega_A(\overleftarrow{\partial})\Psi \, dx_5 dx_6, \end{aligned} \tag{3·2}$$

where A's denote I, II, III, and IV and

$$\left.\begin{aligned} &\Omega_{\rm I} = \gamma_a, \ \Omega_{\rm II} = \gamma_\mu, \ \Omega_{\rm III} = i\gamma_5\gamma_\mu, \ \Omega_{\rm IV} = \gamma_{[\mu}\gamma_{\nu]}\gamma_a \\ &\Omega_{\rm I}(\partial) = \gamma_a\partial_a, \ \Omega_{\rm II}(\partial) = \gamma_\mu\partial_a, \ \Omega_{\rm III}(\partial) = i\gamma_5\gamma_\mu\partial_a, \ \Omega_{\rm IV}(\partial) = \gamma_{[\mu}\gamma_{\nu]}\gamma_a\partial_a, \end{aligned}\right\} \tag{3·3}$$

and a repeated suffix in a term implies a summation as usual. If we take $\Psi_{\rm I}$ of (2·19) as the unified field, then the interaction Hamiltonian is expressed as a linear combination of the interaction terms of elementary particles,

$$g\int dx_5 dx_6 \bar{\Psi}_{\rm I}\Omega\Psi_{\rm I}\bar{\Psi}_{\rm I}\Omega\Psi_{\rm I} = \sum_{\varepsilon_i=\pm}\sum_{l_i m_i} g\int dx_5 dx_6 \bar{\Psi}^{\varepsilon_1}_{l_1 m_1}\Omega\Psi^{\varepsilon_2}_{l_2 m_2}\bar{\Psi}^{\varepsilon_3}_{l_3 m_3}\Omega\Psi^{\varepsilon_4}_{l_4 m_4}, \tag{3·4}$$

where Ω is to represent symbolically arbitrary one of (3·3). It is sufficient, however, for the consideration of the general terms to investigate only the special term of (3·4),

$$H(l_i m_i) = g\int \bar{\Psi}^{+}_{l_1 m_1}\Omega\Psi^{+}_{l_2 m_2}\bar{\Psi}^{+}_{l_3 m_3}\Omega\Psi^{+}_{l_4 m_4} dx_5 dx_6, \tag{3·5}$$

from the following reasons. At first, the other terms of (3·4), for instance $g\int \bar{\Psi}^{-}_{l_1 m_1} \Omega\Psi^{+}_{l_2 m_2}\bar{\Psi}^{+}_{l_3 m_3}\Omega\Psi^{+}_{l_4 m_4} dx_5 dx_6$ can be obtained directly from (3·5), using the relation between $\Psi^{-}_{lm}(x_\mu, x_a)$ and $\Psi^{+}_{lm}(x_\mu, x_a)$ as stated in the preceding section, by substituting $-(l_1+1)$, $\kappa^{-}_{l_1 m_1}$ and $\psi^{-}(x_\mu, \kappa^{-}_{l_1 m_1})$ for l_1, $\kappa^{+}_{l_1 m_1}$ and $\psi^{+}(x_\mu, \kappa^{+}_{l_1 m_1})$ in the every expression derived from (3·5). Secondly, even if we take $\Psi_{\rm II}$ in (2·25) as the unified field in place of $\Psi_{\rm I}$ the interaction Hamiltonian $g\int \bar{\Psi}_{\rm II}\Omega\Psi_{\rm II}\bar{\Psi}_{\rm II}\Omega\Psi_{\rm II}$ consists only of the term of (3·5) as seen from (2·25).

By using the expression (2·20) of $\Psi^{+}_{lm}(x_\mu, x_a)$ and by denoting $\kappa_{l_i m_i}$ as κ_i for brevity, the interaction terms, $H^{(A)}(l_i m_i)$'s and $H^{(A)\prime}(l_i m_i)$'s can be expressed as follows,

$$\begin{aligned} H^{(A)}(l_i m_i) &= \delta(l_1 - l_2 + l_3 - l_4) \\ &\quad \times [g_A(l_i m_i)\bar{\psi}_1\gamma_A\psi_2\bar{\psi}_3\gamma_A\psi_4 + \bar{g}_A(l_i m_i)\bar{\psi}_1\gamma_A\gamma_5\psi_2\bar{\psi}_3\gamma_A\gamma_5\psi_4 \\ &\quad + g_A'(l_i m_i)\bar{\psi}_1\gamma_A\gamma_5\psi_2\bar{\psi}_3\gamma_A\bar{\psi}_4 + \bar{g}_A'(l_i m_i)\bar{\psi}_1\gamma_A\psi_2\bar{\psi}_3\gamma_A\gamma_5\psi_4], \end{aligned} \tag{3·6}$$

$$\begin{aligned} H^{(A)\prime}(l_i m_i) &= \delta(l_1 - l_2 + l_3 - l_4) \\ &\quad \times [f_A(l_i m_i)\bar{\psi}_1\gamma_A\psi_2\bar{\psi}_3\gamma_A\psi_4 + \bar{f}_A(l_i m_i)\bar{\psi}_1\gamma_A\gamma_5\psi_2\bar{\psi}_3\gamma_A\gamma_5\psi_4 \\ &\quad + f_A'(l_i m_i)\bar{\psi}_1\gamma_A\gamma_5\psi_2\bar{\psi}_3\gamma_A\psi_4 + \bar{f}_A'(l_i m_i)\bar{\psi}_1\gamma_A\psi_2\bar{\psi}_3\gamma_A\gamma_5\psi_4], \end{aligned} \tag{3·7}$$

where

$$\left.\begin{aligned}\delta(x)&=1 \quad \text{if } x=0\\ &=0 \quad \text{if } x\neq 0\end{aligned}\right\},$$

$$\gamma_{A=\mathrm{I,II,III,IV}}=(1,\ \gamma_\mu,\ i\gamma_5\gamma_\mu,\ \gamma_{[\mu}\gamma_{\nu]}) \tag{3·8}$$

and ψ_i's are the abbreviated expressions of $\psi^{\pm}(x_\mu,\ \kappa_i)$'s. The explicit forms of the effective couplings $g_A(l_i m_i)$, $\bar{g}_A(l_i m_i)$, $g_A'(l_i m_i)$, $\bar{g}_A'(l_i m_i)$ and $f_A(l_i m_i)$, $\bar{f}_A(l_i m_i)$, $f_A'(l_i m_i)$, $\bar{f}_A'(l_i m_i)$ are given in the appendix II. Here we shall refer only the result from (I) and (I)$'$:

$$\text{(I)}:\ \begin{Bmatrix} g_{\mathrm{I}}\ (l_i m_i) \\ g_{\mathrm{I}}'(l_i m_i)\end{Bmatrix}=-\begin{Bmatrix} \bar{g}_{\mathrm{I}}\ (l_i m_i) \\ \bar{g}_{\mathrm{I}}'(l_i m_i)\end{Bmatrix}=g_{\mathrm{I}}\pi\int r\,dr$$

$$\times[J_{l_1+1}(\kappa_1 r)J_{l_2}(\kappa_2 r)J_{l_3}(\kappa_3 r)J_{l_4+1}(\kappa_4 r)$$

$$\pm J_{l_1}(\kappa_1 r)J_{l_2+1}(\kappa_2 r)J_{l_3+1}(\kappa_3 r)J_{l_4}(\kappa_4 r)], \tag{3·9}$$

$$\text{(I)}':\ \begin{Bmatrix} f_{\mathrm{I}}(l_i m_i) \\ \bar{f}_{\mathrm{I}}(l_i m_i)\end{Bmatrix}=\frac{\pi}{2}\,(f_{\mathrm{I}}\kappa_2\kappa_4+f_{\mathrm{I}}^*\kappa_1\kappa_3)\int r\,dr$$

$$\times[J_{l_1}(\kappa_1 r)J_{l_2}(\kappa_2 r)\pm J_{l_1+1}(\kappa_1 r)J_{l_2+1}(\kappa_2 r)]$$

$$\times[J_{l_3}(\kappa_3 r)J_{l_4}(\kappa_4 r)\pm J_{l_3+1}(\kappa_3 r)J_{l_4+1}(\kappa_4 r)], \tag{3·10}$$

$$\begin{Bmatrix} f_{\mathrm{I}}'(l_i m_i) \\ \bar{f}_{\mathrm{I}}'(l_i m_i)\end{Bmatrix}=\frac{\pi}{2}\,(-f_{\mathrm{I}}\kappa_2\kappa_4+f_{\mathrm{I}}^*\kappa_1\kappa_3)\int r\,dr$$

$$\times[J_{l_1}(\kappa_1 r)J_{l_2}(\kappa_2 r)\mp J_{l_1+1}(\kappa_1 r)J_{l_2+1}(\kappa_2 r)]$$

$$\times[J_{l_3}(\kappa_3 r)J_{l_4}(\kappa_4 r)\pm J_{l_3+1}(\kappa_3 r)J_{l_4+1}(\kappa_4 r)]. \tag{3·11}$$

Now, let us tentatively summarize the main properties of the results derived above.

a) From the appearance of the factor $\delta(l_1-l_2+l_3-l_4)$ in the interaction term, we find here exists the strict conservation of the angular momentun in the new degrees of freedom, that is, $j_1+j_3=j_2+j_4$, where $j_i=l_i+\frac{1}{2}$ or $-(l_i+\frac{1}{2})$ corresponding to ψ^+-field or ψ^--field. As seen from (2·31), this is the direct consequence from the invariance of our theory under the rotation R_M.

b) In general, however, the violation of the parity conservation occurs, with the exception of the case discussed in d). This is shown by the coexistence of the even coupling and the odd coupling. As seen from the appendix II, the mixing ratio of both couplings is definitely determined theoretically from the mass structures of elementary particles interacting on each other.

c) It is remarkable that the effective coupling constants $g_A(l_i m_i)$, $\bar{g}_A(l_i m_i)$, $g_A'(l_i m_i)$ and $\bar{g}_A'(l_i m_i)$ which come from the interactions without derivatives $H^{(\mathrm{I})}$, $H^{(\mathrm{II})}$, $H^{(\mathrm{III})}$ and $H^{(\mathrm{IV})}$ are always real and that $f_A(l_i m_i)$, $\bar{f}_A(l_i m_i)$, $f_A'(l_i m_i)$ and $\bar{f}_A'(l_i m_i)$ which come from the interactions with derivatives $H^{(\mathrm{I})'}$, $H^{(\mathrm{II})'}$, $H^{(\mathrm{III})'}$, and $H^{(\mathrm{IV})'}$ also become real if the elementary coupling constants f_A's are real. Therefore, we find that the theory is invariant under the Wigner time reversal or the combined inversion of the space reflection and the particle conjugation in agreement with the theory of Landau,[4] when the elementary coupling constants g_A's and f_A's of the interactions $(3\cdot1)$ and $(3\cdot2)$ are real, of which the reality of the formers, necessarily comes from the hermiticity of the Hamiltonian. This statement can be also proved directly by using the property of the unified field. In fact, using $(2\cdot19)$, $(2\cdot25)$, $(2\cdot7)$, and $(2\cdot8)$, we can show first that

$$(R_W \Psi_{\mathrm{I}}(x_i,\ -x_4,\ x_a) R_W^{-1})^T = \rho\gamma_5\gamma_4 C\overline{\Psi}_{\mathrm{I}}(x_\mu,\ x_5,\ -x_6)$$

$$(=\rho\gamma_5\gamma_4 \Psi_{\mathrm{I}}{}^A(x_\mu,\ -x_5,\ x_6)), \tag{3·12}$$

$$(R_W \Psi_{\mathrm{II}}(x_i,\ -x_4,\ x_a) R_W^{-1})^T = \rho\gamma_5\gamma_4 C\overline{\Psi}_{\mathrm{II}}(x_\mu,\ x_5,\ -x_6)$$

$$(=\rho\gamma_5\gamma_4 \Psi_{\mathrm{II}}{}^A(x_\mu,\ -x_5,\ x_6)), \tag{3·12$'$}$$

where R_W is the unitary operator concerning the usual Wigner time reversal[10] defined by

$$(R_W \psi^\pm(x_i,\ -x_4,\ \kappa_{lm}) R_W^{-1})^T = \rho\gamma_5\gamma_4 C\overline{\psi}^\pm(x_\mu,\ \kappa_{lm}), \tag{3·13}$$

and ρ is an arbitrary phase factor with the modulus 1. Therefore, by using $(3\cdot12)$ and $(3\cdot12)'$ and the reality of f_A's, we can show the invariance of the theory, that is,[10]

$$\left.\begin{aligned} (R_W H^{(A)}(x_i,\ -x_4) R_W^{-1})^T &= H^{(A)}(x_\mu), \\ (R_W H^{(A)\prime}(x_i,\ -x_4) R_W^{-1})^T &= H^{(A)\prime}(x_\mu). \end{aligned}\right\} \tag{3·14}$$

In the same way, we can show the invariance of our theory under the combined inversion of the space reflection and the particle conjugation. However, our theory cannot as a whole be invariant under the other inversions such as the space reflection, the Pauli time reversal or the strong reflection[11] of the space and time, since the unified field can not have such a simple property as $(3\cdot12)$ and $(3\cdot12)'$ under those inversions. Of course, this does not mean that any interaction term $H(l_i m_i)$ cannot be invariant under those inversions, but it is shown below that the special interaction term can be invariant under any inversions.

d) Let us consider the case where four particles interacting on each other belong to the same kind of particle, that is, they have same quantum numbers $(l,\ m)$. The interactions, $\bar{N}N\bar{N}N$ and $\bar{\Lambda}\Lambda\bar{\Lambda}\Lambda$ in $(1\cdot1)$, which are characterized as the strong interaction, belong to such a case. Therefore, it is desirable that, in such a case, the parity mixing disappears. In fact, taking into account the explicit form of the effective couplings $(\mathrm{A}2\cdot1)$–$(\mathrm{A}2\cdot9)$, we find that the odd coupling constants $\bar{g}_A'$, g_A', $\bar{f}_A'$, f_A' vanish completely in this case if we set f_A's to be real and $g_{\mathrm{II}}=g_{\mathrm{III}}$ and $f_{\mathrm{II}}=f_{\mathrm{III}}$. However, even if such a condition of the coupling constants is assumed, the parity mixing takes place for the interaction as $\bar{\Lambda}\Lambda\bar{N}N$ in $(1\cdot1)$, which is considered to be responsible for the strong coupling concerning K-meson. However, since at the present stage of experiment we have no definite knowledge of K-meson, it seems premature to speak if the violation of the parity conservation in this interaction is undesirable.

e) As seen from $(3\cdot9)$–$(3\cdot11)$ or $(\mathrm{A}2\cdot1)$–$(\mathrm{A}2\cdot9)$, the strength of coupling of the elementary particles is decided by the magnitude of the elementary coupling constants

g_A's, f_A's, and the mass structure of the elementary particles. Therefore, the condition of the overlapping of the mass wave functions and the magnitude of the masses of the four interacting particles may be responsible for the distinction of the strength of coupling. About the first argument, we may say qualitatively that for the interactions as (i)–$(1\cdot 2)$, where several kinds of particles having a large mass difference from each other take place in somewhat irregular way, the strength of coupling may become weak in camparison with that of $(2\cdot 1)$, because of the unfavorable overlapping of the mass wave functions. The detailed argument about this point will be given in the next section.

f) It is worthwhile to notice that the tensor coupling, the existence of which has been confirmed in the β-coupling,[12] is obtained only from the derivative couplings $H^{(1)\prime}$ and $H^{(\mathrm{IV})\prime}$. At first sight, the tensor coupling may be obtained also from the scalar and pseudoscalar couplings in $H^{(1)}(l_i m_i)$, if we exchange the order of the spinor fields in the Fermi coupling following the treatment given by Fierz.[13] However, two tensor couplings derived from the scalar and the pseudoscalar couplings in this way cancel each other since, as seen from $(3\cdot 6)$ and $(3\cdot 9)$, the scalar and the pseudoscalar couplings in $H^{(1)}$ occur in the equal weight and the opposite sign.

§ 4. Summary and outlook

With the intention to explain the inner relation which may exist between the strength of couplings and the violation of the parity conservation, the six dimensional equation was proposed in place of the usual four dimensional Dirac equation. The elementary particle is represented as a special solution of this basic equation, which is characterized by the mass and the angular momentum in the new degrees of freedom. The two component neutrino is obtained as the special case when the mass spectrum involves the vanishing mass. From the Fermi interactions of elementary particles, which is constructed basing on the symmetry in the new degrees of freedom, we find that, in general, the parity mixing occurs in a definite rate and that, if we impose the simple condition to the elementary coupling constants, then the theory becomes invariant under the Wigner time reversal and in the interaction among the identical particles, the parity mixing disappears.

Now, at a moment, let us focus our attention on the strength of interactions. We can empirically classify the interactions into the strong and the weak couplings as $(1\cdot 1)$ and $(1\cdot 2)$, excluding the electromagnetic interaction. As remarked in the introduction, we should, however, distinguish the weak interactions of the first and the second kinds. From the starting point of view of our theory, it is expected that at least the weakness of the interaction of the first kind in comparison with the strong interaction of $(1\cdot 1)$ can be explained by using the theoretical distinction between the baryon and lepton fields. In fact, we can distinguish the baryon and the lepton, for instance, by the magnitude of the angular momentum j, and masses in our theory. Then, as seen from the argument in e) in the preceding section, the difference of the magnitude of the angular momentum and of the masses between baryon and lepton or among leptons themselves affects to make the overlapping of the mass wave function in the weak interaction of the first kind to be unfavorable and to weaken the effective coupling compared with the strong interaction. Further, if we adopt the derivative interactions of (3·2), the factors concerning the mass appears explicitly in the effective coupling constants as shown in (3·10) and (3·11) or (A2·5)–(A2·10), and they also, in addition to the above cause, affect to weaken the strength of the weak interaction of the first kind compared with that of the strong interaction of (1·1), by the order of (κ_l/κ_h) or $(\kappa_l/\kappa_h)^2$, where κ_l and κ_h represent the masses of the lepton and the baryon in an average sense. There seem to exist more reasons which make it preferable to adopt the derivative couplings rather than the non-derivative ones. At first, the mass factor in the effective couplings characteristic to the derivative coupling makes it possible to weaken the strength of the interactions such as $\bar{\mu}\mu\bar{\mu}\mu$, $\bar{e}e\bar{e}e$, or $\bar{\mu}\mu\bar{e}e$, which are experimentally undesirable interactions, by the order of $(\kappa_l/\kappa_h)^2$ compared with that of the strong interactions while otherwise the strength of the former interactions becomes comparable to that of the latter ones so far as we take into account only the effect of the overlapping of the mass wave functions. Secondly, as stated in f) in § 3, only the derivative couplings, $H^{(\mathrm{I})'}$ and $H^{(\mathrm{IV})'}$, enable to give the tensor coupling.

Now, consider the weak interaction of the second kind, which is concerned only with the baryon fields contrary to that of the first kind. Therefore, in order to explain theoretically the distinction of the coupling strength between the strong interactions of (1·1) and the weak interaction of the second kind, both of which are concerned with the baryon fields only, we have to provide an unequivocal difference between the nucleon and the Λ-particle in our theory itself. However, taking into account the conservation of the angular momentum j in our theory, we find that the nucleon and the Λ-field must have a same angular momentum and consequently they have no definite qualitative difference beside the slight mass difference. On the other hand, in the theories[7)] of Nishijima and Gell-Mann, the nucleon and the Λ-particle are distinguished by the strangeness or the isotopic spin, and the weak interaction of the second kind is just the case where the conservation of the strangeness or the isotopic spin is violated. It is, however, evident from the beginning as remarked in the introduction that the present theory involves such a limit of applicability, so far as we do not introduce some entity concerned with the strangeness or isotopic spin beside the parity. We are now investigating several possibilities to generalize our basic equation so as to involve the isotopic spin according to a suggestion by Dr. T. Nakano.

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Appendix I

In this appendix, we shall consider another specification of the elementary particles by imposing a square periodic boundary upon the unified field in place of the circular one treated in the text. Then, the elementary particle can be represented as a plane wave solution $\Psi_{mn}(x_\mu, x_a)$ characterized by

$$\left.\begin{aligned}\frac{\partial}{\partial x_5}\Psi_{mn}(x_\mu,\ x_a)&=-2\pi i\frac{m}{a}\Psi_{mn}(x_\mu,\ x_a),\\ \frac{\partial}{\partial x_6}\Psi_{mn}(x_\mu,\ x_a)&=-2\pi i\frac{n}{b}\Psi_{mn}(x_\mu,\ x_a),\end{aligned}\right\}\qquad(\mathrm{A1\cdot 1})$$

$$(m,\ n=0,\ \pm 1,\ \pm 2,\ \cdots)$$

where a and b denote periodicities of x_5 and x_6 variables, respectively. Thus, the mass of the elementary particle, κ_{mn}, is given as follows,

$$(\partial^2/\partial x_5^2+\partial^2/\partial x_6^2)\Psi_{mn}(x_\mu,\ x_a)=-\kappa_{mn}^2\Psi_{mn}(x_\mu,\ x_a),\qquad(\mathrm{A1\cdot 2})$$

$$\kappa_{mn}=2\pi\{(m/a)^2+(n/b)^2\}^{1/2}.\qquad(\mathrm{A1\cdot 3})$$

Now, from (A1·1) $\Psi_{mn}(x_\mu,\ x_a)$ has the form

$$\Psi_{mn}(x_\mu,\ x_a)=A_{mn}(x_\mu)\exp[-2\pi i(mx_5/a+nx_6/b)].\qquad(\mathrm{A1\cdot 4})$$

Substituting the above expression for the basic equation (2·1), we have

$$(\gamma_\mu\cdot\partial/\partial x_\mu-2\pi i\gamma_5\, m/a+2\pi\, n/b)A_{mn}(x_\mu)=0.\qquad(\mathrm{A1\cdot 5})$$

By using (A1·3) and

$$\left.\begin{aligned}\cos\alpha_{mn}&\equiv n/b\sqrt{(m/a)^2+(n/b)^2},\\ \sin\alpha_{mn}&\equiv m/a\sqrt{(m/a)^2+(n/b)^2},\end{aligned}\right\}\qquad(\mathrm{A1\cdot 6})$$

(A1·5) can be rewritten as

$$(\gamma_\mu\cdot\partial/\partial x_\mu+\kappa_{mn}\,e^{-i\gamma_5\alpha_{mn}})A_{mn}(x_\mu)=0.\qquad(\mathrm{A1\cdot 7})$$

On the other hand, one finds easily

$$(\gamma_\mu\cdot\partial/\partial x_\mu+\kappa_{mn}\,e^{-i\gamma_5\alpha_{mn}})\,e^{i\gamma_5\alpha_{mn}/2}\psi(x_\mu,\ \kappa_{mn})=0,\qquad(\mathrm{A1\cdot 8})$$

where $\psi(x_\mu,\ \kappa_{mn})$ is a usual Dirac field satisfying the equation

$$(\gamma_\mu\cdot\partial/\partial x_\mu+\kappa_{mn})\psi(x_\mu,\ \kappa_{mn})=0.\qquad(\mathrm{A1\cdot 9})$$

By comparing (A1·7) with (A1·8), we obtain the final expression of $\Psi_{mn}(x_\mu,\ x_a)$,

$$\Psi_{mn}(x_\mu,\ x_a)=e^{i\gamma_5\alpha_{mn}/2}\psi(x_\mu,\ \kappa_{mn})\exp[-2\pi i(mx_5/a+nx_6/b)].\qquad(\mathrm{A1\cdot 10})$$

The general solution $\Psi_{\mathrm{III}}(x_\mu,\ x_a)$ can be represented as a linear superposition of $\Psi_{mn}(x_\mu,\ x_a)$,

$$\Psi_{\mathrm{III}}(x_\mu,\ x_a)=\sum_{mn}\Psi_{mn}(x_\mu,\ x_a).\qquad(\mathrm{A1\cdot 11})$$

Before investigating the interaction in detail, we shall consider the transformation property of the unified fields $\Psi_{\mathrm{III}}(x_\mu,\ x_a)$ under several inversions. Contrary to the unified fields Ψ_{I} and Ψ_{II}, Ψ_{III} has a simple transformation property under the particle conjugation, but not under the Wigner time reversal. In fact, under the particle conjugation R_A which is defined by[14)]

$$R_A\psi(x_\mu,\ \kappa_{mn})R_A^{-1}=C\overline{\psi}(x_\mu,\ \kappa_{mn}), \tag{A1·12}$$

$\Psi_{mn}(x_\mu,\ x_a)$ and $\Psi_{\text{III}}(x_\mu,\ x_a)$ are transformed as follows,

$$R_A\Psi_{mn}(x_\mu,\ x_a)R_A^{-1}=C\overline{\Psi}_{mn}(x_\mu,\ -x_a)\ (=\Psi^A_{mn}(x_\mu,\ x_a)), \tag{A1·13}$$

and

$$R_A\Psi_{\text{III}}(x_\mu,\ x_a)R_A^{-1}=C\overline{\Psi}_{\text{III}}(x_\mu,\ -x_a)\ (=\Psi^A_{\text{III}}(x_\mu,\ x_a)). \tag{A1·14}$$

Therefore, we find that the interactions of elementary particles coming from either non-derivative interactions of the form of (3·1) or derivative interactions of the form of (3·2) with a real coupling constants ($f_A=f_A{}^*$) are invariant under the particle conjugation. On the other hand, as mentioned in § 1, the experiments show the violation of the invariance under the particle conjugation as well as the space reflection. Thus, in order to violate the invariance under the particle conjugation, we have to adopt the derivative interactions for the interaction of Ψ_{III} and to take the complex elementary coupling constants f_A's.

Here, we like to notice that if we assume the plane wave solution with the square boundary of $(x_5,\ x_6)$-space which is not invariant under R_M, it is necessary to modify the fundamental postulate adopted in the text by replacing the requirement of the invariance of the theory under the continuous transformation R_M by that of the especial angles under which the boundary remains invariant.

In the similar way as treated in § 3, we can obtain the coupling terms, $H^{(A)\prime}(m_in_i)$'s coming from the derivative interactions $H^{(\text{I})\prime}$, $H^{(\text{II})\prime}$, $H^{(\text{III})\prime}$, and $H^{(\text{IV})\prime}$ of (3·2) concerned with the unified field Ψ_{III}. The result is given as follows:

$$\begin{aligned}H^{(A)\prime}(m_in_i)&=\delta(m_1-m_2+m_3-m_4)\,\delta(n_1-n_2+n_3-n_4)\\&\times[h_A(m_in_i)\overline{\psi}_1\gamma_A\psi_2\overline{\psi}_3\gamma_A\psi_4+\overline{h}_A(m_in_i)\overline{\psi}_1\gamma_A\gamma_5\psi_2\overline{\psi}_3\gamma_A\gamma_5\psi_4\\&+h_A'(m_in_i)\overline{\psi}_1\gamma_A\gamma_5\psi_2\overline{\psi}_3\gamma_A\psi_4+\overline{h}_A'(m_in_i)\overline{\psi}_1\gamma_A\psi_2\overline{\psi}_3\gamma_A\gamma_5\psi_4].\end{aligned} \tag{A1·15}$$

The effective coupling constants $h_A(m_in_i)\cdots$, $\overline{h}_A'(m_in_i)$ are given in the appendix II.

Appendix II

(α) The effective coupling constants in (3·6), (3·7).

(I)
$$\begin{Bmatrix}g_{\text{I}}\ (l_im_i)\\ g_{\text{I}}'(l_im_i)\end{Bmatrix}=-\begin{Bmatrix}\bar{g}_{\text{I}}\ (l_im_i)\\ \bar{g}_{\text{I}}'(l_im_i)\end{Bmatrix}$$
$$=\pi g_{\text{I}}\int r\,dr\,(J_{l_1+1}J_{l_2}J_{l_3}J_{l_4+1}\pm J_{l_1}J_{l_2+1}J_{l_3+1}J_{l_4}). \tag{A2·1}$$

(II) and (III) (A=II, III)

$$\begin{Bmatrix}g_A(l_im_i)\\ \bar{g}_A(l_im_i)\end{Bmatrix}=\frac{\pi}{2}g_A\int r\,dr$$
$$\times(J_{l_1+1}J_{l_2+1}\pm J_{l_1}J_{l_2})(J_{l_3+1}J_{l_4+1}\pm J_{l_3}J_{l_4}). \tag{A2·2}$$

$$\left\{\begin{matrix} g_A'(l_i m_i) \\ \bar{g}_A'(l_i m_i) \end{matrix}\right\} = -\frac{\pi}{2} g_A \int r\,dr$$

$$(J_{l_1+1}J_{l_2+1} \mp J_{l_1}J_{l_2})(J_{l_3+1}J_{l_4+1} \pm J_{l_3}J_{l_4}). \tag{A2·3}$$

(IV) $$g_{\mathrm{IV}}(l_i m_i) = \bar{g}_{\mathrm{IV}}(l_i m_i) = g_{\mathrm{IV}}'(l_i m_i) = \bar{g}_{\mathrm{IV}}'(l_i m_i) = 0. \tag{A2·4}$$

(I)′ $$\left\{\begin{matrix} f_{\mathrm{I}}(l_i m_i) \\ \bar{f}_{\mathrm{I}}(l_i m_i) \end{matrix}\right\} = \frac{\pi}{2}(f_{\mathrm{I}}\kappa_2\kappa_4 + f_{\mathrm{I}}^*\kappa_1\kappa_3)\int r\,dr$$

$$\times (J_{l_1}J_{l_2} \pm J_{l_1+1}J_{l_2+1})(J_{l_3}J_{l_4} \pm J_{l_3+1}J_{l_4+1}). \tag{A2·5}$$

$$\left\{\begin{matrix} f_{\mathrm{I}}'(l_i m_i) \\ \bar{f}_{\mathrm{I}}'(l_i m_i) \end{matrix}\right\} = \frac{\pi}{2}(-f_{\mathrm{I}}\kappa_2\kappa_4 + f_{\mathrm{I}}^*\kappa_1\kappa_3)\int r\,dr$$

$$\times (J_{l_1}J_{l_2} \mp J_{l_1+1}J_{l_2+1})(J_{l_3}J_{l_4} \pm J_{l_3+1}J_{l_4+1}). \tag{A2·6}$$

(II)′ and (III)′ (A=II, III)

$$\left\{\begin{matrix} f_A(l_i m_i) \\ \bar{f}_A(l_i m_i) \end{matrix}\right\} = -\frac{\pi}{4}\kappa_2\kappa_4 f_A \int r\,dr$$

$$\times \{(J_{l_1}J_{l_2-1} \pm J_{l_1+1}J_{l_2})(J_{l_3}J_{l_4+1} \pm J_{l_3+1}J_{l_4+2})$$

$$+ (J_{l_1}J_{l_2+1} \pm J_{l_1+1}J_{l_2+2})(J_{l_3}J_{l_4-1} \pm J_{l_3+1}J_{l_4})\}$$

$$-\frac{\pi}{4}\kappa_1\kappa_3 f_A^* \int r\,dr$$

$$\times \{(J_{l_1-1}J_{l_2} \pm J_{l_1}J_{l_2-1})(J_{l_3+1}J_{l_4} \pm J_{l_3+1}J_{l_4+1})$$

$$+ (J_{l_1+1}J_{l_2} \pm J_{l_1+2}J_{l_2+1})(J_{l_3-1}J_{l_4} \pm J_{l_3}J_{l_4+1})\}. \tag{A2·7}$$

$$\left\{\begin{matrix} f_A'(l_i m_i) \\ \bar{f}_A'(l_i m_i) \end{matrix}\right\} = -\frac{\pi}{4}\kappa_2\kappa_4 f_A \int r\,dr$$

$$\times \{(J_{l_1}J_{l_2-1} \mp J_{l_1+1}J_{l_2})(J_{l_3}J_{l_4+1} \pm J_{l_3+1}J_{l_4+2})$$

$$+ (J_{l_1}J_{l_2+1} \mp J_{l_1+1}J_{l_2+2})(J_{l_3}J_{l_4-1} \pm J_{l_3+1}J_{l_4})\}$$

$$-\frac{\pi}{4}\kappa_1\kappa_3 f_A^* \int r\,dr$$

$$\times \{(J_{l_1-1}J_{l_2} \mp J_{l_1}J_{l_2+1})(J_{l_3+1}J_{l_4} \pm J_{l_3+2}J_{l_4+1})$$

$$+ (J_{l_1+1}J_{l_2} \mp J_{l_1+2}J_{l_2+1})(J_{l_3-1}J_{l_3} \pm J_{l_3}J_{l_4+1})\}. \tag{A2·8}$$

(IV)′ $$\left\{\begin{matrix} f_{\mathrm{IV}}(l_i m_i) \\ f_{\mathrm{IV}}'(l_i m_i) \end{matrix}\right\} = \frac{\pi}{2}(\pm\kappa_2\kappa_4 f_{\mathrm{IV}} + \kappa_1\kappa_2 f_{\mathrm{IV}}^*)\int r\,dr$$

$$\times \{(J_{l_1}J_{l_2} - J_{l_1+1}J_{l_2+1})(J_{l_3}J_{l_4} \mp J_{l_3+1}J_{l_4+1})$$

$$+ (J_{l_1}J_{l_2} + J_{l_1+1}J_{l_2+1})(J_{l_3}J_{l_4} \pm J_{l_3+1}J_{l_4+1})\}. \tag{A2·9}$$

$$\left\{\begin{matrix} \bar{f}_{\mathrm{IV}}(l_i m_i) \\ \bar{f}_{\mathrm{IV}}'(l_i m_i) \end{matrix}\right\} = 0. \tag{A2·10}$$

In the above expressions we omit to write the arguments $\kappa_i r$'s of the Bessel functions for brevity. Complete expressions are found, for instance, in Eqs. (3·9)–(3·11) in the text.

(β) The effective coupling constants in (A1·15).

(I)′ and (IV)′: (A=I, IV) $(\alpha_i \equiv \alpha_{m_i n_i})$

$$\left.\begin{matrix} h_A\,(m_i n_i) \\ h_A'\,(m_i n_i) \end{matrix}\right\} = 16\pi^2 ab \left\{\begin{matrix} h_A\left(\dfrac{m_2}{a}\sin\dfrac{\alpha_1+\alpha_2}{2}+\dfrac{n_2}{b}\cos\dfrac{\alpha_1+\alpha_2}{2}\right) \\ -ih_A\left(\dfrac{m_2}{a}\cos\dfrac{\alpha_1+\alpha_2}{2}-\dfrac{n_2}{b}\sin\dfrac{\alpha_1+\alpha_2}{2}\right) \end{matrix}\right\}$$

$$\times\left(\frac{m_4}{a}\sin\frac{\alpha_3+\alpha_4}{2}+\frac{n_4}{b}\cos\frac{\alpha_3+\alpha_4}{2}\right)$$

$$+16\pi^2 ab \left\{\begin{matrix} h_A{}^*\left(\dfrac{m_1}{a}\sin\dfrac{\alpha_1+\alpha_2}{2}+\dfrac{n_1}{b}\cos\dfrac{\alpha_1+\alpha_2}{2}\right) \\ -ih_A{}^*\left(\dfrac{m_1}{a}\cos\dfrac{\alpha_1+\alpha_2}{2}-\dfrac{n_1}{b}\sin\dfrac{\alpha_1+\alpha_2}{2}\right) \end{matrix}\right\}$$

$$\times\left(\frac{m_3}{a}\sin\frac{\alpha_3+\alpha_4}{2}+\frac{n_3}{b}\cos\frac{\alpha_3+\alpha_4}{2}\right). \tag{A2·11}$$

$$\left.\begin{matrix} \bar{h}_A\,(m_i n_i) \\ \bar{h}_A'\,(m_i n_i) \end{matrix}\right\} = -16\pi^2 ab \left\{\begin{matrix} h_A\left(\dfrac{m_2}{a}\cos\dfrac{\alpha_1+\alpha_2}{2}-\dfrac{n_2}{b}\sin\dfrac{\alpha_1+\alpha_2}{2}\right) \\ ih_A\left(\dfrac{m_2}{a}\sin\dfrac{\alpha_1+\alpha_2}{2}+\dfrac{n_2}{b}\cos\dfrac{\alpha_1+\alpha_2}{2}\right) \end{matrix}\right\}$$

$$\times\left(\frac{m_4}{a}\cos\frac{\alpha_3+\alpha_4}{2}-\frac{n_4}{b}\sin\frac{\alpha_3+\alpha_4}{2}\right)$$

$$-16\pi^2 ab \left\{\begin{matrix} h_A{}^*\left(\dfrac{m_1}{a}\cos\dfrac{\alpha_1+\alpha_2}{2}-\dfrac{n_1}{b}\sin\dfrac{\alpha_1+\alpha_2}{2}\right) \\ ih_A{}^*\left(\dfrac{m_1}{a}\sin\dfrac{\alpha_1+\alpha_2}{2}+\dfrac{n_1}{b}\cos\dfrac{\alpha_3+\alpha_4}{2}\right) \end{matrix}\right\}$$

$$\times\left(\frac{m_3}{a}\cos\frac{\alpha_3+\alpha_4}{2}-\frac{n_3}{b}\sin\frac{\alpha_3+\alpha_4}{2}\right). \tag{A2·12}$$

(II)′ and (III)′ (A=II, III)

$$\left.\begin{matrix} h_A(m_i n_i) \\ \bar{h}_A(m_i n_i) \end{matrix}\right\} = -16\pi^2 ab\left[h_A\left(\frac{m_2 m_4}{a^2}+\frac{n_2 n_4}{b^2}\right)+h_A{}^*\left(\frac{m_1 m_3}{a^2}+\frac{n_1 n_3}{b^2}\right)\right]$$

$$\times\left\{\begin{matrix} \cos\dfrac{\alpha_2-\alpha_1}{2}\cos\dfrac{\alpha_4-\alpha_3}{2} \\ -\sin\dfrac{\alpha_2-\alpha_1}{2}\cos\dfrac{\alpha_4-\alpha_3}{2} \end{matrix}\right\}. \tag{A2·13}$$

$$\begin{Bmatrix} h_A'(l_i m_i) \\ \bar{h}_A'(l_i m_i) \end{Bmatrix} = -16 i \pi^2 ab \left[h_A \left(\frac{m_2 m_4}{a^2} + \frac{n_2 n_4}{b_2} \right) + h_A^* \left(\frac{m_1 m_3}{a^2} + \frac{n_1 n_3}{b^2} \right) \right]$$

$$\times \begin{Bmatrix} \sin \dfrac{\alpha_2 - \alpha_1}{2} \cos \dfrac{\alpha_4 - \alpha_3}{2} \\ \cos \dfrac{\alpha_2 - \alpha_1}{2} \sin \dfrac{\alpha_4 - \alpha_3}{2} \end{Bmatrix} . \qquad (A2\cdot 14)$$

Appendix III

In this appendix we shall show the validity of the solution of (2·7) and (2·8) following the orthodox way given by T. Nakano. Using the polar coorninate of (x_5, x_6),

$$x_5 = r \sin\varphi, \quad x_6 = r \cos\varphi, \qquad (A3\cdot 1)$$

we can express the angular momentum j and the square of the mass operator M^2 as follows,

$$j = -i\left(x_5 \frac{\partial}{\partial x_6} - x_6 \frac{\partial}{\partial x_5} \right) - i\gamma_5\gamma_6/2 = i\frac{\partial}{\partial\varphi} + \gamma_5/2, \qquad (A3\cdot 2)$$

$$M^2 = -\left(\frac{\partial^2}{\partial x_5^{\,2}} + \frac{\partial^2}{\partial x_6^{\,2}} \right) = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r} \right). \qquad (A3\cdot 3)$$

Then the following commutation relations are easily found:

$$\begin{aligned} &[j,\ M^2] = 0, \\ &[j,\ \Lambda(\partial)] = -\gamma_5 \Lambda(\partial), \qquad (A3\cdot 4) \\ &[M^2,\ \Lambda(\partial)] = 0, \end{aligned}$$

where $\Lambda(\partial)$ is the differential operator of the basic equation (2·1), i. e.

$$\begin{aligned} \Lambda(\partial) &= \gamma_\mu \frac{\partial}{\partial x_\mu} + \gamma_5 \frac{\partial}{\partial x_5} + \gamma_6 \frac{\partial}{\partial x_6} \\ &= \gamma_\mu \frac{\partial}{\partial x_\mu} + i e^{-i\varphi\gamma_5} \left(\frac{\partial}{\partial r} + \frac{1}{2r} - \frac{1}{r}\gamma_5 j \right). \qquad (A3\cdot 5) \end{aligned}$$

If we take account of the relations (A3·4), it is possible to make the solution of (2·1) the simultaneous eigenstate of j and M^2, that is,

$$j\Psi = \mu\Psi, \qquad (A3\cdot 6)$$

$$M^2\Psi = \kappa^2\Psi. \qquad (A3\cdot 7)$$

Let us define projection operators $\frac{1}{2}(1 \pm \gamma_5)$. Then we get

$$\Psi = \Psi^{(+)} + \Psi^{(-)}, \qquad (A3\cdot 8)$$

where

$$\Psi^{(\pm)} = \tfrac{1}{2}(1 \pm \gamma_5)\Psi, \qquad (A3\cdot 9)$$

and

$$\gamma_5 \Psi^{(\pm)} = \pm \Psi^{(\pm)}. \qquad (A3\cdot 10)$$

Operating $\frac{1}{2}(1\pm\gamma_5)$ upon (A3·6) and (A3·7) and using the expressions of j and M^2 given by (A3·2) and (A3·3), we obtain

$$i\frac{\partial}{\partial\varphi}\Psi^{(\pm)}=\left(\mu\mp\frac{1}{2}\right)\Psi^{(\pm)}, \tag{A3·11}$$

$$\left.\begin{aligned}&-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\left(\mu\mp\frac{1}{2}\right)^2\right)\Psi^{(\pm)}=\kappa^2\Psi^{(\pm)},\\ \text{or}\quad&\frac{\partial^2}{\partial\rho^2}\Psi^{(\pm)}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\Psi^{(\pm)}+\left(1-\left(\mu\mp\frac{1}{2}\right)^2/\rho^2\right)\Psi^{(\pm)}=0,\end{aligned}\right\} \tag{A3·12}$$

where $\rho=\kappa r$. Thus, we find that $\Psi^{(\pm)}$'s have a following form

$$\Psi^{(\pm)}=C^{(\pm)}(x_\mu)J_{\mu\mp1/2}(\kappa r)\,e^{-i(\mu\mp1/2)\varphi}. \tag{A3·13}$$

Further, in order that the solution is a one-valued function of the coordinates (x_5, x_6), μ must be confined to a positive or negative half integer, i. e.

$$\mu=\pm\left(l+\tfrac{1}{2}\right),\quad l=0,\ 1,\ 2,\ \cdots \tag{A3·14}$$

On the other hand, operating the projection operators $\frac{1}{2}(1\pm\gamma_5)$ upon the basic equation (2·1), we have

$$\gamma_\mu\frac{\partial}{\partial x_\mu}\Psi^{(\mp)}+\left(\gamma_5\frac{\partial}{\partial x_5}+\gamma_6\frac{\partial}{\partial x_6}\right)\Psi^{(\pm)}=0. \tag{A3·15}$$

Then, using (A3·13), (A3·10) and the expression of (A3·5), we can rewrite the above equation as follows,

$$\begin{aligned}&\gamma_\mu\partial_\mu C^{(\mp)}(x_\mu)J_{\mu\pm1/2}(\kappa r)\,e^{-i(\mu\mp1/2)\varphi}\\ &\qquad+ie^{\mp i\varphi}\left\{\frac{\partial}{\partial r}+\frac{1}{2r}(1\mp2\mu)\right\}C^{(\pm)}(x_\mu)J_{\mu\mp1/2}(\kappa r)\,e^{-i(\mu\mp1/2)\varphi}=0.\end{aligned} \tag{A3·15$'$}$$

Further, on account of the relation

$$\left\{\frac{\partial}{\partial r}+\frac{1}{2r}(1\mp2\mu)\right\}J_{\mu\mp1/2}(\kappa r)=\mp\kappa J_{\mu\pm1/2}(\kappa r), \tag{A3·16}$$

(A3·16) becomes

$$\begin{aligned}&\gamma_\mu\frac{\partial}{\partial x_\mu}C^{(\mp)}(x_\mu)J_{\mu\pm1/2}(\kappa r)\,e^{-i(\mu\pm1/2)\varphi}\\ &\qquad\mp i\kappa C^{(\pm)}(x_\mu)J_{\mu\pm1/2}(\kappa r)\,e^{-i(\mu\pm1/2)\varphi}=0.\end{aligned} \tag{A3·15$''$}$$

Thus, $C^{(\pm)}$'s have to satisfy the equations

$$\gamma_\mu\frac{\partial}{\partial x_\mu}C^{(\mp)}(x_\mu)\mp i\kappa C^{(\pm)}(x_\mu)=0, \tag{A3·17}$$

and

$$\tfrac{1}{2}(1\pm\gamma_5)C^{(\pm)}(x_\mu)=C^{(\pm)}(x_\mu), \tag{A3·18}$$

where the latter equation (A3·18) comes from (A3·9). The solution of (A3·17) and (A3·18) can be given as follows,

$$\left.\begin{aligned} C^{(+)}(x_\mu) &= \frac{i}{2}(1+\gamma_5)\psi(x_\mu,\ \kappa,\ \mu), \\ C^{(-)}(x_\mu) &= \frac{1}{2}(1-\gamma_5)\psi(x_\mu,\ \kappa,\ \mu), \end{aligned}\right\} \tag{A3·19}$$

where $\psi(x_\mu,\ \kappa,\ \mu)$ is a usual Dirac field satisfying the equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa\right)\psi(x_\mu,\ \kappa,\ \mu) = 0. \tag{A3·20}$$

In consequence we obtain the final expression of the solution,

$$\begin{aligned} \Psi &= \Psi^{(+)} + \Psi^{(-)} \\ &= \tfrac{1}{2}\{i(1+\gamma_5)J_{\mu-1/2}(\kappa r)\, e^{-i(\mu-1/2)\varphi} \\ &\quad + (1-\gamma_5)J_{\mu+1/2}(\kappa r)\, e^{-i(\mu+1/2)\varphi}\}\psi(x_\mu,\ \kappa,\ \mu), \\ &\qquad \mu = \pm(l+\tfrac{1}{2}),\ l = 0,\ 1,\ 2,\ \cdots \end{aligned} \tag{A3·21}$$

One may find easily that this expression coincides with (2·7) and (2·8) except the trivial numerical factor, if one identifies

$$\psi(x_\mu,\ \kappa,\ \mu = \pm(l+\tfrac{1}{2})) = \psi^{\mp}(x_\mu,\ \kappa,\ l). \tag{A3·22}$$

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type an standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On the Production of Mesons in Hydrogen and Carbon above 10 GeV (II)

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July 6, 1957

As the continuation of the previous work[1)] on the production of mesons in hydrogen and carbon, a cloud chamber which contained four pairs of paraffin (3.14 g/cm^2) and graphite (2.68 g/cm^2) layers and three lead plates of 1 cm thickness was operated with a counter hodoscope system at Mt. Norikura Cosmic-Ray Laboratory, University of Tokyo, (2840 m), in 1955. The arrangement is shown in Fig. 1. The thickness of each graphite layer was chosen in such a way that the number of carbon nuclei in it is equal to that contained in the paraffin layer. Those events which formed coincidences $(S\geqq 2)+(B\geqq 3)+(C\geqq 1)$ were photographed. Scanning about a half of the total cloud chamber photographs obtained, 156 nuclear interactions were found to be produced in paraffin and 120 interactions to be produced in graphite originated by both neutral and charged primary particles. Assuming that the cross-section of carbon is 320 mb, these rates give 53 ± 24 mb for the cross-section of nucleon-proton interactions, the energies of which are shown to be a few to several tens of GeV from the angular distribution of secondary particles. Therefore, taking weighted average of the previous value of 67 ± 25 mb[1)] and the present one, we obtain the value of 60 ± 17 mb. This value seems slightly larger than 42 mb at 2.75 GeV[2)] and 29.5 mb at 5.3 GeV.[3)]

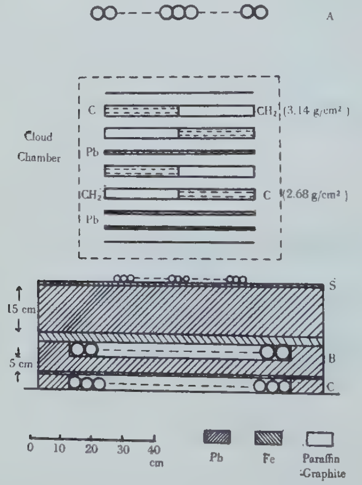

Fig. 1 The experimental arrangement

Then, the energy distributions of photons originating from neutral pions produced by nuclear interactions in paraffin, graphite and lead were compared, estimating the energy of photons from the cascade development through lead layers by using

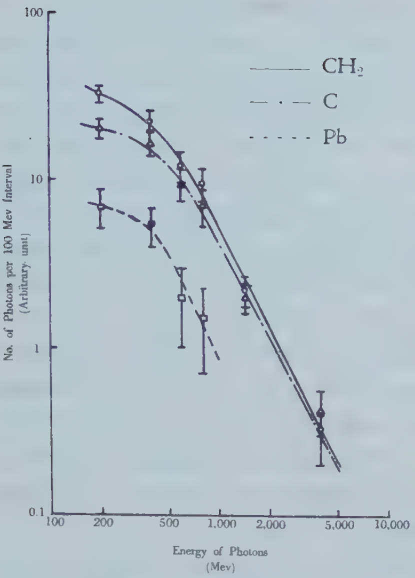

Fig. 2 The integral energy spectra of photons originating from neutral pions.

Snyder's curves.[4)]

The integral energy spectra for these three cases are shown in Fig. 2. The slope of the energy spectrum for paraffin is rather steeper than that for carbon in the low energy region, whereas the frequencies of photons of high energies are nearly the same in those two cases. But there seems to be no significant difference in the shapes of the energy spectra of photons between carbon and lead. These facts may be explained by an assumption that neutral mesons of low energies produced inside a carbon nucleus are considerably absorbed by secondary collisions with nucleons in the same nucleus.

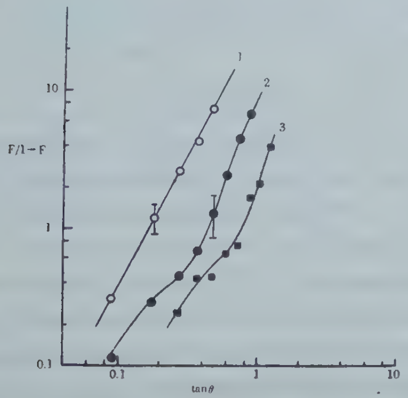

Fig. 3 The F-plots of secondary charged particles.

The angular distribution of the secondary particles produced in paraffin and carbon, with a charged primary particle and without neutral secondary pions, was studied. The latter restriction is imposed because otherwise some charged secondary particles behind the cascade showers might be missed. The so-called F-plot[5)] is shown in Fig. 3 for three groups tabulated in Table I which includes interpretation of the figure. These results may show that the angular distribution of the secondary particles in the center-of-mass system is nearly isotropic in the nucleon-nucleon collision, while the successive collision in the same nucleus tend to reduce the energy of secondary particles

Table I

	Group 1	Group 2	Group 3
mean angle of secondary particles	0～20°	20°～30°	30°～40°
mean multiplicity	3.8	3.6	4.5
Lorentz factor in C.M. system of primary	5.9	2.3	1.7
Average velocity of secondary in C. M. system	≈1	<1	<1
Angular distribution of the secondary particles in C. M. system	nearly isotropic	rather sharp in forward and backward directions	rather sharp in forward and backward directions

and also to produce particles at wider angles so that these simulate the angular distribution which is rather sharp in forward and backward directions in the center-of-mass system deduced by assuming the collision as the nucleon-nucleon collision.

To examine the azimuthal angular distribution of the secondary particles with respect to the direction of a primary particle, we took the azimuthal angular separation Φ between each secondary particle and its nearest neighbour. If $P(\Phi)d\Phi$ is the frequency of cases with azimuthal angular separation in the range $d\Phi$ at Φ, $W(\Phi)=\int_{\Phi}^{2\pi}P(\Phi)d\Phi/\int_{0}^{2\pi}P(\Phi)d\Phi$ vs. Φ represents the characteristic feature of the angular distribution. In Fig. 4 and 5 two examples of $W(\Phi)$ are shown for the events with three and six secondary particles together with theoretically expected curves in the cases of isotropic (dotted curve) and $\cos^2\varphi$ (dot-dashed curve) distributions. From these figures, it may be seen that at least in cases with many secondary particles, the azimuthal angular distribution is different from isotropy. This difference may be explained partly in terms of fluctuation of secondary interactions in the nucleus, although we could not draw nothing more quantitative due to insufficient statistics in this analysis.

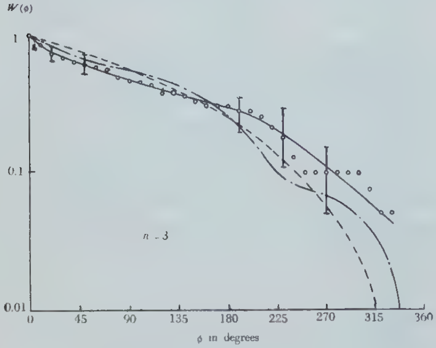

Fig. 4 $W(\Phi)$ vs. Φ for the case of three secondary particles.

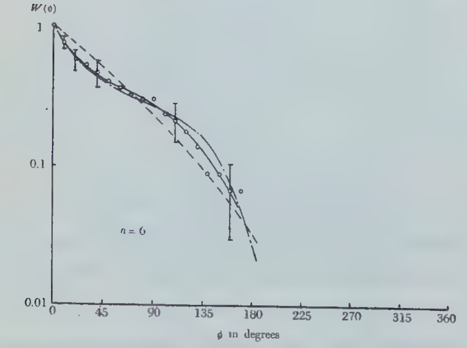

Fig. 5 $W(\Phi)$ vs. Φ for the case of six secondary particles.

Full accounts of this letter will soon be published in the Journal of the Physical Society of Japan.

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On the Pressure Shift of the Inversion Frequency of Ammonia

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July 23, 1957

Recently those who are working with Maser have found a definite amount of shift in the inversion frequency of ammonia which is proportional to pressure and to the width of resonance.1),2),3) In this letter we note that this shift is due to the second order non-adiabatic effect of the electric dipole-dipole interaction between ammonia molecules.

In our previous letter,4) which will be referred to as I, we discussed the equivalence between the line width and the rate of energy relaxation by using the relaxation function method.5) According to this method the relaxation function $\phi_\alpha(t)$ of the electric dipole moment of the molecule having the inversion frequency ω_α is given by

$$\phi_\alpha(t)=\exp\{i\omega_\alpha t-\psi_\alpha(t)\}, \qquad (1)$$

and

$$\psi_\alpha(t)=\sum_\Upsilon\sum_M\langle\sigma^2_{\alpha M}(b) \times\int_0^t (t-\tau)\, e^{i\omega_\Upsilon\tau} f_M(v\tau/b)\, d\tau\rangle_b \qquad (2)$$

where $\sigma^2_{\alpha M}(b)$ is the second moment of the intermolecular interaction (eq. (5) of I), $f_M(v\tau/b)$ is the correlation function of the radial part of it (eq. (6) of I). v is the average velocity of the molecule, b is the impact parameter, and $\langle\quad\rangle_b$ denotes the summation over the number and type of collision (eq. (4) of I). In the case of self-broadening the energy change $\hbar\omega_\Upsilon$ associated with the second order effect is equal to $\hbar(\omega_\alpha\pm\omega_{\alpha'})$, where $\omega_{\alpha'}$ is the inversion frequency of the perturbing molecule.

When the pressure is not very high, say below 10 mm Hg, we have a condition $\sigma_{\alpha M}(b)\ll v/b$ which assures the validity of the impact approximation. In this case $\psi_\alpha(t)$ becomes proportional to t, and is written as

$$\psi_\alpha(t)=t\sum_\Upsilon\sum_M\langle\sigma^2_{\alpha M}(b)\,(b/v)F_M(b\omega_\Upsilon/v)\rangle_b$$
$$=t(\Phi_2-i\Delta), \qquad (3)$$

where

$$F_M(b\omega_\Upsilon/v)=\int_0^\infty d\tau'\, e^{i(b\omega_\Upsilon/v)\tau'} f_M(\tau'). \qquad (4)$$

Here Φ_2 is the damping constant which is measured as line width, and Δ denotes the persistent cloud effect which should be detected as shift in the resonance frequency. This latter effect was dropped out in Anderson's treatment6) when he assumed the commutability of the interactions at different times, however it is not really negligible in the case of ammonia as we show in the following. If we assume $f_M(\tau')$ to be given by $\exp\{-(v\tau'/b)\}$ for a fixed value of the impact parameter b, then we have

$$a\equiv\Delta/\Phi_2=b\omega_\Upsilon/v\simeq b(\text{Å})\times(0.035). \qquad (5)$$

This is not actually negligible as compared with unity for the value of b around 10 Å for which we have the dominant contribution to the width. In view of this situation we have performed a straightforward calculation of the functions $f_M(\tau')$ by adopting the dipole-dipole interaction as the perturbation. In order to perform the integration over the impact parameter b, we can resort to the characteristic situation in the case of pressure broadening, i. e. that the spectrum is fairly insensitive to the behaviour of the radiator and the perturber inside a certain range. The reason is that the radiation emitted and absorbed by the radiator is seriously disturbed and is not observed as resonance inside an intermolecular distance b_r which is far greater than the kinetic radius b_k. Based on this consideration we have introduced a method of cutting off the contribution of the irrelevant region to the shift, which is similar to that proposed for width by Anderson6). The cut-off radii are 12.6 Å for the width and 5.1 Å for the shift.

The final results are listed in Table I together with the observed value of shift and width.

Table I. The width and the shift of 3−3 inversion line of ammonia (in the unit of megacycles per mm Hg)

	Theory	Experiment
Width	31.9	27.
Shift (to higher frequency side)	5.9	1.0∼0.5

The ratio a of shift to width is found to be 0.18 which is definitely larger than the observed ratio of 0.04∼0.02. The reason for this discrepancy may be traced back to the assumption of the straight line motion with uniform velocity. Although it should be allowed for the calculation of width, it may not be so in the case of shift, for the cut-off distance is actually comparable with the kinetic radius, i. e. 4.4 Å. It seems, therefore, that we have to take into account the effect of deflection and velocity change in the relative motion of the radiator and perturber. The details and the possible refinement will be published soon.

The author expresses his deep appreciation to Professors I. Takahashi, K. Shimoda, and Dr. Y. Sugiura for their kind communication on the experimental data. Also his thanks are due to Mr. A. Hirai who has drawn his attension to the existence of the phenomenon and assisted him in the numerical work.

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Pion Production in Nucleon-Nucleon Collisions in Bev Range

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June 29, 1957

The consistency of the cross section of pion production in nucleon-nucleon collisions in Bev range with the cross section of the corresponding pion-nucleon scattering is examined in this note.

Since we have not yet a reliable theory of the pion-nucleon interaction in this energy range, it seems meaningful for us to understand experimental results theoretically basing on phenomenological consideration. We shall show in the following that, if single pion exchange between the colliding two nucleons is predominant, the total cross section for the reaction $N+N\rightarrow N+N+\pi$ (N donotes a nucleon) can be expressed as an integral of the total cross sections for pion-nucleon scattering $N+\pi\rightarrow N+\pi$ over a certain energy region.

For the low energy pion-nucleon scattering we have a powerful method, so called the static meson theory, which have been successfully developed by Chew and Low.[1)]

If the kinetic energy of the incident nucleon is less than 1 Bev (in the laboratory

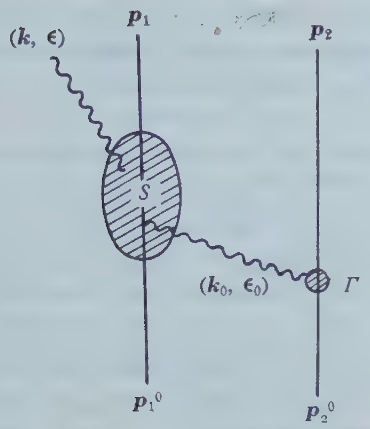

Fig. 1

system), we may be allowed to apply the static theory to describe the final nucleon-pion system in $N+N \rightarrow N+N+\pi$. Because the relative energy between the pion and the nucleon which emitted the pion remains still less than 500 Mev.[2)]

If we call S and Γ the scattering part and the vertex part, respectively, (cf. Fig. 1) the total cross section of pion production in nucleon-nucleon collissons is as follows,

$$\sigma_N = \frac{2\pi}{v_r} \iint \left| \frac{\langle \boldsymbol{k} | S | \boldsymbol{k}_0 \rangle \langle \boldsymbol{p}_2 | \Gamma | \boldsymbol{p}_2^0 \rangle}{\epsilon_0} \right|^2 \times \frac{d\boldsymbol{k}\, d\boldsymbol{k}_0}{(2\pi)^6} \delta(E_1 + E_2 + \epsilon - E_0), \tag{1}$$

where $\boldsymbol{k}$, $\boldsymbol{k}_0$ and ϵ, ϵ_0 denote respectively the momenta and energies of the produced pion and the exchanged pion between the colliding nucleons, and $\boldsymbol{P}_2$, $\boldsymbol{P}_2{}^0$ and E_1, E_2 indicate the momenta of the nucleon (2) after and before the collision, and energies of the nucleons in the final state, respectively. E_0 is the tatal energy in the initial state.

Adopting the static meson theory of Chew and Low, we can express the scattering part S in terms of the differential cross section for pion-nucleon scattering, i. e.

$$\langle \boldsymbol{k} | S | \boldsymbol{k}_0 \rangle = (2\pi)^2 / \epsilon^2 \cdot d\sigma_\pi / d\Omega_K . \tag{2}$$

On the other hand, the vertex part Γ has the following form to a good approximation, provided that a nucleon makes little change in its direction of motion through the collision,

$$\langle \boldsymbol{p}_2 | \Gamma | \boldsymbol{p}_2{}^0 \rangle \simeq i \frac{f}{\mu} \frac{\sigma \cdot \boldsymbol{k}_0}{\sqrt{2\epsilon_0}}, \tag{3}$$

where f denotes the renormalized coupling constant between the pion and the nucleon. Substituting (2) and (3) in (1), we obtain a formula for the total cross section of the reaction $N+N \rightarrow N+N+\pi$ as a function of kinetic energy T of the incident nucleon in the center of mass system :

$$\sigma_N(T) = \int_\mu^T d\epsilon' \sigma_\pi(\epsilon') f(\epsilon'), \tag{4}$$

where $\sigma_\pi(\epsilon')$ and $f(\epsilon')$ respectively indi-

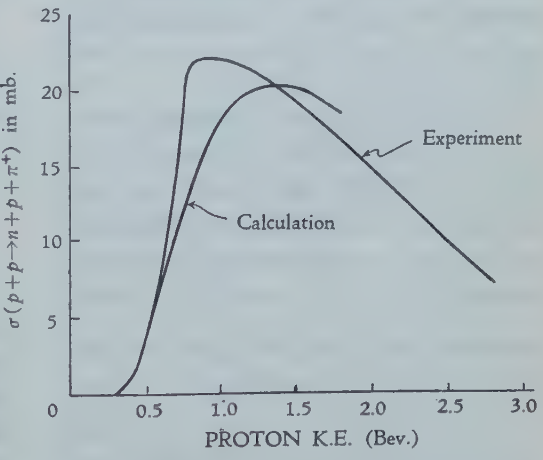

Fig. 2. Total cross section of pion production in proton-proton collisions *vs.* the incident kinetic energy of the bombarding proton.

cate the total cross section of pion-nucleon scattering and a slowly varying function with energy which is to be determined by the model assumed for the interaction between pion and nucleon.

For $p+p\to n+p+\pi^+$,[3] writing $f(\epsilon)$ explicitely, we obtain

$$\sigma_N(T)=\frac{1}{2\pi}\cdot\frac{f^2}{4\pi}(M/\mu)^2$$

$$\times\int_\mu^T\frac{d\epsilon'}{\epsilon'}\left(1-\frac{\mu^2}{\epsilon'^2}\right)^{3/2}\sqrt{1-\frac{\epsilon'}{T}}\,\sigma_\pi(\epsilon'). \tag{5}$$

Making use of the experimental results of Lindenbaum and Yuan[4] for $\sigma_\pi(\epsilon')$ we are able to calculate the total cross section $\sigma_N(T)$ for the reaction $p+p\to n+p+\pi^+$.

As is shown in Fig. 2, our results with $f^2/4\pi=0.08$ are consistent with the recent Cosmotron experiments of Shutt et al.

A more comprehensive analysis is now in progress, where calculation is based on the rigorous dispersion relations of pion-nucleon system. There the processes of the double pion production are also taken into account by using the experimental cross section for the processes.

One of qualitatively interesting results of our analysis is that the shape of the curve of the total cross section for pion production in nucleon-nucleon collisions *vs.* the incident energy of the bombarding nucleon reflects the shape of integral spectrum $\int_\mu^T d\epsilon'\sigma_\pi(\epsilon')f(\epsilon')$ for the pion-nucleon scattering cross section.

It is worthy of remark that there exists such relation as connecting the production and scattering.

The author would like to express his sincere thanks to Dr. D. Ito for his kind guidance and valuable discussions throughout the course of this work.

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On the Cut-off Theory in Quantum Field Theory

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For the investigation of the problem of internal consistency of current quantum field theory, the cut-off method is usually adopted. Among others, the cut-off method proposed by Gell-Mann and Low[1] has very frequently been used and was very useful for finding the high energy behaviours of quantum field theory. However, no one to this date has ever made certain whether this cut-off method was really a consistent cut-off theory, or, in other words, whether this cut-off method reduced to the original divergent theory in the nomenclature of Gell-Mann and Low when the cut-off mementum was made to go to infinity. In this note we shall consider this problem in Lee's model[2] and up to the fourth order perturbation approximations in quantum electro-dynamics. In discussing this problem, we will choose the cut-off formalism given by Umezawa et al.,[3] which is proved to be equivalent to the formalism of Gell-Mann and Low.

In the paper of Umezawa et al., it was shown that Gell-Mann and Low's theory consists in modifying the original V-particle Green's function in Lee's model;

$$(P_0-m_V-\sum(P_0)+\sum(m_V))\,S_V'(P_0)=1 \tag{1}$$

into the following form

$$(P_0-m_V-\sum(P_0)+\sum(m_V)+(P_0-m_V)\frac{\sum(\Lambda)-\sum(m_V)}{\Lambda-m_V}S_{\Lambda V}'(P_0)=1 \tag{2}$$

where, Λ is the cut-off mementum and m_V is the V-particle mass. Hence the condition which should be satisfied by fields is that

$$\lim_{\Lambda\to\infty}\frac{\sum(\Lambda)-\sum(m_V)}{\Lambda-m_V}=0. \tag{3}$$

In Lee's model, $\sum(\Lambda)$ is given by

$$\sum(\Lambda)=g^2\sum_k\frac{1}{2\omega\Omega(\Lambda-m_N-\omega-i\varepsilon)}, \tag{4}$$

where g^2 is the unrenormalized coupling constant. If we take the imaginary part of eq. (4) and insert this into the left-hand side of eq. (3), we get

$$\frac{ig^2}{4\pi}\lim_{\Lambda\to\infty}\sqrt{\frac{(\Lambda-m_N)^2-\mu^2}{\Lambda-m_V}}=\frac{i}{4\pi}g^2, \tag{5}$$

here μ is the θ-particle mass. Thus we can see that there exists a finite imaginary part in eq. (3) when the cut-off momentum Λ is made to tend to infinity, and we can conclude that in Lee's model the Gell-Mann and Low's cut-off method can not reduce to the original divergent theory when $\Lambda\to\infty$. On the other hand, Umezawa et al. took in eq. (2) as the counterterm

$$\frac{Re[\sum(\Lambda)]-\sum(m_V)}{\Lambda-m_V} \tag{6}$$

instead of

$$\frac{\sum(\Lambda)-\sum(m_V)}{\Lambda-m_V}.$$

Then we can get rid of the above difficulty, but in this case a new difficulty will arise in connection with the boundary condition which requires that the particle does behave as a free particle when $\Lambda\to\infty$. Now, if in eq. (2) we insert Λ for P_0 and make Λ tend to infinity, we shall obtain as the proper self-energy part a linearly divergent imaginary part $i/4\pi\cdot g\lim_{\Lambda\to\infty}\Lambda$, which just destroys the boundary condition. The same discussion as above can be carried through in quantum electrodynamics and we get the following condition for eq. (3) in Källen's notation.[4]

$$\lim_{\Lambda^2\to\infty}\pi(\Lambda^2)=0. \tag{7}$$

However, we were unable to obtain any decisive conclusion on the question whether quantum electrodynamics really satisfied eq. (7), but we can test this in the perturbation theory up to the fourth order. Using the Källen and Sabry's result,[5] we get

$$\lim_{\Lambda^2\to\infty} \pi^{\text{2nd order}}(\Lambda^2) = e^2/12\pi^2 \qquad (8)$$

$$\lim_{\Lambda^2\to\infty} \pi^{\text{4th order}}(\Lambda^2) = \frac{e^4}{72\pi^2}\log\infty. \qquad (9)$$

where e is the renormalized coupling constant.

Finally we note that eq. (7) is sufficient but not necessary to insure the internal consistency of the renormalization theory.[5]

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Preliminary Note on the Indirect (*p*, *t*) Process

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As for the indirect (p, t) process suggested by Hess and Moyer[1], two kinds of possible modes, i. e., the successive pick-up mode and the paired pick-up mode of nucleons, are expected to give important contributions. The former mode is defined as the pick-up process by the deuteron which has been already formed in the indirect (p, d) process[1] and the latter as the pick-up process in which the pair of nucleons correlating with each other in the target nucleus are picked-up at the same time by the nucleon scattered inside the nucleus. In the present note, using the analogous method to indirect (p, d) process,[2] we shall actually calculate the indirect (p, t) process according to the successive pick-up mode and discuss also the contributions from the paired pick-up mode.

By means of eq. (1) in our previous note,[2] the successive pick-up probability to form a triton along the direction of secondary nucleon beam is effectively given by

$$P_{\text{eff}} = \int_0^d \frac{l_d}{l_p}(1-e^{-x/l_d}) \times \frac{dx}{l_d'} e^{-(d-x)}/l_t \qquad (1)$$

where the notations of l_p, l_d and d follow the definition in the previous note. l_d' denotes the mean free path of the deuteron allowing for the forward pick-up process only, where the deuteron formed in the first step picks up one more neutron of the nucleus before leaving the nuclear surface and l_t the mean free path of produced triton, which should be estimated by the similar way as l_d. With the approximate relations, $d/l_d \gg 1$ and $d/l_t \gg 1$, eq. (1) is reduced as

$$P_{\text{eff}} = \frac{l_d}{l_p} \cdot \frac{l_t}{l_d'}. \qquad (2)$$

To obtain l_d', we have calculated the cross section of the direct (d, t) process by the similar method to the theory of Chew and Goldberger in the (n, d) process,[3] where the triton wave function has been assumed to be Gaussian type according to Heidmann.[4] The obtained differential cross section per neutron is

$$d\sigma_{(d,t)}/d\Omega_t = \frac{9}{2} A_d^2 A_t^2 [I(\alpha) - I(\beta)]^2$$

$$\times (\pi/2\kappa)^3 \frac{k_t}{k_d} F(\vec{k}_t, \vec{k}_d), \tag{3}$$

where

$$F(\vec{k}_t, \vec{k}_d) = \Big[\frac{4m}{3\hbar^2} (\varepsilon_t - \varepsilon_d)$$

$$- \Big(\frac{2}{3} \vec{k}_t - \vec{k}_d \Big)^2 \Big]^2$$

$$\times N(\vec{k}_t - \vec{k}_d)\, e^{-1/4\kappa (\frac{2}{3}\vec{k}_t - \vec{k}_d)^2}, \tag{4}$$

$$I(\alpha) = \frac{1}{3\kappa} \Big[1 - \alpha \sqrt{\pi/6\kappa}$$

$$\times e^{\alpha^2/6\kappa} \Big(1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\alpha^2/6\kappa}} e^{-z^2} dz \Big) \Big]. \tag{5}$$

Assuming the 16 Mev-Gaussian for the momentum distribution of picked-up neutron $N(\vec{k}_t - \vec{k}_d)$, we can see that the above differential cross section shows the strongly forward peak. So we can safely make use of the total cross section in evaluation of l_d'. The integration of triton angle has been performed as follows:

$$\int F(\vec{k}_t, \vec{k}_d)\, d\Omega_t = \frac{32}{9} (2\pi)$$

$$\times (\sqrt{4\pi}/a)^3 \frac{1}{\Big(\frac{2}{a^2} + \frac{1}{3\kappa} \Big)^3}$$

$$\times \frac{\exp\Big[-\frac{1}{a^2}(k_t^2 + k_d^2) - \frac{1}{4\kappa}\Big(\frac{4}{9} k_t^2 + k_d^2 \Big) \Big]}{k_t k_d}$$

$$\times [(\xi^2 + \eta^2 + 2\xi + 2) \sinh \eta$$

$$- 2\eta(\xi + 1) \cosh \eta], \tag{6}$$

where

$$\left.\begin{aligned} \xi &= \frac{3}{4} \Big(\frac{2}{a^2} + \frac{1}{3\kappa} \Big) \\ &\quad \times \Big[-\frac{4m}{3\hbar^2} (\varepsilon_t - \varepsilon_d) + \frac{4}{9} k_t^2 + k_d^2 \Big], \\ \eta &= \Big(\frac{2}{a^2} + \frac{1}{3\kappa} \Big) k_t k_d. \end{aligned}\right\} \tag{7}$$

In the course of numerical evaluation of the indirect (p, t) cross sections, we have made use of the calculated deuteron spectra of the indirect (p,d) process, normalizing the magnitude of the differential cross sections to the observed values. The obtained final results for various nuclei are listed in Table I. The theoretical values for successive pick-up mode are considerably lower than the experimental one even if our crude approximations are allowed. It is suggested that the other modes may play an important role in the indirect (p, t) process.

In view of these circumstances, we have performed the preliminary estimation of the contribution from the paired pick-up mode, based on the Fermi gas model. As the internal wave function of the correlating pair of nucleons, we have assumed the Hulthén type wave function in triplet state, which should coincide with the quasi-deuteron wave function in short range. The numerical results obtained by the use of the expression for pick-up probability analogous to eq. (1) in the previous note are also listed in Table I. The absolute magnitude of theoretical cross section may become too large, when we take account of the singlet state contribution. We might expect to have the satisfactory

Table I. Differential cross sections, $E_t \geqq 58$ Mev (mb/ster)

element	angle	theoretical$_{\text{I}}$	theoretical$_{\text{II}}$	experimental
C	26°	0.020	0.076	0.120±0.068
	40°	0.027	0.096	0.148±0.069
	60°	0.021	0.088	0.024±0.041
Al	26°	0.045	0.176	0.150±0.108
	40°	0.071	0.208	0.415±0.108
Cu	26°	0.074	0.304	0.316±0.179
	40°	0.11	0.332	0.260±0.189
Cd	40°	0.15	0.40	0.492±0.296
Pb	26°	0.19	0.276	1.22 ±0.550
	40°	0.27	0.52	2.035±0.596
U	40°	0.28	0.56	0.191±0.248

theoretical$_{\text{I}}$ and theoretical$_{\text{II}}$ correspond to the successive pick-up mode and the paired pick-up mode, respectively.

results by the use of the original quasi-deuteron wave function instead of the Hulthén type function for correlating pair system. The detailed study is now in progress. Finally the authors wish to thank Prof. T. Muto for his encouragement and advice.

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On the Many-Body Pseudopotential for a System of Hard-Spheres

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A many-body pseudopotential is obtained as a generalization of Huang and Yang's binary pseudopotential.[1] First we consider the following differential equation,

$$(\nabla_1^2+\nabla_2^2+2k^2)\,\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2)=f(\boldsymbol{x}_1,\boldsymbol{x}_2), \quad (1)$$

and investigate the nature of the singularity of the solution $\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2)$ at $|\boldsymbol{x}_1-\boldsymbol{x}_2|=0$ when the boundary condition $\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2)=0$ $|\boldsymbol{x}_1-\boldsymbol{x}_2|=a$ is imposed, where a is the hard-sphere diameter. As can be seen easily a multipole is induced at $|\boldsymbol{x}_1-\boldsymbol{x}_2|=0$ by the function $f(\boldsymbol{x}_1,\boldsymbol{x}_2)$ and this multipole will give rise to the pseudopotential $\delta(r_{12})K_{12}f(\boldsymbol{x}_1,\boldsymbol{x}_2)$ which is to be added to Huang und Yang's pseudopotential $\delta(r_{12})\cdot P_{12}^{(b)}\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2)$.

Hence eq. (1) together with the boundary condition at $|\boldsymbol{x}_1-\boldsymbol{x}_2|=a$ turns out to be equivalent to the equation

$$(\nabla_1^2+\nabla_2^2+2k^2)\,\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2)=f(\boldsymbol{x}_1,\boldsymbol{x}_2)+\delta(r_{12})P_{12}^{(b)}\Psi+\delta(r_{12})K_{12}f \quad (2)$$

without the above condition.

Now consider a system of N hard-spheres. The pseudopotentials which replace the condition that the wave function $\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots\boldsymbol{x}_N)$ vanishes at $|\boldsymbol{x}_i-\boldsymbol{x}_j|=a$

may be expressed as

$$\sum_{i=1}^{N}(\nabla_i^2+k^2)\Psi$$
$$=\sum_{i<j}\delta(r_{ij})(P_{ij}^{(b)}+P_{ij}^{(t)})\Psi, \qquad (3)$$

where $P_{ij}^{(t)}$ represents the pseudopotential corresponding to the triple and higher order collisions. To determine $P_{ij}^{(t)}$ write eq. (3) in the following from

$$(\nabla_1^2+\nabla_2^2+2k^2)\Psi=-\sum_{n\neq 1,2}(\nabla_n^2+k^2)\Psi$$
$$+\sum_{i<j}{}'\delta(r_{ij})(P_{ij}^{(b)}+P_{ij}^{(t)})\Psi$$
$$+\delta(r_{12})(P_{12}^{(b)}+P_{12}^{(t)})\Psi$$

and consider the first and the second terms on the right side as corresponding to the $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in eq. (2). This correspondence leads to the equation for $P_{ij}^{(t)}$

$$P_{ij}^{(t)}=-K_{ij}\sum_{n\neq i,j}(\nabla_n^2+k^2)$$
$$+K_{ij}\sum_{k<l}{}'\delta(r_{kl})P_{kl}^{(b)}$$
$$+K_{ij}\sum_{k<l}{}'\delta(r_{kl})P_{kl}^{(t)}, \qquad (4)$$

where the prime on $\sum$ means that the pair (ij) is excluded from the summation. Since K_{ij} is proved to be an integral operator, eq. (4) is an integral equation for $P_{ij}^{(t)}$ which may be solved by an iteration procedure. It turns out that the first term of this solution is of the order a^4 and further terms are of the higher orders.

By a straightforward calculation it can be shown that the lowest term in the expansion of the ground state energy for the Bose system which results from the triple collisions is given by

$$-(8\pi^2a^4\hbar^2/3m)\rho^2. \qquad (5)$$

The appearance of a^4 in this expression is consistent with the suggestion given by Huang and Yang.

The lowest order term of the binary pseudopotential,

$$8\pi a\sum_{i<j}\delta(\boldsymbol{x}_i-\boldsymbol{x}_j)\frac{\partial}{\partial r_{ij}}r_{ij}, \qquad (6)$$

also gives a term of the order a^4. In the calculation of this term the direct perturbational treatment of (6) as was performed by Huang and Yang is inappropriate because such a procedure yields an unreasonable result that the energy per particle for the ground state diverges if one makes the particle number N tend to ∞ keeping the density as constant. To avoid this divergence the use of the ρ_k-representation[2) for the Bose system seems to be suitable. The computation gives

$$E_0/N=\frac{2\pi a\hbar^2\rho}{m}\Big(1+\frac{128(\rho a^3)^{1/2}}{15\sqrt{\pi}}$$
$$+\alpha\rho a^3+\beta(\rho a^3)^{3/2}+\cdots\Big), \qquad (7)$$

where the terms $\alpha\rho a^3$ and $\beta(\rho a^3)^{3/2}$ are contributions from the phonon-phonon interactions. By a simple dimensional argument each term of eq. (7) is proved to be of the form $a\hbar^2\rho(\rho a^3)^{n/2}/m\,(n=0, 1, 2, \cdots)$, indicating that the expansion (7) is the expansion in powers of $(\rho a^3)^{1/2}$ and that the divergence difficulty just mentioned no longer appears. These facts are in agreement with Lee and Yang's[3) conclusions drawn from the binary collision expansion method or Lee, Yang and Huang's[4) conclusions drawn from the explicit consideration of the eigenvalue problem of the pseudopotential. A full detail of this article will be published in the near future.

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Ferromagnetism of a Gas of Hard Sphere Fermions with Spin 1/2

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In his recent paper with a similar title to this note, Ehrman[1] has presented the result that a gas of hard sphere fermions with spin 1/2 will show the ferromagnetism at high densities and at low temperatures. We have reached the same conclusion from another point of view, with more intuitive interpretation. We shall present it in the following, § 1 to § 3.

§ **1.** Let us start from the state of the system where fermions fill the Fermi sphere up to the Fermi top* and consider the hard sphere interaction as the perturbation**. This perturbation causes an increase in the energy of the system. Because of the Pauli principle the probability that two fermions come near is smaller for a pair with parallel spins than for an anti-parallel pair, and so the increase of energy due to the presence of the repulsive potential is larger for the latter. It follows, therefore, that there would be a gain of energy if we could reverse one of spins of an anti-parallel pair. At low densities this gain will not be sufficient to bring a fermion to a state with reversed spin outside the Fermi sphere owing to the larger loss in the kinetic energy. As the density increases the gain associated with a reverse of spin becomes larger, since particles come to interact with each other more frequently and the repulsive potential acts more effectively for anti-parallel pairs. Hence, above some critical density fermions in the Fermi sphere can be brought outside it, with a net gain of energy. These fermions should have parallel spins to minimize the total energy, thus the ferromagnetism appears. At the lower densities where the ferromagnetism does not occur, the spin paramagnetic susceptibility is expected at least to be larger than that of free fermions because of the tendency for the spins to become parallel with each other.

§ **2.** To get some quantitative estimation, we calculate the zero point energy of the system by using the pseudo-potential method presented by Huang and Yang[2]. With the Hamiltonian

$$\mathscr{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial \boldsymbol{r}_i^{\,2}} + \frac{4\pi a\hbar^2}{m} \times \sum_{i>j}\delta(\boldsymbol{r}_i - \boldsymbol{r}_j)\frac{\partial}{\partial \boldsymbol{r}_{ij}} r_{ij}, \tag{1}$$

where a is the diameter of a hard sphere, the energy in the first order perturbation is

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* In this note the state at 0°K will be discussed exclusively.

** It might be better to consider firstly the case where the potential is finite.

$$E(\boldsymbol{k}_1^+, \cdots, \boldsymbol{k}_{N^+}^+, \boldsymbol{k}_1^-, \cdots, \boldsymbol{k}_{N^-}^-)$$

$$= \frac{\hbar^2}{2m}\left(\sum_{i=1}^{N^+} \boldsymbol{k}_i^{+2} + \sum_{i=1}^{N^-} \boldsymbol{k}_i^{-2}\right)$$

$$+ \frac{4\pi a\hbar^2}{mV} N^+ N^-; \tag{2}$$

(at the presence of the constant magnetic field H, $-(N^+ - N^-)\mu H$ should be added). This becomes minimum if N^+, N^- and N^+ $\boldsymbol{k}_i^-$'s and N^- $\boldsymbol{k}_i^-$'s are determined by the following equations

$$|\boldsymbol{k}_i^+| \leqq k_0^+, \quad |\boldsymbol{k}_i^-| \leqq k_0^-,$$

$$\frac{\hbar^2}{2m}(k_0^{+2} - k_0^{-2})$$

$$- \frac{4\pi a\hbar^2}{mV}(N^+ - N^-) = 0,$$

$$k_0^\pm = 2\pi\left(\frac{3}{4\pi}\frac{N^\pm}{V}\right)^{1/3}, \quad N^+ + N^- = N.$$

It is easily shown that in the range of densities satisfying

$$a^3 N/V > \pi/24$$

the above equations have the solution, $N^+ \neq N^-$, so that the spontaneous magnetization appears as expected, and also that when $a^3 N/V \geqq 9\pi/128$ the magnetization saturates; the energy is minimum when $N^+ = N^-$ and $N^- = 0$ (or $N^- = N$ and $N^+ = 0$). It is found that the correction term for the energy of order a^2 is positive when $N^+ = N^-$ and zero when $N^+ = N$ and $N^- = 0$, which favors the occurrence of the ferromagnetism *a fortiori*. With the aid of eq. (2), the paramagnetic susceptibility for $a^3 N/V < \pi/24$ is shown to be

$$\chi = \chi_0 [1 - 2(3a^3 N/\pi V)^{1/3}]^{-1},$$

where $\chi_0 = 3m\mu^2 N/\hbar^2 k_0^2$ is the value for free fermions: surely $\chi > \chi_0$ as was expected.

§ **3.** Next, we take into account the possibility that the increase of energy of a pair of fermions depends on the distance between them in the momentum space, which has been neglected in § 2. If the distance is large, the difference between the increase of energy for an antiparallel pair and that for a parallel one will not be so large as was evaluated in § 2. Hence the density at which the ferromagnetism occurs is expected to be higher than that evaluated in § 2.

In this way, a possibility of the ferromagnetism of a gas of hard sphere fermions with spin 1/2 has been concluded in the first order perturbation. Of cours there remain needs to discuss the problem in better approximations.

The similar reasoning as used above leads us to the conclusion that the "anti-ferromagnetic order" will appear for a gas of attractive fermions with spin 1/2. And this is nothing but the reasoning used in the nucler shell model to explain the rule that even number of nucleons have total spin 0 at the ground state[3].

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Dispersion Relations in Nucleon-Nucleon Scattering*

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The dispersion relation in quantum field theory has been successfully applied to pion-nucleon scattering. When applied to the case of nucleon-nucleon scattering, it leads to a new relation between the scattering amplitudes and a quantity which is intimately connected with the two-body nuclear potential. A brief account of this formula will be presented in this letter.

Let $M_\alpha(E)$ be the forward amplitude of the nucleon-nucleon scattering in a definite spin and charge state α, where E is the energy of the incident nucleon in the laboratory system. Making use of the dispersion relation, one can express the real part of $M_\alpha(E)$ in terms of its imaginary part, which, for $E>M$ and $E<-M$, is related by the so-called optical theorem to the total cross sections of nucleon-nucleon and nucleon-antinucleon scattering respectively.

On has thus

$$\Re e\, M_\alpha(E)=U_\alpha(E)+\frac{1}{2\pi}\mathfrak{P}\int_M^\infty\frac{v'\sigma_\alpha(E')}{E'-E}dE'+\frac{1}{2\pi}\mathfrak{P}\int_M^\infty\frac{v'(\bar{\sigma}_{\alpha'}(E')-\bar{\sigma}_{\alpha'}^{an}(E'))}{E'+E}dE' \tag{1}$$

where $U_\alpha(E)$ is a quantity defined by

$$U_\alpha(E)=\frac{1}{\pi}\mathfrak{P}\int_{-M}^{M}\frac{\Im m\, M_\alpha(E')}{E'-E}dE'+\frac{1}{2\pi}\mathfrak{P}\int_M^\infty\frac{v'\bar{\sigma}_{\alpha'}^{an}(E')}{E'+E}dE'.$$

In these equations, M is the rest mass of the nucleon, $\sigma_\alpha(E')$ means the total cross section of nucleon-nucleon scattering in the state α, $\bar{\sigma}_{\alpha'}(E')$ that of nucleon-antinucleon scattering in α', the state in which the target nucleon is the same as in α, while the incident nucleon is replaced by an antinucleon, $\bar{\sigma}_{\alpha'}^{an}(E')$ is the annihilation cross section in α', v' means the velocity of the incident nucleon or antinucleon corresponding to the energy E'. If the state α is the neutron and proton in triplet state, another term

$$C/(E-E_d)$$

$$\left(E_d:\ \begin{array}{l}\text{the energy corresponding}\\ \text{to deuteron formation}\end{array}\right)$$

must be added to the right-hand side of (1), C being a constant which can be approximately expressed in terms of the triplet scattering length and the triplet effective range.

The main result obtained is that in the static approximation $(E-M)/M\to0$ and $\mu/M\to0$, where μ is the pion mass, $U(E)$ is proportional to the Fourier transform of the adiabatic nuclear potential $V(p)$ except for an additive constant.

$$\lim_{M\to\infty}U_\alpha(E)=-V_\alpha(p)+\text{const.},$$

$$p^2=E^2-M^2.$$

The proof of this relation was done by comparing every term in the perturbation expansion of both sides in the approximation $M \rightarrow \infty$. Thus the static limit of (1) becomes

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* Dispersion relations for nucleon-nucleon scattering have also been investigated by many authors. See Goldberger, Nambu and Oehme, preprint.

$$V_\alpha(p) = -\Re\mathrm{e}\, M_\alpha(E) + \frac{1}{2\pi}\mathfrak{P}\int_M^\infty \frac{v'\sigma_\alpha(E')}{E'-E}\,dE' + \left(\frac{C}{E_d - E}\right) + \text{const.} \tag{2}$$

This relation may be very useful for the analysis of nucleon-nucleon scattering. Moreover eq. (2) provides a means of determining the nuclear potential if the scattering data is known.

It may happen that the integrals in the right-hand side of (1) do not converge. In that case, the subtraction procedure well-known in dispersion theory must be performed.

A more detailed account will soon appear in a later issue of this journal.

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Progress of Theoretical Physics, Vol. 18, No. 4, October 1957

Nuclear Matrix Element Relations in the Fermi Theory of Beta Decay

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The purpose of the following is largely consolidatory. A common basis is provided for the formal relations between forbidden nuclear matrix elements. It is shown how numerical estimates of the coefficients of proportionality can subsequently be made upon introducing on the one hand less on the other hand more detailed nuclear force characteristics. Coarse structure correction factors are derived differing from similar ones appearing in the literature. Finally, it is indicated how theoretical nuclear matrix element ratios may serve in the interpretation of beta spectra.

§ 1. Introduction and summary

Feenberg, Primakoff and this author used commutation relations and simple nuclear model arguments to establish numerical relations between nuclear beta decay matrix elements.[1,2] Peaslee, in addition, drew upon somewhat explicit nuclear force considerations, modifying two of the relations and introducing others.[3]

Feenberg et al. saw no need for bringing but broad features of nuclear forces into the picture, features as are reproduced by the semi-empirical nuclear energy formula, the statistical model and the shell model. Peaslee's method requires from the beginning at least partial specification of a force ; in fact, Peaslee employed a charge symmetric pseudoscalar two-body force.

It is one purpose of this note to spell out a common basis for the formal nuclear matrix element relations. This basis can be provided by two sets of relativistic commutation relations. One set contains terms multiplied by the nucleon rest mass, the other does not. Numerical estimates of the coefficients entering the various nuclear matrix element relations can be made subsequent to the introduction of nuclear force characteristics.

It is another purpose of this note to use the derived nuclear matrix element relations in the simplification of "coarse structure" first and second forbidden correction factors. The expression "coarse structure" is meant to imply the presence only of terms proportional to the square of the Coulomb barrier. For medium and heavy nuclei the effect of the other terms is in general small.[4] The resulting correction factors differ from similar ones presented in the literature.[3,5] One consequence of this change is that certain second forbidden spectra previously interpreted as favoring, though weakly, the minus sign in the linear combination of scalar and tensor interactions no longer render such a service.[3] At most two of the spectra still indicate a preference for including pseudoscalar potentials in the matrix element relation considerations.

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Appendix A contains a sample calculation demonstrating semi-explicitly how the numerical coefficients relating the various nuclear matrix elements to one another may be divided rather definitely into two categories.

Appendix B contains a few clarifying remarks concerning a more limited topic, that of the pseudoscalar beta interaction.[2,6,7,8)]

§ 2. Theory

As described in detail in a previous publication covering the pseudoscalar case, the starting point for the derivation of each commutation relation consists of an initial and an Hermitian conjugate final-state, many-particle Dirac equation.[7)] The first equation is multiplied by a judiciously chosen beta operator, the latter by the corresponding Hermitian conjugate operator. Subtraction and integration of the products yield the basic exact form of the desired relation. Thus, if the chosen operators are XR, X being shorthand for the sum over all nucleons, $\sum_n'(Q\chi)_n$, and if X equals, e.g., γ_5, $\sigma\cdot\boldsymbol{r}$, α and $\boldsymbol{r}$, respectively, the following relations are obtained

$$0=(W_i-W_f)\,(f|\sigma\cdot\boldsymbol{r}R|i)+(f|[V,\sigma\cdot\boldsymbol{r}]R|i)$$

$$+3i(f|\gamma_5R|i)+i(f|[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{p}),R]_+|i)-(f|\gamma_5[\boldsymbol{r}\cdot\boldsymbol{p},R]|i), \tag{1a}$$

$$0=(W_i-W_f)\,(f|\boldsymbol{r}R|i)+(f|[V,\boldsymbol{r}]R|i)$$

$$+i(f|\alpha R|i)-(f|\boldsymbol{r}[\alpha\cdot\boldsymbol{p},R]|i), \tag{1a$'$}$$

$$2M(f|\beta\gamma_5L|i)=(W_i-W_f)\,(f|\gamma_5R|i)+(f|[V,\gamma_5]R|i)$$

$$-(f|[\sigma\cdot\boldsymbol{p},R]|i), \tag{1b}$$

$$2M(f|\beta\alpha L|i)=(W_i-W_f)\,(f|\alpha R|i)+(f|[V,\alpha]R|i)$$

$$-i(f|[\sigma\times\boldsymbol{p},R]_+|i)-(f|\boldsymbol{p},R]|i). \tag{1b$'$}$$

V is a general potential energy operator. R signifies the appropriate radial lepton covariant and Q the charge transformation operator. The terms in eqs (1) are ordered so as to call to mind their mutual correspondence. Apart from being supposed to render relations between certain nuclear matrix elements, the choice of the operators XR is determined by the desire to exhibit explicitly the division of the relations into two categories. These categories are immediately recognizable by the presence on the one hand, and absence on the other, of matrix elements multiplied by the nucleon mass.

Eq.s (1a) and (1b′) are not yet in a sufficiently tractable form. They become so when the elements

$$(f|\gamma_5[\sigma\cdot(\boldsymbol{r}\times\boldsymbol{p}),R]_+|i) \quad \text{and} \quad (f|[\sigma\times\boldsymbol{p},R]_+|i)$$

are approximated by

$$-2(f|\gamma_5R|i) \quad \text{and} \quad 2iM(f|[H_0,\sigma\times\boldsymbol{r}]R|i)$$

respectively, a procedure permissible as long as errors actually of order $1/M^2$ are not considered to be serious.

$$(f|[H_0, \sigma\times\boldsymbol{r}]|i)$$

is in turn equated to

$$(W_f-W_i)(f|\sigma\times\boldsymbol{r}|i)-(f|[V, \sigma\times\boldsymbol{r}]|i).$$

The last two elements as well as the two pairs

$$(W_f-W_i)(f|\sigma\cdot\boldsymbol{r}|i) \quad \text{and} \quad -(f|[V, \sigma\cdot\boldsymbol{r}]|i),$$

$$(W_f-W_i)(f|\boldsymbol{r}|i) \quad \text{and} \quad -(f|[V, \boldsymbol{r}]|i)$$

are subsequently treated according to a previously given prescription.$^{1)}$ The potential energy commutator matrix elements are first expanded in accordance with the expansion theorem of matrix algebra. The expanded forms

$$\sum_{f'}(f|V|f')(f'|X|i)-\sum_{i'}(f|X|i')(i'|V|i)$$

X standing in this particular case for $\sigma\cdot\boldsymbol{r}$ and $\boldsymbol{r}$, respectively, are then approximated by

$$[(f|V|f)-(i|V|i)](f|X|i).$$

The difference of the potential energy expectation values is evaluated after calling upon the semi-empirical nuclear energy formula;$^{9)}$ the contribution of the potential energy to the symmetry energy term in this formula is assumed to be given adequately by the statistical model.

While there are reasonable plausibility arguments in favor of following this prescription when dealing with operators such as $\sigma\cdot\boldsymbol{r}$, $\boldsymbol{r}$ and $\sigma\times\boldsymbol{r}$, the same arguments can, judging from the probable nature of nuclear forces, easily become unreasonable when brought forth to justify a similar treatment in the case of operators like γ_5 and α. There is the well-known possibility as regards some nuclear properties of describing the nucleus by a non-relativistic Schroedinger equation, representing the effects of the nuclear interactions on a nucleon as though they were largely due to a uniform potential well. This leads one to expect little obstruction of the recommended path for the evaluation of potential energy commutator elements if the associated operators do not mix positive and negative energy states. The approximation

$$(f|[V, X]|i)\simeq[(f|V|f)-(i|V|i)](f|X|i)$$

looks well on the basis of shell model reasoning. The nuclear part of V can be replaced by a non-relativistic potential with little configuration interaction. Thus any attempts by X to contribute significantly with large overlapping between pairs of neighboring configurations are frustrated as long as one member of the pairs of states is always an intermediate state.* Nor is the approximation likely to be greatly in error as far as the Coulomb interaction is concerned. Non-diagonal Coulomb matrix elements are small compared with the Coulomb energy difference of neighboring isobars. An endangering buildup of non-diagonal elements can in general not take place because of the selection rules connected with X. One cannot expect this simple scheme to apply to the evaluation of potential energy commutator matrix elements when the associated operators do mix positive and negative energy states. In fact, it is unlikely that the evaluation can at all proceed without invoking some details of the nuclear interaction.

---

* An exception is in general provided by transitions between states whose leading components cannot be connected by the operator under consideration.

A semi-explicit sample calculation is presented in Appendix A to demonstrate how such a differentiation between two types of commutator elements may come about.

The difference of the potential energy expectation values has been given as $\simeq \alpha Z/2\rho = B$, α being the fine structure constant and ρ the distance of the point of the lepton covariant evaluation from the center of the nucleus.[1] By virtue of the presented arguments one can consequently write

$$\begin{aligned} (f|[V, \sigma\cdot\boldsymbol{r}]|i) &\simeq B\int \sigma\cdot\boldsymbol{r}, \\ (f|[V, \boldsymbol{r}]|i) &\simeq B\int \boldsymbol{r}, \\ (f|[V, \sigma\times\boldsymbol{r}]|i) &\simeq B\int \sigma\times\boldsymbol{r}. \end{aligned} \tag{2}$$

As for the other two operators one must in general be satisfied with stating

$$\begin{aligned} (f|[V, \gamma_5]|i) &= -\mathcal{H}B\int \gamma_5, \\ (f|[V, \alpha]|i) &= -\mathcal{H}B\int \alpha \end{aligned} \tag{3}$$

where $\mathcal{H}$ can easily assume a value of the order of 10 and perhaps even of 100 (see appendix A). The assignment of a definite value, as said before, appears to require a considerably more detailed knowledge of the nuclear forces than did the evaluation of B.[3,6]

Making use of all the hitherto described simplifications one can write on the basis of eq.s (1)

$$\int \gamma_5 = iB\lambda \int \sigma\cdot\boldsymbol{r}, \tag{4a}$$

$$\int \alpha = iB\lambda \int \boldsymbol{r}, \tag{4a$'$}$$

$$\int \beta\gamma_5 = -iB\lambda\delta \int \sigma\cdot\boldsymbol{r}, \tag{4b}$$

$$\int \beta\alpha = -iB\lambda' t \int \boldsymbol{r} \tag{4b$'$}$$

with

$$\delta = \mathcal{H}B/2M, \qquad \lambda' = \lambda(1+\delta/t), \qquad \lambda \simeq 1 + (W_i - W_f)/B^{1)}.$$

In all instances the small momentum commutator matrix elements have been neglected. As far as eq. (1b) is concerned this is only permissible if $|\mathcal{H}| \gg 1$ (see appendix B);

after all, eq. (1b) does not contain a term equivalent to the momentum anti-commutator element in eq. (1b′).

Eq.s (4a) are not of direct value if interest is confined to those beta interactions that alone appear to be admitted by experimental evidence. Present information seems to dictate against the vector and axial vector interactions. Eq.s (4a) are, however, of indirect value, inasmuch as they are invoked in the derivation of eq.s (4b).

The second forbidden relations corresponding to (4a′) and (4b′) are

$$\int A_{ij}=iB\lambda\int R_{ij}, \tag{5a$'$}$$

$$\int \beta A_{ij}=-iB\lambda'(t/2)\int R_{ij} \tag{5b$'$}$$

with

$$t=-i\int T_{ij}/\int R_{ij}, \quad \lambda'=\lambda(1+2\partial/t)$$

and $\mathcal{H}$, which enters ∂ just as in the first forbidden case, defined by second forbidden equations equivalent to eq.s (3).

§ 3. Application

A sufficient amount of material has now been amassed to proceed with a simplification of first and second forbidden correction factors.

The linear combination $G_sS+G_tT+G_pP$ presents one in general with the essential nuclear matrix elements

$$\int\beta\sigma\cdot\boldsymbol{r}\simeq-\int\sigma\cdot\boldsymbol{r},\ \int\beta\gamma_5,\ \int\beta\boldsymbol{r}\simeq-\int\boldsymbol{r},\int\beta\sigma\times\boldsymbol{r}\simeq-\int\sigma\times\boldsymbol{r},\int\beta\boldsymbol{\alpha},$$
$$\int\beta R_{ij}\simeq-\int R_{ij},\ \int\beta T_{ij}\simeq-\int T_{ij},\ \int\beta A_{ij}$$

as long as one confines his attention to first and second forbidden non-unique transitions. First forbidden spin change zero spectra consequently leave four adjustable " experimental " parameters, first forbidden spin change one and second forbidden spin change two spectra each leave two. The theory provides larger numbers of parameters: 7, 4, and 4. However, the values which the " theoretical " parameters can assume are considerably restricted by their underlying theory. An interpretation of experimental data in terms of a certain linear combination and the presence or absence of pseudoscalar nuclear forces is thus facilitated.

It is, of course, differential information which may lead to a relatively unambiguous assignment of values to the experimental parameters. Integral information, i.e. knowledge of comparative half lives, may perhaps also be drawn upon, but never except on a highly statistical basis;[5)] after all, the theory's power lies largely in avoiding the use of absolute nuclear matrix element magnitudes.

As far as spectra are concerned it is those of non-allowed shape arising from non-unique transitions which can serve the desired purpose. Second forbidden non-unique transitions exhibit a non-allowed shape in general, first forbidden non-unique transitions only in particular cases.

Thus far one knows of one first forbidden non-unique decay of non-allowed shape, which permits a relatively definite assignment of values to the minimum of theoretical parameters required to make a statement concerning the linear combination of interactions and the possible presence of pseudoscalar forces. This decay, $Bi^{210} \to Po^{210}$, has found an interpretation$^{10)}$ which, with reasonable but not conclusively convincing assumptions underlying the derivation of a value for $t = -i\int \sigma \times \boldsymbol{r} / \int \boldsymbol{r}$, leads to a positive ratio of tensor to scalar coupling constants. It so happens, though, that the experimentally permissible range of values for λ' overlaps the range realizable by the theory with $0 \lesssim \delta \lesssim 1$. Consequently, one is not in a position to draw any conclusions from this decay regarding the presence or absence of pseudoscalar nuclear forces. A drastic reassignment of values to the theoretical parameters, particularly to t, could change the picture.

The decays of Sb^{124} and Re^{186} also appear to be first forbidden with $\Delta I = 1^{11,12)}$. Both of these decays make an assignment to t difficult and untrustworthy. In view of the type of calculation necessary for such assignments the advantage of the present method of spectral interpretation is diminished whenever nuclei are involved that are well removed from shell edges. In the presence of several strongly interacting nucleons outside the core the nuclear wavefunction will not be adequately given by terms, all of which attribute only one orbital and one total spin to each particle. Instead, one will be presented with appreciable configuration mixing, and the nuclear matrix elements $\int \sigma \times \boldsymbol{r}$ and $\int \boldsymbol{r}$ need be expanded if a relation is to be established between them. The expansions read

$$\int \sigma \times \boldsymbol{r} \equiv (f|\sigma \times \boldsymbol{r}|i) = \sum_{j,k} a_j b_k (f_j|\sigma \times \boldsymbol{r}|i_k)$$

and

$$\int \boldsymbol{r} \equiv (f|\boldsymbol{r}|i) = \sum_{j,k} a_j b_k (f_j|\boldsymbol{r}|i_k). \tag{7}$$

Employing

$$(f_j|\sigma \times \boldsymbol{r}|i_k) = it_{jk}(f_j|\boldsymbol{r}|i_k)$$

where

$$t_{jk} = (f_j|\sigma \cdot \boldsymbol{l}|f_j) - (i_k|\sigma \cdot \boldsymbol{l}|i_k) \tag{8}$$

one obtains

$$t = \frac{1 + \alpha\{1 + \beta[1 + \gamma(1 + \cdots}{1 + a\{1 + b[1 + c(1 + \cdots} \tag{9}$$

where α and a are the "first corrections"

$$\alpha = \frac{a_1 b_2 t_{12}(f_1|\boldsymbol{r}|i_2) + a_2 b_1 t_{21}(f_2|\boldsymbol{r}|i_1)}{a_1 b_1 t_{11}(f_1|\boldsymbol{r}|i_1)}$$

$$a = \frac{a_1 b_2(f_1|\boldsymbol{r}|i_2) + a_2 b_1(f_2|\boldsymbol{r}|i_1)}{a_1 b_1(f_1|\boldsymbol{r}|i_1)} \tag{10a}$$

β and b the "second corrections"

$$\beta=\frac{a_2b_2t_{22}(f_2|\boldsymbol{r}|i_2)+a_1b_3t_{13}(f_1|\boldsymbol{r}|i_3)+a_3b_1t_{31}(f_3|\boldsymbol{r}|i_1)}{a_1b_2t_{12}(f_1|\boldsymbol{r}|i_2)+a_2b_1t_{21}(f_2|\boldsymbol{r}|i_1)}$$

$$b=\frac{a_2b_2(f_2|\boldsymbol{r}|i_2)+a_1b_3(f_1|\boldsymbol{r}|i_3)+a_3b_1(f_3|\boldsymbol{r}|i_1)}{a_1b_2(f_1|\boldsymbol{r}|i_2)+a_2b_1(f_2|\boldsymbol{r}|i_1)}, \tag{10b}$$

γ and c the " third corrections ", and so on. A remaining bright spot is that mistakes in the choice of a radial dependence for the functions ψ_{fj} and ψ_{ik} will enter the corrections in ratios.

The final state of the Sb^{124} decay is excited, an aggravating circumstance in the determination of t, as it makes even the evaluation of t_{11} speculative. The same holds, as the nomenclature already suggests, for the major inner group of the Re^{186} decay; for the outer group one has probably $t_{11}=-1$. Disregarding the corrections, a procedure which cannot be recommended for the decay in question, the $Re^{186}\rightarrow Os^{186}$ ground state transition can be explained with positive as well as with negative ratios of the scalar to tensor coupling constants. Again, within this context, the experimental material does not lend itself to illuminate the question of the presence of pseudoscalar forces. If one applies the same procedure to the major inner group transition and also uses the same t_{11}, which is justified to a good extent by evidence that the 137 Kev level of Os^{186} involved is interpretable as close to purely rotationally excited,12) the coupling constant ratio comes out positive. As before, the data appear neither to be for nor against pseudoscalar forces.

Lee-Whiting10) actually used part of a first correction,

$$\alpha=b_2t_{12}(f_1|\boldsymbol{r}|i_2)/b_1t_{11}(f_1|\boldsymbol{r}|i_1)$$

and

$$a=b_2(f_1|\boldsymbol{r}|i_2)/b_1(f_1|\boldsymbol{r}|i_1),$$

α in this specific instance being evaluated as

$$\alpha=-10b_2(f_1|\boldsymbol{r}|i_2)/b_1(f_1|\boldsymbol{r}|i_1)$$

or

$$t=(1-10a)/(1+a).$$

It is questionable, whether the groundstate of Po^{210} can be adequately represented by the spectroscopically pure $(h_{9/2}, h_{9/2}; 0, 0)$; the first two places refer to protons, the latter two to neutrons. All second forbidden non-unique decays can in principle be used for the purpose of assigning numbers to the parameters. The second forbidden $\Delta I=2$ correction factor due to the interaction G_sS+G_tT can in general be approximated by the expression*

---

* Eq. (11) has a coefficient v^2 differing by the sign of λ from the coefficient presented in reference 3. The same disagreement exists with reference 5, as regards the coarse structure first forbidden correction factor, which is here

$$C^y{}_{0,1}\simeq B^2\{G_p{}^2[(G_t/G_p)^2-\lambda\delta]^2|\int\boldsymbol{\sigma}\cdot\boldsymbol{r}|^2+G_t{}^2[(G_s/G_t)^2+t-\lambda(t+\delta)]^2|\int\boldsymbol{r}|^2\}.$$

The present form is chosen to make the equivalent mathematical origin of the terms in the $\Delta I=0$ and $\Delta I=0$, 1 parts apparent.
$G_t\int\boldsymbol{\sigma}\cdot\boldsymbol{r}$ corresponds to $G_s\int\boldsymbol{r}$ and $G_p\int\beta\gamma_5$ to $G_t\int\beta\alpha$; $G_t\int\sigma\times\boldsymbol{r}$ has, of course, no counterpart in the pseudoscalar case. The disagreement in sign stems from an inconsistently taken non-relativistic limit of $\int\beta$ in references 3 and 5. The authors of references 3 and 5 concurred with this author in this conclusion.

$$C_2^n \simeq (1/2) B^2 t^2 [(1/2)(1-\lambda')+x]^2 \cdot \\ \{p^2+k^2[(1/2)(2-\lambda')+2x]^2/[(1/2)(1-\lambda')+x]^2\} G_t^2 \left| \int B_{ij} \right|^2 . \tag{11}$$

The approximation consists in keeping only the terms proportional to the square of the Coulomb barrier. The subscript on the abbreviation of the correction factor signifies the spin change, the superscript the presence, y, or absence, n, of a parity change. The parameter x is equal to $(G_s/G_t)(1/t)$. Electron and neutrino momentum are as usual denoted by p and k, respectively. Three well-known groups of $\Delta I=2$ transitions[3)] are consistent with $v^2 \approx 0.6$, $v^2 \approx 2$ and $v^2 \lesssim 10$;

$$v=[(1/2)(2-\lambda')+2x]/[(1/2)(1-\lambda')+x]$$

assigning $t=0$, 2 and -2 to the three groups, and in that order, gives rise to the following cases:

$$\begin{array}{llll} v^2=.6 & t=0 & G_s/G_t=1 & \lambda\delta=1.6 \\ & & =-1 & =-1.6 \\ =2 & =2 & =1 & \lambda(1+\delta)=2.8 \\ & & =-1 & =0 \\ \geq 10 & =-2 & =1 & \lambda(1-\delta)=0 \\ & & =-1 & 1.1 \leq \lambda(1-\delta) \leq 2.0. \end{array} \tag{12}$$

On the basis of present thoughts on the matter λ is expected to be positive and of about one or two in value.[1,13)] δ, if present at all, could well be small enough to escape detection; it is unlikely to exceed ~ 1.

The first of the three second forbidden non-unique spectrum groups can only be interpreted with the presence of pseudoscalar potentials. The second group does not definitely require their presence if the relative signature of tensor and scalar interactions is positive, and neither does the third group, by any means, if the signature is negative. Since the above theory makes no explicit prediction as to the sign of the parameter δ, the present considerations involving second forbidden non-unique spectra do not yield any definite information in regard to the relative signature of the linear combination. The approach taken here will not favor one signature over the other until spectra have been analyzed that exhibit an energy dependence and consequent nuclear matrix element ratios resulting in combinations of λ and δ whose values are clearly in discord with the theory for either $G_s/G_t \approx 1$ or $G_s/G_t \approx -1$. Any further development in the specification of the ranges of values acceptable to the parameters λ and δ would, of course, permit an earlier tipping of the scales in favor of one or the other relative signature; thus, e.g., the solidification of the appealing notion that the magnitude of

$$[(f|H^\nu|f)-(i|H^\nu|i)](f|X^b|i)$$

is enhanced by the rest of the terms,

$$\sum_{f'}(f|H^{\nu}|f')(f'|X^{b}|i)-\sum_{f'}(f|X^{b}|i')(i'|H^{\nu}|i)$$

making up the commutator matrix element $(f|[H^{\nu}, X^{b}]|i)$ would permit positive δ's only. The cases (12) would then provide an argument, however crude, against the combination S-T.

Be it mentioned at this point that in (12) the value selected for each $\lambda\delta$ is the theoretically more plausible member of a pair of solutions. The same holds in the case of $\lambda(1+\delta)$ when $G_s/G_t=1$. It also holds for every value of $\lambda(1-\delta)$ in the range $1.1-2.0$. The remaining two cases possess only one solution each.

Evidently the values of certain combinations of λ and δ exhibited in (12) will change with improving rigor of the calculations of t; the use of spectroscopically pure assignments in (12) can at best be considered a rough approximation, in view of the structure of the nuclides in question. Thus in Cl^{36}, e.g., the 17th proton and the 19th neutron are probably not well described by pure $1d_{3/2}$ states but necessitate the admixing of $2s_{1/2}$ components. Furthermore, whenever experimental accuracy warrants this, linear terms in B and B independent terms in the correction factors should also be taken into account in second forbidden spectral calculations$^{14)}$; one great advantage of such detail is the appearance of two adjustable experimental parameters.

Appendix A

Consider the potential (cs=charge symmetric, p=pseudoscalar, 2=two body)

$$V_2^{\mathrm{cs,p}}=\sum_{n<n'}\tau_n\cdot\tau_{n'}(\beta\gamma_5)_n(\beta\gamma_5)_{n'}V_{nn'}.$$

Let X^a be an operator commuting and X^b an operator anti-commuting with $\beta\gamma_5$. The two classes of commutator elements

$$(f|[V_2^{\mathrm{cs,p}}, X^a]|i) \quad \text{and} \quad (f|[V_2^{\mathrm{cs,p}}, X^b]|i)$$

can differ considerably as regards the factor of proportionality relating one to the nuclear matrix element $\int X^a$, the other to $\int X^b$. That this is so can already be noticed by merely evaluating the two classes of commutators:

$$[V_2^{\mathrm{cs,p}}, X^a]=-\sum_{n\neq n'}(Q_n\tau_{\zeta n'}-Q_{n'}\tau_{\zeta n})(\chi_n^a-\chi_n^{a'})(\beta\gamma_5)_n(\beta\gamma_5)_{n'}V_{nn'} \qquad \text{(AIa)}$$

$$[V_2^{\mathrm{cs,p}}, X^b]=-\sum_{n\neq n'}(Q_n\chi_{n'}^b+Q_{n'}\chi_n^b)(\beta\gamma_5)_n(\beta\gamma_5)_{n'}V_{nn'}\,. \qquad \text{(AIb)}$$

The right side of eq. (AIa) can be simplified appreciably if certain assumptions are made about the initial and final nuclear wave functions between which it lies in its matrix element. Let the nuclear wave function, Ψ, be represented by an infinite series of independent particle determinantal wave functions,

$$(A!)^{-1/2}\sum_t c_t\|\psi_n^q\|_t$$

q denoting one of the quantum states appearing in the t-th term and n, as before, one of the A nucleons. The integral over a two particle operator, $\sum_{n\neq n'}O_{nn'}$, sandwiched between any one term of the initial and any one term of the final nuclear wave function expansions, reduces to one of the three well-known forms

$$\int \sum_{q \neq q'} (\psi_n^{q'} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'})^{+} O_{nn'} (\psi_n^{q'} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'})\, dv_n dv_{n'}\,,$$

$$\int \sum_{q} (\psi_n^{q'f} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'f})^{+} O_{nn'} (\psi_n^{q'i} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'i})\, dv_n dv_{n'}\,,$$

$$\int (\psi_n^{q'f} \psi_{n'}^{qf} - \psi_n^{qf} \psi_{n'}^{q'f})^{+} O_{nn'} (\psi_n^{q'i} \psi_{n'}^{qi} - \psi_n^{qi} \psi_{n'}^{q'i})\, dv_n dv_{n'}$$

or O. It reduces to the first if the initial and final terms employed differ in no states, to the second if they differ in one state, q', and to the third if they differ in two states, q and q'. It reduces to 0 if more than two states in the initial and final terms differ from one another. In the case at hand, contributions of the second form dominate ; e.g., if the nuclear wave functions are sufficiently approximated by one independent particle term, sufficiently approximated in particular as regards the selection rules of the decay under consideration. With such a situation given one can, by dropping the no state change and two state change terms in the expansions, write

$$\begin{aligned}
&(f|\sum_{n\neq n'}^{A} (Q_n \tau_{\zeta n'} - Q_{n'} \tau_{\zeta n}) (\chi_n^a - \chi_{n'}^a) (\beta\gamma_5)_n (\beta\gamma_5)_{n'} V_{nn'}|i) \\
&\approx \int \sum_{q}^{A-Z} (\psi_n^{q'f} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'f})^{+} (Q_n - Q_{n'}) (\chi_n^a - \chi_{n'}^a) \cdot \\
&\qquad \cdot (\beta\gamma_5)_n (\beta\gamma_5)_{n'} V_{nn'} (\psi_n^{q'i} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'i})\, dv_n dv_{n'} \qquad \text{(AII)} \\
&- \int \sum_{q}^{Z} (\psi_n^{q'f} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'f})^{+} (Q_n - Q_{n'}) (\chi_n^a - \chi_{n'}^a) \\
&\qquad \cdot (\beta\gamma_5)_n (\beta\gamma_5)_{n'} V_{nn'} (\psi_n^{q'i} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'i})\, dv_n dv_{n'}\,.
\end{aligned}$$

The right side of eq. (AII) can, for medium weight and heavy nuclei, be approximated by

$$\begin{aligned}
&[(A-2Z)/A] \int \sum_{q}^{A} (\psi_n^{q'f} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'f})^{+} (Q_n - Q_{n'}) (\chi_n^a - \chi_{n'}^a) \cdot \\
&\qquad \cdot (\beta\gamma_5)_n (\beta\gamma_5)_{n'} V_{nn'} (\psi_n^{q'i} \psi_{n'}^{q} - \psi_n^{q} \psi_{n'}^{q'i})\, dv_n dv_{n'}\,.
\end{aligned}$$

The last integral is in turn, by virtue of the situation under discussion, an approximation to the matrix element

$$(f|\sum_{n\neq n'}^{A} (Q_n - Q_{n'}) (\chi_n^a - \chi_{n'}^a) (\beta\gamma_5)_n (\beta\gamma_5)_{n'} V_{nn'}|i)\,.$$

The different effective behaviour of the two classes of operators is thus indicated.

Peaslee employs the potential $V_2^{\mathrm{cs,p}}$ also in the direct evaluation of $\int\beta\gamma_5$. In his treatment only terms up to first order in the small components of the final and initial states and in the potential energy function are included. It is felt that consistency demands the same treatment for $\int\gamma_5$. In this case one obtains

$$\int \gamma_5 \approx -(1/2M)\,(f|\sum_n [(\sigma\cdot \boldsymbol{p})_n,\ Q_n(1-(1/2M)\sum_{n'} V_{nn'})]_+|i), \quad n \neq n' \tag{AIII}$$

$$\sum_{n'} V_{nn'} = \rho(2M/\mu)^2 H^{nr,\nu} \quad (nr = \text{non-relativistic} \quad \nu = \text{nuclear} \quad \mu = \pi\text{-meson rest mass}). \tag{AIV}$$

Approximating $(\sigma' = \text{Pauli's } \sigma)$

$$\int [\sigma'\cdot \boldsymbol{p},\ H^{nr,\nu}]_+ \simeq [(f|H^{nr,\nu}|f) + (i|H^{nr,\nu}|i)](f|\sigma'\cdot \boldsymbol{p}|i) \tag{AV}$$

by assuming the off-diagonal non-relativistic energy matrix elements to be comparatively small, and, moreover, of varying signature, allows one to write

$$\int \gamma_5 \approx -(1/M)[1+\rho(2M/\mu)\,(\bar{V}/\mu)\int \sigma\cdot \boldsymbol{p}]. \tag{AVI}$$

$\bar{V}$ stands for the average nuclear potential energy of a single neutron. Calling upon eq. (18) of ref. 3

$$\int \beta\gamma_5 \approx (\rho/\mu^2)\int [\sigma'\cdot \boldsymbol{p},\ H^{nr,\nu}] \approx (7/5)\,(B\rho/\mu^2)\int \sigma\cdot \boldsymbol{p} \tag{AVII}$$

one obtains

$$\int \beta\gamma_5 = -\varepsilon\int \gamma_5,\ \varepsilon \approx (7/5)\,MB\rho/(\mu^2+2M\bar{V}\rho) \tag{AVIII}$$

rather than

$$\int \beta\gamma_5 = -\delta\int \gamma_5,\ \delta \approx (7/5)\,MB\rho/\mu^2. \tag{AIX}$$

Eq. (AIX) results from combining eq.s (7) and (20) of ref. 3. They read

$$\int \gamma_5 = iB\lambda\int \sigma\cdot \boldsymbol{r} \tag{AX}$$

$$\int \beta\gamma_5 = -iB\lambda\delta\int \sigma\cdot \boldsymbol{r} \tag{AXI}$$

allowing for a slight change in notation.

According to Peaslee ρ takes care of the fact that the effective potential in the nucleus does not arise from two-body pseudoscalar forces alone; it includes the uncertainties in the relation (AIV). $|\rho|$ is not expected to differ by an order of magnitude from 1.

Appendix B

Since 1952 various approaches have been taken in formulating a theory of the pseudoscalar beta interaction.[2),3),6),7),8)] The last paper, essentially of review nature, provides a handy medium for some clarifying remarks. While often availing themselves of different formalisms, the authors of that paper follow in principle the lines established in refs. 2 and 7.

Some confusion appears to have arisen during the course of interpreting refs. 2 and 7. Eq.s (6)—(9) in ref. 7

$$\begin{aligned} 2M(f|\beta\gamma_5 L_p|i) &= (W_i-W_f)(f|\gamma_5 L_p|i)+(f|[V,\gamma_5 L_p]|i) \\ &\quad +i(f|\sigma\cdot\{\nabla L_p\}|i) && \text{(BI)} \\ &\simeq (W_i-W_f)(f|\gamma_5\theta_p|i)R_p]_\rho \\ &\quad +(f|[V,\gamma_5\theta_p]|i)R_p]_\rho+(f|\gamma_5\theta_p[V,R_p]|i) && \text{(BII)} \\ &\quad +i(f|\sigma\cdot\{\nabla\theta_p\}|i)R_p]_\rho+i(f|\theta_p\sigma\cdot\{\nabla|i)R_p\}_{\mathrm{p}}, \\ 2M(f|\beta\gamma_5\theta_p|i) &= (W_i-W_f)(f|\gamma_5\theta_p|i)+(f|[V,\gamma_5\theta_p]|i) \\ &\quad +i(f|\sigma\cdot\{\nabla\theta_p\}|i), && \text{(BIII)} \\ (f|\beta\gamma_5 L_p|i) &\simeq (f|\beta\gamma_5\theta_p|i)R_p]_\rho \\ &\quad +(1/2M)(f|\gamma_5\theta_p[V,R_p]|i) \\ &\quad +(i/2M)(f|\theta_p\sigma'\cdot\{\nabla|i)R_p\}_{\mathrm{p}} && \text{(BIV)} \end{aligned}$$

$L_p=\theta_p R_p$, p=pseudoscalar

are relativistic.* Only one sample calculation, eqs. (11) and (12) in ref. 7, resulted in the partially relativistic approximation

$$(f|\beta\gamma_5\theta_p|i)R_p]_\rho+i(f|\theta_p\sigma^1\cdot\{\nabla|i)R_p\}_{\mathrm{p}}/2M. \qquad \text{(BV)}$$

Its specialization to $\Delta I=0$ (yes) transitions is

$$(f|\beta\gamma_5|i)L_p]_\rho+i(f|\sigma'\cdot\{\nabla|i)L_p\}_{\mathrm{p}}/2M, \quad L_p\propto R_p \qquad \text{(BVI)}$$

eq. (1) in ref. 2. Be it mentioned, though, that eq. (BVI) is of relativistic form, as becomes obvious by glancing at eq. (BIV). All one needs to do to make it actually relativistic is to replace Pauli's σ' by Dirac's σ. Eq. (1) in ref. 2 is called partially relativistic due to its particular mode of derivation, represented by eqs. (11) and (12) in ref. 7.

There are terms in the $\Delta I=0$ (yes) PP, TP and AP correction factors, whose evaluation the authors of ref. 8 do not consider necessary for presumed reasons of negligibility, which, however, can in general not be discarded. In fact, indications are, that these terms may not only approximately equal those kept by the authors of ref. 8 but may actually overwhelm them.[3),6)] The terms in question are the ones containing $\int\beta\gamma_5$. The complete correction factors may be found in ref. 7.

In the absence of such features as are provided by present day pseudoscalar meson theory the commutator of the potential energy operator, $V=H^c+H^\nu$——c denoting the Coulomb interaction and ν the specifically nuclear interaction——with γ_5 has a matrix element approximately equal to $B\int\gamma_5$. (See also eq.s (2)). Eq. (1b) allows one consequently to write

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* The notation $(f|\theta_p\sigma\cdot\{\nabla|i)R_p\}_{\mathrm{p}}$ implies the gradient operation on R_p, and R_p alone, followed by the evaluation at ρ.

$$\int\beta\gamma_5 \simeq (1/2M)\,B\int\gamma_5\,{}^*. \tag{BVII}$$

Using eq. (4a) one obtains

$$\int\beta\gamma_5 \simeq (i/2M)\,B^2\int\sigma\cdot\boldsymbol{r}. \tag{BVIII}$$

This nuclear matrix element is by no means negligible when compared to

$$[i/2MR_p]_\rho\,(f|\sigma\cdot\{\nabla|i)\,R_p\}_\rho = [iR_p'/2MR_p]_\rho\int\frac{\sigma\cdot\boldsymbol{r}}{r} \tag{BIX}$$

$$\simeq (i/M)\,B^2\int\sigma\cdot\boldsymbol{r}$$

contrary to the contention of the authors of ref. 8. The approximate equality results from neglecting higher order radial functions and cutting the remaining expression for $|R_p'R_p^{-1}|_\rho$,

$$\{[1+(W+\alpha Z\rho^{-1})^2](f_0^2+g_{-2}^2)+2(W+\alpha Z\rho^{-1})(f_0^2-g_{-2}^2)\}^{1/2}(g_0^2+f_{-2}^2)^{-1/2}$$

down to the lowest order in the nuclear radius, ρ^{-1}.

Abandoning the particularization to $\Delta I=0$ (yes) transitions one can state that, within the realm of pseudoscalar beta interaction magnitudes, it is in general not permissible to neglect the commutator matrix element, $(f|[V,\gamma_5 L_p]|i)$**; in some cases one may not even neglect the term $(W_i-W_f)(f|\gamma_5 L_p|i)$. Should it prove possible to devise a reasonable model making $(f|[V,\gamma_5 L_p]|i)$ and $(W_i-W_f)(f|\gamma_5 L_p|i)$ always much smaller than $(f|\sigma\cdot\{\nabla L_p\}|i)$ then

$$(f|\beta\gamma_5\theta_p|i)\,R_p]_\rho \simeq (i/2M)\,(f|\sigma\cdot\{\nabla\theta_p\}|i)\,R_p]_\rho \tag{BX}$$

enabling one to state $2M(f|\beta\gamma_5 L_p|i) \simeq i(f|\sigma\cdot\{\nabla L_p\}|i)$. (BXI)

Were such a procedure permissible the correction factors appearing in reference 8 would, as expected, turn out to be special cases of those in ref. 7.

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* The term $(W_i-W_f)(f|\gamma_5|i)$ need only be considered in addition to $(f|[V,\gamma_5]|i)$ if the total energy difference, W_i-W_f, is comparable in magnitude to B.

** $(f|[V,\gamma_5 L_p]|i)=(f|[V,\gamma_5\theta_p]R_p|i)+(f|\gamma_5\theta_p[V,R_p]|i)$ The last term is identically equal to 0 unless, and this would entail an extension to the Dirac theory, V contains operators not commuting with the radial coordinate. Such an extension shall not be considered here nor was it in the calculations in ref. 7; in eq. (9), ref. 7, eq. (B IV), which is an approximate rather than an actual equality because of the evaluation of the radial lepton covariant near the nuclear boundary—and for no other reasons—the term is still present.

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Cluster Development Method in the Quantum Mechanics of Many Particle System, II

—*Saturation of Nuclear Forces*—

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The cluster development method formulated in the first paper is applied to the discussion on the saturation of nuclear forces. We assume Serber type two-body forces with hard cores, in which central potentials are assumed to be spin independent (eqs. (1) and (2)). The potential depth and range are determined from the scattering length and the effective range in the spin singlet state, and the hard core radius is taken as $D=0.6\times10^{-13}$ cm. The Coulomb forces are neglected. The usual variational method is used with a simple trial correlation function (eq. (6)). By using the one- and two-nucleon clusters only, the energy minimum per nucleon is found to be -5.2 Mev at $r_0=1.1\times10^{-13}$ cm (for r_0 see eq. (5)). At this minimum point the three-nucleon cluster terms are evaluated. They are one order of magnitude smaller than the leading term, and the convergency of the development seems to be fairly good. It is seen that the effective potential for the single particle motion in the nuclear matter is much weakened after separating the short range correlations between nucleons.

§ 1. Nuclear forces

In the first paper[1] (hereafter refered to as I) we formulated the cluster development method which seems useful for the treatment of a system of particles interacting through strong but short range forces. In this paper we apply the method to a large volume of nuclear matter, in order to see whether the saturation property can be explained by the repulsive cores in the two-body forces. In I we already referred to the reason why the saturation phenomena are expected to occur, although the core radius is much smaller than the mean distance between nucleons. This problem has been investigated by many researchers, but it seems still far from the real solution. It is so important and fundamental for the nuclear physics that we choose it for the first application of our cluster method.

We take the charge independent two-body forces with Serber type exchange character :

$$V(x_i,\ x_j)=\tfrac{1}{2}(1+P_{ij}^M)\{V_C(r_{ij})+S_{ij}V_T(r_{ij})\},\tag{1}$$

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where P_{ij}^M is the Majorana exchange operator and S_{ij} is the tensor operator. At present the tensor part is ambiguous in the spin triplet state.[2] Then we assume that the central force in the spin triplet state is the same as in the spin singlet state, which is not inconsistent with the experiments. For the central potential we assume the exponential shape with a hard core:

$$V_C(r)=\begin{cases}\infty & , r<D\\ -Ae^{-\alpha(r-D)}, & r>D,\end{cases} \tag{2}$$

where the force constants A and α are determined from the scattering length and the effective range in the spin singlet state. We confine ourselves to the calculation of the nuclear volume energy with neglect of the Coulomb forces.

§ 2. Trial function and the energy expectation value

Let us consider a system of N nucleons in a large volume Ω and $N/4$ nucleons in each spin-isotopic spin state. For the ground state of the system we take the trial function

$$\Psi(x_1,\cdots,x_N)=[\prod_{(ij)}f(r_{ij})]\cdot(N!)^{-1/2}\sum_P\varepsilon_P\prod_i u_{p_i}(\alpha_i)\,e^{i\boldsymbol{k}_{p_i}\boldsymbol{r}_i}/\sqrt{\Omega}, \tag{3}$$

in which the single particle functions are taken to represent the degenerate Fermi gas in the volume Ω. α_i's are the spin and isotopic spin variables and u_i's are their eigen functions.

If we denote the Fermi wave number by k_F, the well-known expression for the density can be written:

$$N/\Omega=16\pi k_F^3/3(2\pi)^3. \tag{4}$$

The following parameter is usually used to represent the mean radius of the volume occupied by a nucleon.

$$r_0=(3\Omega/4\pi N)^{1/3}=1.523/k_F. \tag{5}$$

The two particle function $f(r)$ should be zero inside the hard core and should approach unity as the particle separation r becomes large, as is shown in Fig. 1. Here we assume its form to be as follows:

$$f(r)=\begin{cases}0 & , r<D\\ 1-e^{-\mu(r-D)}, & r>D.\end{cases} \tag{6}$$

μ and r_0 are variational parameters which are adjusted so as to give the minimum of the energy expectation value. According to relation (5) we may consider k_F or N/Ω as a variational parameter instead of r_0.

The correlations we are concerned with are of short range. It is expected that the probability of finding a number of particles in such a small range at the same time will decrease rapidly with the number. In I, therefore, we expanded the energy expectation value in the power of $N\omega/\Omega$ (ω is the volume where the function f is appreciably different from unity):

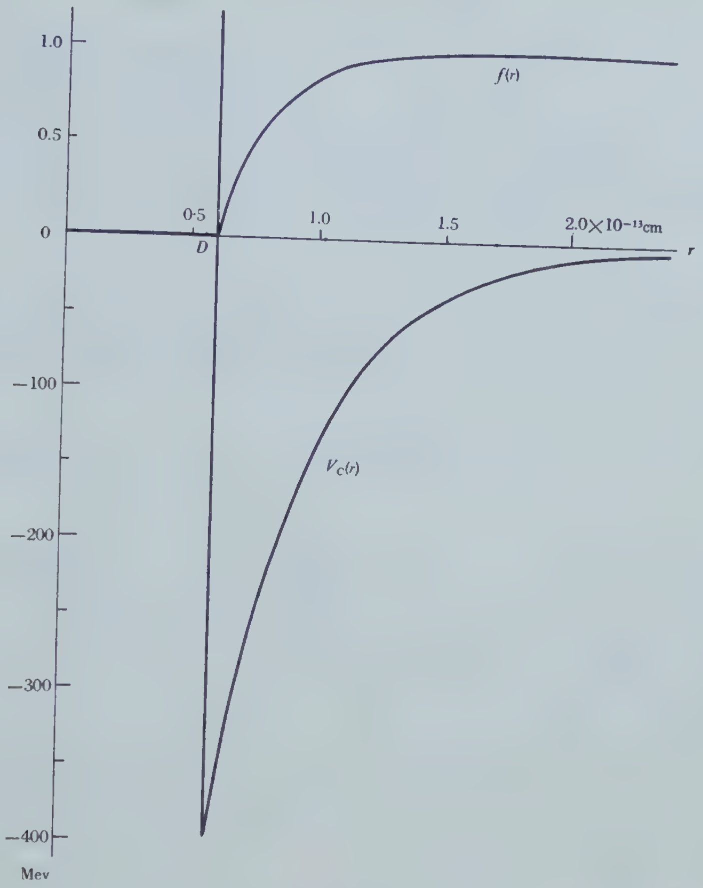

Fig. 1

$$\mathcal{E}=(1/N)\cdot(\Psi,\ H\Psi)/(\Psi,\ \Psi)=\mathcal{E}^{(0)}+\mathcal{E}^{(1)}+\cdots \tag{7}$$

The zeroth order term $\mathcal{E}^{(0)}$ comes from the first sums in (I·31) and in (I·32). For the potential (1) and the trial function (3), it is written as follows:

$$\mathcal{E}^{(0)}=\frac{3}{5}\frac{\hbar^2 k_F{}^2}{2m}+\frac{1}{2N\Omega}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}^{<k_F}\int\{v_D(\boldsymbol{r})-v_X(\boldsymbol{r})\,e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}}\}\,d\boldsymbol{r}, \tag{8}$$

in which spin and isotopic spin summations are already performed. m is the nucleon mass and the mass difference of neutron and proton is neglected. $\sum^{<k_F}$ means that the sum of $\boldsymbol{k}$ should be taken inside the Fermi sphere. v_D and v_X, which are the direct and the exchange terms respectively, are given by the formulas

$$\begin{aligned}v_D(\boldsymbol{r})=&6f^*(r)\,V_C(r)\,f(r)\\&-16(\hbar^2/m)\,f^*(r)\,\nabla^2 f(r)-16i(\hbar^2/m)\,f^*(r)\,(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\nabla f(r),\end{aligned}$$

$$v_X(\boldsymbol{r}) = -6f^*(r)\,V_C(r)\,f(r)$$
$$-4(\hbar^2/m)\,f^*(r)\,\nabla^2 f(r) - 4i(\hbar^2/m)\,f^*(r)\,(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\nabla f(r). \tag{9}$$

The tensor forces do not contribute to the energy expectation value for the trial function (3) which has no directional correlation.

The first order term $\mathcal{E}^{(1)}$ is conveniently separated in four parts according to the types of the correlations between nucleons:

$$\mathcal{E}^{(1)} = \mathcal{E}_\Lambda + \mathcal{E}_\Delta + \mathcal{E}_H + \mathcal{E}_Y \tag{10}$$

$\mathcal{E}_\Lambda$ consists of the second sum and a part of the third sum in (I·32), in which the third particle is connected to the $H(1,2)g(1,2)$ through the $h(2,3)$. After spin and isotopic spin sums are performed, it is written as follows:

$$\mathcal{E}_\Lambda = \frac{1}{N\Omega^3}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}^{<k_F}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_3\, h(r_{23})$$
$$\times\{v_X(\boldsymbol{r}_{12})\,e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{12}+\boldsymbol{k}_2\cdot\boldsymbol{r}_{23}+\boldsymbol{k}_3\cdot\boldsymbol{r}_{31})} - v_D(\boldsymbol{r}_{12})\,e^{i\boldsymbol{k}_{31}\cdot\boldsymbol{r}_{31}}\}. \tag{11}$$

$\mathcal{E}_\Delta$, which comes from the part of the third sum in (I·32) where the third particle is connected to the $H(1,2)g(1,2)$ by the $h(2,3)h(3,1)$, is written as

$$\mathcal{E}_\Delta = \frac{1}{2N\Omega^3}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}^{<k_F}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_3\, h(r_{23})\,h(r_{31})$$
$$\times\{v_D(\boldsymbol{r}_{12})\,(4-e^{i\boldsymbol{k}_{23}\cdot\boldsymbol{r}_{23}}-e^{i\boldsymbol{k}_{31}\cdot\boldsymbol{r}_{31}})$$
$$-v_X(\boldsymbol{r}_{12})\,(4\,e^{i\boldsymbol{k}_{12}\cdot\boldsymbol{r}_{12}}-e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{13}+\boldsymbol{k}_2\cdot\boldsymbol{r}_{21}+\boldsymbol{k}_3\cdot\boldsymbol{r}_{32})}-e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{12}+\boldsymbol{k}_2\cdot\boldsymbol{r}_{23}+\boldsymbol{k}_3\cdot\boldsymbol{r}_{31})})\}. \tag{12}$$

$\mathcal{E}_H$ corresponds to the fourth sum in (I·32), where $h(3,4)$ is connected to the $H(1,2)\cdot g(1,2)$ by an exchange. It is given by

$$\mathcal{E}_H = \frac{1}{4N\Omega^4}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4}^{<k_F}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_3 d\boldsymbol{r}_4\, h(r_{34})\,e^{i(\boldsymbol{k}_{13}+\boldsymbol{k}_{24})\cdot\boldsymbol{r}_{24}}$$
$$\times\{v_D(\boldsymbol{r}_{12})\,(e^{i\boldsymbol{k}_{13}\cdot(\boldsymbol{r}_{12}-\boldsymbol{r}_{34})}+e^{i(\boldsymbol{k}_{14}\cdot\boldsymbol{r}_{12}+\boldsymbol{k}_{32}\cdot\boldsymbol{r}_{34})})$$
$$-v_X(\boldsymbol{r}_{12})\,(e^{i(\boldsymbol{k}_{13}\cdot\boldsymbol{r}_{12}+\boldsymbol{k}_{32}\cdot\boldsymbol{r}_{34})}+e^{i(\boldsymbol{k}_{14}\cdot\boldsymbol{r}_{12}+\boldsymbol{k}_{31}\cdot\boldsymbol{r}_{34})})\}. \tag{13}$$

Finally (I·33), which arises from $\nabla f\cdot\nabla f$ terms, is written as $\mathcal{E}_Y$:

$$\mathcal{E}_Y = -\frac{2}{N\Omega^3}\frac{\hbar^2}{m}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}^{<k_F}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_3\frac{\boldsymbol{r}_{23}\cdot\boldsymbol{r}_{13}}{r_{23}r_{13}}\,f^*(r_{12})\,f(r_{12})\,f^*(r_{23})\,f'(r_{23})\,f^*(r_{31})\,f'(r_{31})$$
$$\times(16+e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{13}+\boldsymbol{k}_2\cdot\boldsymbol{r}_{21}+\boldsymbol{k}_3\cdot\boldsymbol{r}_{32})}+e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{12}+\boldsymbol{k}_2\cdot\boldsymbol{r}_{23}+\boldsymbol{k}_3\cdot\boldsymbol{r}_{31})}$$
$$-4\,e^{i\boldsymbol{k}_{12}\cdot\boldsymbol{r}_{12}}-4e^{i\boldsymbol{k}_{23}\cdot\boldsymbol{r}_{23}}-4\,e^{i\boldsymbol{k}_{31}\cdot\boldsymbol{r}_{31}}), \tag{14}$$

where the prime on f means the derivative of the function f. In these formulas

$$h(r) = f^*(r)\,f(r) - 1,$$

and

$$\boldsymbol{r}_{ij}=\boldsymbol{r}_i-\boldsymbol{r}_j, \quad \boldsymbol{k}_{ij}=\boldsymbol{k}_i-\boldsymbol{k}_j.$$

The second and the third sums in (I·31), which are also of first order in $N\omega/\Omega$, vanish for the present trial function (3).

$\mathcal{E}_\Lambda$ and $\mathcal{E}_\Pi$ have no counterparts in the expression (I·A4) of the energy expectation value for a boson system. They are essentially due to the exchange effect.

Integrations of the above formulas are carried out with the explicit forms (2) and (6) for the V_C and the f. The resulting expressions are given in the Appendix.

§ 3. Results of the calculation

We choose the radius of the hard core in the potential (2) as

$$D=0.6\times10^{-13}\,\text{cm}.$$

Then the remaining force constants A and α are given by

$$A=397.3\ \text{Mev},$$

$$\alpha=2.627\times10^{13}\,\text{cm}^{-1}$$

from Table I in reference 3) for the singlet effective range of 2.7×10^{-13} cm.

With these constants $\mathcal{E}^{(0)}$ is calculated for various values of parameters μ and r_0. The result is shown in Fig. 2. We can see that the minimum of the energy expectation

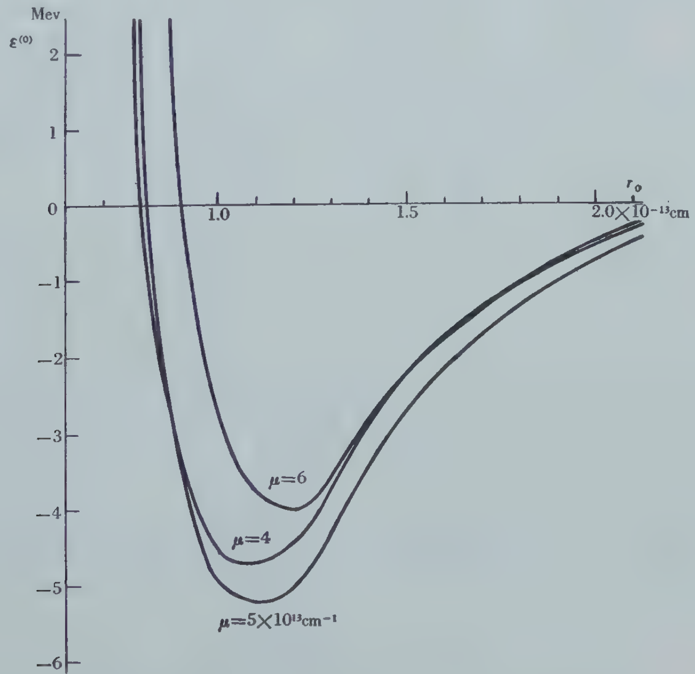

Fig. 2

value per nucleon is found to be -5.2 Mev at $r_0 \doteq 1.1\times10^{-13}$ cm and $\mu=5.0\times10^{13}\,\mathrm{cm}^{-1}$.

At this minimum point the contributions of the terms in (8) and (9) to $\mathcal{E}^{(0)}$ are shown in Table I.

Table I. Contributions to $\mathcal{E}^{(0)}$ at $r_0=1.1\times10^{-13}$ cm and $\mu=5.0\times10^{13}\,\mathrm{cm}^{-1}$

	Energy per nucleon in Mev
Kinetic energy as a Fermi gas	24.2
Effective potential energy	-29.4
Pure potential part	-77.5
Kinetic part due to correlations	48.1

In this table the first row is the contribution from $(3/5)(\hbar^2k_F^2/2m)$ term in (8), which represents the average kinetic energy as a Fermi gas. The contribution from the integral in (8) is given in the second row. This may be regarded as the average effective potential energy to the single particle motion. It consists of two parts: One is the pure potential energy (f^*V_Cf term in (9)), and the other is the kinetic energy due to correlations ($-(\hbar^2/m)f^*\nabla^2f$ and $-i(\hbar^2/m)f^*(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\nabla f$ terms in (9)). They are given in the third and the last rows.

At the point where $\mathcal{E}^{(0)}$ is minimum the various contributions to $\mathcal{E}^{(1)}$ are calculated, since they are important in order to see how rapidly the cluster development converges. The results are summarized in Table II.

Table II. Contributions to $\mathcal{E}^{(1)}$ at $r_0=1.1\times10^{-13}$ cm and $\mu=5.0\times10^{13}\,\mathrm{cm}^{-1}$

Type of term	Energy per nucleon in Mev	Potential part in Mev	Kinetic part in Mev
Λ	-2.8	-11.5	8.7
Δ	0.1	-3.4	3.5
H	0.6	2.6	-2.0
Y	1.5	—	1.5

Sum of the values shown in the second column, each of which (except Y type term appearing from $\nabla f\cdot\nabla f$) consists of potential and kinetic parts written in the third and the last columns, gives $\mathcal{E}^{(1)}=-0.6$ Mev. However, in the values of the third and last columns there may be errors of less than 10% in Λ and of less than 30% in Δ, H, and Y terms, since they are calculated by a certain expansion described in the Appendix. Therefore, the values in the second column should not be regarded with too much confidence, since they are obtained after rather a large cancellation of potential and kinetic parts.

§ 4. Discussion

We have calculated the energy expectation value of large nuclear matter by the cluster development method assuming a reasonable two-body force with hard core.

In order to see the convergency of our development, let us compare $\mathcal{E}^{(1)}$ with $\mathcal{E}^{(0)}$ at the minimum point. The ratio is 1/9. However, there may be accidental cancellation in $\mathcal{E}^{(1)}$ due to the errors mentioned before. To see the relations more precisely, we compare the expectation values of $f^* V_C f$, one of which is given in the third row of Table I and the other is the sum of the values in the third column of Table II. Their ratio is 1/6. Also the similar ratio of the expectation value of $-(\hbar^2/m)f^* \nabla^2 f$ and $-i(\hbar^2/m)f^*(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\nabla f$ is 1/5. Therefore, even if it may not be a very good approximation to neglect the first order term $\mathcal{E}^{(1)}$ entirely, our development will still be a powerful method for the treatment of nuclear problems. Even if we do not include all the terms in $\mathcal{E}^{(1)}$, the inclusion of rather a large Λ type term improves the situation very much.

It is interesting to compare the expressions (8) and (9) with the usual Hartree approximation. The first term in (8) is just the kinetic energy of the degenerate Fermi gas. If we put $f(r)=1$ in (9), there appears $\pm 6V_C(r)$ alone instead of $v_D(\boldsymbol{r})$ and $v_X(\boldsymbol{r})$ in the integral in (8). This is the well-known Hartree approximation. In this case the integral diverges to infinity when the potential $V_C(r)$ has a hard core.

However, in the present case $V_C(r)$ appears in a form $f^* V_C f$ in (9). That is to say, it is averaged with the weight of the actually correlated functions. $f^* V_C f$ is of the form $0\times\infty$ in the core, $r<D$. However, if we solve a two-body problem with a core potential of finite height and then let the height tend to infinity, we can easily see that the above form tends to zero. It is likely that such a property still continues to hold in the case of the many-body problem. The second term $-(\hbar^2/m)f^*\nabla^2 f$ in (9) means that there is a kinetic energy due to the correlations between nucleons. The last term $-i(\hbar^2/m)f^*(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\nabla f$ is the cross term in the kinetic energy between correlations and the single particle motions.

The above relations are illustrated in Fig. 3 for $r_0=1.1\times10^{-13}$ cm and $\mu=5.0\times 10^{13}$ cm^{-1}. We can see that the $v(r)$, which may be regarded as the effective potential to the "single particle motion" in the nucleus, is much weaker than the real potential due to the separation of correlations between nucleons. This may be a reason why the independent particle model explains some properties of nuclei fairly well in spite of the strong interactions between nucleons.

The result of the present calculation suggests that the system does not collapse into high density but is stable at about the density observed experimentally. The binding energy per nucleon obtained in this paper is smaller than the actual volume energy of $\sim$15 Mev. Although we shall be able to get a better agreement in the binding energy by taking a little deeper central potential in the spin triplet state, it does not seem worthwhile to discuss this problem in detail before we improve the approximation in both formulation and numerical calculation. Especially our formulation cannot take account of the effect of the tensor force. It is averaged to zero in the present calculation, but actually is not, although it is supposed not to be so important in the many-particle system as in the two-body system. It is also to be noted that the result may depend on the constants of the two-body forces rather sensitively, since the $\mathcal{E}$ is obtained after the large

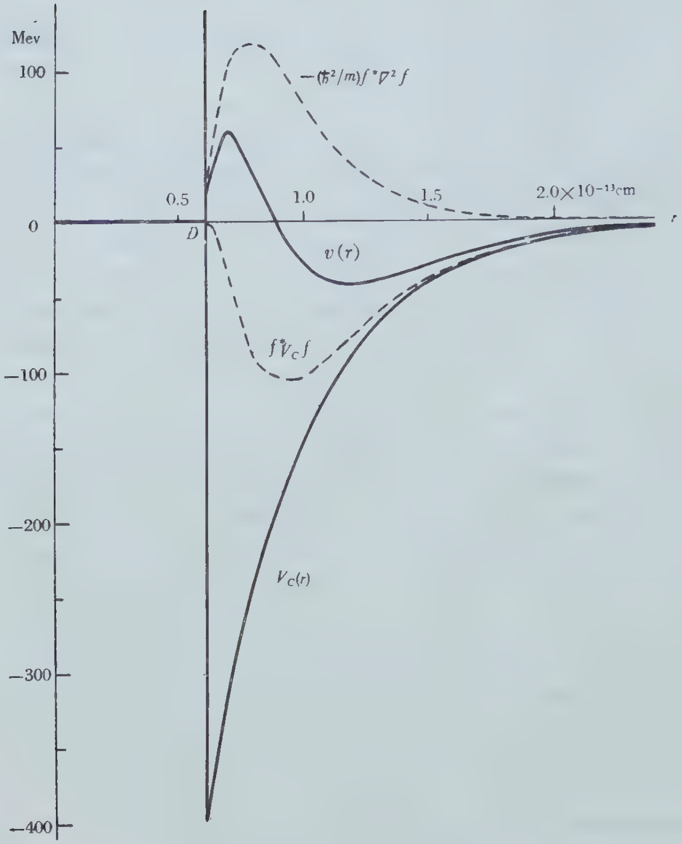

Fig. 3. $v(r)=f^*V_cf-(\hbar^2/m)f^*\nabla^2f$, which represents $(1/12)(v_D(\boldsymbol{r})-v_X(\boldsymbol{r}))$ for $\boldsymbol{k}_1=\boldsymbol{k}_2$. $V(r)$ should be read as $v(r)$ in the figure.

cancellation of the large potential and kinetic energies. Anyhow, it is very probable from the present calculation that the saturation phenomena can be explained by a two-body force only, which is compatible with the two-body data, although we cannot say it conclusively if we consider the crudeness of our calculation and the ambiguities in our present knowledge of the two-body forces. This is the same answer as was found by Brueckner et al.[4] with their method.

It is to be remarked that in the boson case the cluster development is just the expansion in powers of the density N/Ω, as is seen from (I·A4). If we calculate only the zeroth order term in N/Ω, it gives no minimum of the energy expectation value with respect to density. If we further calculate up to first order, there may be a minimum point. However, at this point the contribution from the first order term is a half of that of the zeroth order term. Therefore, in this case the development is not expected to be so useful. On the other hand, there are many other dependences on k_F in the fermion case, giving rise to the results described above. From these considerations we can see that the Pauli principle also plays an important role in the saturation problem.*

It is desirable to generalize the trial function so as to include many effects neglected in the present calculation. The general formulation will be given in a later issue of this series. On the other hand it will be interesting to construct the equations for φ and f just as the usual Hartree-Fock equation coming from the variational principle.[6] Such equations will give us a different point of view from that of the straight-forward calculation with variational parameters.

Appendix I. Details of calculation

In this paper $\mathcal{E}$'s are mainly calculated in the following way : $\boldsymbol{k}$ sums are taken inside the Fermi sphere, the resulting Bessel functions are expanded in the power of $k_F r$, and then space integrations are performed.

However, the expansions of the Bessel functions are not powerful for the terms involving $e^{-\alpha r}$ or $f^*(r)f(r)$, since they do not damp so rapidly at large r. (It is very useful for the terms including $e^{-\mu r}$, $e^{-(\alpha+\mu)r}$, $\cdots$.) Therefore, the integrals for the $e^{-\alpha r}$ terms in $\mathcal{E}^{(0)}$, $\mathcal{E}_\Delta$, and $\mathcal{E}_H$ are rewritten as

$$\int_D^\infty dr = \int_0^\infty dr - \int_0^D dr,$$

and the expansions are applied only to the domain $(0, D)$, while the $(0, \infty)$ integrations are performed analytically. For $\mathcal{E}_\Delta$ and $\mathcal{E}_Y$, the r_{12} integrations are carried out numerically.

$\mathcal{E}_H$ is calculated with the following approximation. Integrating $\mathcal{E}_H$ with respect to $\boldsymbol{r}_1$ and $\boldsymbol{r}_{24}$, it is expressed as

$$\mathcal{E}_H = \frac{1}{4N\Omega^2} \sum_{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4}^{<k_F} \delta(\boldsymbol{k}_1+\boldsymbol{k}_2,\ \boldsymbol{k}_3+\boldsymbol{k}_4) \int d\boldsymbol{r}_{12} d\boldsymbol{r}_{34} h(r_{34}) \{v_X(\boldsymbol{r}_{12})\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{12}+\cdots)} - \cdots\}.$$

Here, $\delta(\boldsymbol{k}_1+\boldsymbol{k}_2,\ \boldsymbol{k}_3+\boldsymbol{k}_4)$ is 1 for $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$ and otherwise 0. The exact calculation of $\mathcal{E}_H$ is difficult to perform. Therefore, we employ the approximation

$$\mathcal{E}_H \simeq \frac{1}{4N\Omega^2} w(4/N)^4 \sum_{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4}^{<k_F} \int d\boldsymbol{r}_{12} d\boldsymbol{r}_{34} h(r_{34}) \{v_X(\boldsymbol{r}_{12})\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_{12}+)\cdots} - \cdots\},$$

where

$$w = \sum_{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4}^{<k_F} \delta(\boldsymbol{k}_1+\boldsymbol{k}_2,\ \boldsymbol{k}_3+\boldsymbol{k}_4) = (34/105)\,(N/4)^3.$$

---

* It should be noted that the Pauli principle which is one of the central subject of the method of Bethe[5] and Brueckner is "the Pauli principle in the intermediate states" in their terms, and is not the same as "the Pauli principle" used more comprehensively in this paper. We have not yet set up the one-to-one correspondence between their theory and ours.

This approximation means that the average value is taken in the Fermi sphere without the restriction of $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$, and then multiplied by the volume of the region where $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$.

Appendix II. Explicit formulas

We use the following abbreviations:

$$a_n=(n-1)!\,\Big[e^{\alpha D}/\alpha^n-\sum_{i=1}^{n}D^{n-i}\{\alpha^{-i}-2(\alpha+\mu)^{-i}+(\alpha+2\mu)^{-i}\}/(n-i)!\Big],$$

$$B(x)=x+2\tan^{-1}2/x-x(3/2+x^2/4)\ln(1+4/x^2),$$

$$B'(x)=x(1-x^2/4)+(1/3-5x^2/4)\tan^{-1}2/x$$
$$-x(1/3-x^2/2-x^4/16)\ln(1+4/x^2),$$

$$p=\mu D,$$

and $j_l(x)$'s are the usual Bessel functions of half-integer order.

i) $$\mathcal{E}^{(0)}=(\hbar^2k_F^2/10\pi m)\{3\pi+(5k_F/2\mu)(1+2p+2p^2)$$
$$+(k_F^3/6\mu^3)(3+6p+6p^2+4p^3+2p^4)$$
$$-(k_F^5/140\mu^5)(45+90p+90p^2+60p^3+30p^4+12p^5+4p^6)+\cdots\}$$
$$-(A/\pi)\{(3e^{\alpha D}/4)B(\alpha/k_F)$$
$$+(e^{\alpha D}/\alpha^3-a_3)k_F^3+a_5k_F^5/10-3a_7k_F^7/350+\cdots\}.$$

ii) $$\mathcal{E}_\Lambda=(k_F^3/36\pi^2\mu^3)E_3-(k_F^5/300\pi^2\mu^5)E_5+\cdots.$$

iii) $$\mathcal{E}_H=-(17k_F^3/3780\pi^2\mu^3)E_3+\cdots.$$

In ii) and iii)

$$E_3=(45+42p+18p^2+4p^3)$$
$$\times\big[(\hbar^2k_F^3/\mu m)\{(1+2p+2p^2)-(k_F^2/10\mu^2)(51+54p+30p^2+12p^3+4p^4)$$
$$+(3k_F^4/350\mu^4)(1935+1950p+990p^2+340p^3+90p^4+20p^5+4p^6)-\cdots\}$$
$$-4A\{(3e^{\alpha D}/2)B(\alpha/k_F)-a_3k_F^3+a_5k_F^5/5-3a_7k_F^7/175+\cdots\}\big],$$

$$E_5=(945+930p+450p^2+140p^3+30p^4+4p^5)$$
$$\times\big[(\hbar^2k_F^3/\mu m)\{(5+2p+2p^2)$$
$$-(k_F^2/315\mu^2)(2319+2478p+1398p^2+572p^3+196p^4)+\cdots\}$$
$$-4A\{(9e^{\alpha D}/2)B'(\alpha/k_F)-a_3k_F^3+26a_5k_F^5/105-\cdots\}\big].$$

iv) $$\mathcal{E}_\Delta = (k_F^6/9\pi^2)\int_D^\infty F(r)\,\{G(r)+\cdots\}\,r\,dr,$$

where

$$\begin{aligned}
F(r) = {} & 3f^*(r)V_C(r)f(r)\,\{1-(3/k_Fr)\,j_1(k_Fr)+(18/k_F^2r^2)\,j_1^2(k_Fr)\} \\
& -(2\hbar^2/m)\,(f^*(r)\nabla^2 f(r))\,\{4+(3/k_Fr)\,j_1(k_Fr)-(18/k_F^2r^2)\,j_1^2(k_Fr)\} \\
& -(6\hbar^2/m)\,(f^*(r)f'(r)/r)\,\{(3k_Fr/10)\,j_1(k_Fr)-j_2(k_Fr) \\
& \qquad\qquad +(12/k_Fr)\,j_1(k_Fr)\,j_2(k_Fr)\},
\end{aligned}$$

$$\begin{aligned}
G(r) &= \iint\limits_{|r-r_{23}|<r_{31}<r+r_{23}} h(r_{23})\,h(r_{31})\,r_{23}r_{31}\,dr_{23}\,dr_{31} \\
&= \begin{cases} G'(r)-G''(r), & D<r<2D \\ G'(r)-G'''(r), & r>2D \end{cases}
\end{aligned}$$

with

$$\begin{aligned}
G'(r) = {} & (2e^{-\mu r}/27\mu^4)\,\{2(59+42p+9p^2)+3\mu r(11+6p)\} \\
& -(e^{-2\mu r}/432\mu^4)\,\{(77+96p+36p^2)+3\mu r(13+12p)\}, \\
G''(r) = {} & (1/48\mu^4)\,\{3(137+96p+20p^2)-8\mu r(45+42p+18p^2+4p^3) \\
& +12\mu^2r^2(7+6p+2p^2)-2\mu^4r^4\}, \\
G'''(r) = {} & (2e^{-\mu(r-2D)}/3\mu^4)\,\{2(3-12p+3p^2-2p^3)+6\mu r(1+p^2) \\
& -3\mu^2r^2-\mu^3r^3\} \\
& +(e^{-2\mu(r-2D)}/48\mu^4)\,\{(219-204p+84p^2-16p^3)+6\mu r(13-12p+4p^2) \\
& -6\mu^2r^2-4\mu^3r^3\}.
\end{aligned}$$

v) $$\mathcal{E}_Y = (\hbar^2k_F^6/9\pi^2m)\int_D^\infty K(r)\,\{r^2L_0(r)-2L_2(r)+\cdots\}\,r\,dr,$$

where

$$K(r) = f^*(r)f(r)\,\{4+(3/k_Fr)\,j_1(k_Fr)-(18/k_F^2r^2)\,j_1^2(k_Fr)\},$$

$$\begin{aligned}
L_0(r) &= \iint\limits_{|r-r_{23}|<r_{31}<r+r_{23}} f^*(r_{23})\,f'(r_{23})\,f^*(r_{31})\,f'(r_{31})\,dr_{23}\,dr_{31} \\
&= \begin{cases} L_0'(r), & D<r<2D \\ L_0'(r)+L_0''(r), & r>2D, \end{cases}
\end{aligned}$$

$$\begin{aligned}
L_2(r) &= \iint\limits_{|r-r_{23}|<r_{31}<r+r_{23}} f^*(r_{23})\,f'(r_{23})\,f^*(r_{31})\,f'(r_{31})\,r_{23}^2\,dr_{23}\,dr_{31} \\
&= \begin{cases} L_2'(r)+L_2''(r), & D<r<2D \\ L_2'(r)+L_2'''(r), & r>2D \end{cases}
\end{aligned}$$

with

$$
\begin{aligned}
L_0'(r) &= (1/12\mu^2)\,(3-4\,e^{-\mu r}+e^{-2\mu r}),\\
L_0''(r) &= (1/4\mu^2)\,\{1+4\,(1+2\,p-\mu r)\,e^{-\mu(r-2D)}-(5-4\,p+2\mu r)\,e^{-2\mu(r-2D)}\},\\
L_2'(r) &= -\,(e^{-\mu r}/54\mu^4)\,\{2\,(26+24\,p+9\,p^2)+3\mu r\,(11+6\,p)+9\mu^2 r^2\}\\
&\qquad +(e^{-2\mu r}/432\mu^4)\,\{2\,(19+30\,p+18\,p^2)+3\mu r\,(13+12\,p)+18\mu^2 r^2\},\\
L_2''(r) &= (1/8\mu^4)\,(7+6\,p+2\,p^2),\\
L_2'''(r) &= -\,(e^{-\mu(r-2D)}/3\mu^4)\,\{(6+12\,p+3\,p^2+2\,p^3)-3\mu r\,(2+p^2)+3\mu^2 r^2 p-\mu^3 r^3\}\\
&\qquad +(e^{-2\mu(r-2D)}/24\mu^4)\,\{(69-66\,p+30\,p^2-8\,p^3)+2\mu r\,(21-18\,p+6\,p^2)\\
&\qquad +6\mu^2 r^2\,(3-2\,p)+4\mu^3 r^3\}.
\end{aligned}
$$

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A Lattice Model of Liquid Helium, III

—*Equilibrium Properties of Liquid* He^4 *and Mixtures of* He^4 *and* He^3—

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The partition function of the lattice model of liquid He^4 developed in the preceding papers[1),2)] is shown to be derived by determining the form of a *universal function* of one variable so far as one admits three assumptions previously made. Instead of determining the function by using Kikuchi's approximation as before, the determination is made here by making use of the experimental values of heat capacity at various temperatures under saturated vapour pressure together with two other thermodynamical data at the λ-point. Comparison of the values thus derived with experimental ones is made for various thermodynamical quantities of pure liquid He^4 and of mixtures of He^4 and He^3. The agreements are good, and a consistent explanation of various peculiar properties of this liquid is obtained.

§ 1. Introduction

Numerous investigations have been carried out to elucidate the peculiar equilibrium properties of liquid helium from the atomic theoretical point of view. They may be roughly divided into three classes according to the methods of approach. In the first class atoms in the liquid are assumed to be almost free.[3)] The theory fails to include the effect of correlations between the positions of atoms sufficiently. In the second class each atom is assumed to be localized in a cell.[4)] The theory cannot take full account of the quantum-statistical mechanical effect, accordingly of the cooperative character of Bose-Einstein condensation.

Actual He atoms are neither completely localized nor free. Such an intermediary character of atoms in the assembly makes the usual approximation invalid, and yet this character is supposed to be the main origin of the peculiarities of this liquid. The third class, therefore, should be the one which takes account of these circumstances.

Feynman[5)] and Butler-Friedman[6)] independently proposed with some success the theories which may be classified into the third class. Their theories directly start from the well-known exact hamiltonian of the assembly of He atoms and in this sense they are based on the first principles. In turn, however, they are rather complicated and the singular character of this liquid cannot be derived easily from them.

At this stage we may turn our attention to the important roles of the model method in the theory of classical liquid. They served to explain both the equilibrium and non-equilibrium properties of liquids and have brought many a fruitful result. In quantum liquid, too, it would be desirable to have a model, which should belong to the third class, simple but sufficient to understand the striking features of the liquid.

Along this line of thought Matsubara and Matsuda[1),2)] have developed a lattice model of liquid helium modifying the model previously proposed by S. Koide.[7)] (These papers 1) and 2) will hereafter be referred to as I and II respectively.) They carried out approximate calculations taking into account the quantum mechanical and quantum-statistical effects and the effect due to the rigid core of an atom, and succeeded to explain qualitatively some of the peculiar properties of pure He^4 liquid.

In I the lattice model was discussed in terms of a spin system. In order to accomplish the calculation, the molecular field approximation was used. Then the negative thermal expansion coefficient inherent to the substance could not be obtained, while the correct λ-line was obtained by adopting for the mass of the model the true mass of He^4 atom. Further elaboration to the usual Bethe's approximation does not seem to improve this situation.

In II the partition function of the lattice model was directly calculated by counting the number of pertinent closed polygons drawn in the lattice space applying Kikuchi's approximation besides several simplifying assumptions. In this case, although both the correct form of the λ-line and the negative thermal expansion coefficient was obtained, it was necessary to use an effective mass equal to one seventh of the true mass of He^4 atom, which is too small. Moreover, the calculated values of the specific heat are rather small. All these discrepancies are, as we have pointed out in II, supposed to be mainly due to the too rough application of Kikuchi's approximation. Notwithstanding the fact that the restriction imposed on the types of fundamental graphs in II was too severe, the formidable complications will arise if one tries to remove this restriction.

Now, one may notice the fact that in II although the number of fundamental graphs $g(\{x_i\},\{y_i\})$ involves as its arguments the set of a great many parameters $\{x_i\}$, $\{y_i\}$ fixing the type of F. G.'s, only the value of $g(\{x_i\}, \{y_i\})$ which gives the maximum contribution to the partition function is of predominant importance. This maximum value depends on the independent variables specifying the thermodynamical state of the system (density, temperature and parameters determining hamiltonian of the system). If this dependence is characterized through only few variables which are the functions of the above-mentioned independent variables, we shall have relations between various thermodynamical quantities without knowing all the values of $g(\{x_i\}, \{y_i\})$. Fortunately, in our case, we shall see in the next section that the partition function contains only one *universal function* of one variable besides the independent variables. It would be worthwhile, then, to determine the *universal function* empirically and try to understand the relations between various thermodynamical quantities. This is the main technique taken in this paper. We shall see that all salient thermodynamical phenomena in liquid He^4 and in the mixture of liquid He^4 and He^3 are interrelated ones, explicable by a simple physical consideration.

In I and II our discussion was restricted to pure He^4. The basic physical idea was that as the density increases the assembly changes its state from nearly free state to nearly localized one, and the approximation in the first class becomes worse while that in the second class becomes better. These circumstances, which is absent in the assembly of ideal particles, will be essential to understand the behavior of real He atoms. It will most significantly be reflected on the process involving a change of density. The waves with longer wave lengths essential to the λ-transition are then more and more quenched as the density increases, resulting in the correct density dependence of the λ-line and the possibility of the occurrence of negative thermal expansion coefficient.

This idea will also be applicable when there are He^3 atoms contained as a solute. The peculiar properties of He^4-He^3 mixture, e. g. large deviation from Raoult's law, have been discussed phenomenologically by many authors,[8)] and yet the clear-cut explanation without any *ad hoc* assumptions does not seem to exist. Making use of analogous considerations as in the case of pure He^4, we shall be able to explain in the later sections the properties of the mixture within the frame of our theory when the concentration of He^3 is low.

In § 2 by the aid of three assumptions introduced in II, various thermodynamical quantities are expressed in terms of a universal function and independent thermodynamical variables. In § 3 the universal function is determined by using the experimental data of heat capacity under saturated vapour pressure and two other data at the λ-point. In § 4 various thermodynamical quantities are calculated by the expressions derived in § 2 and the universal function determined in § 3, and comparison is made between the theory and experiments.

§ 2. Reduction of the various thermodynamical quantities

First we discuss the case of pure He^4 in the same model as in II. We shall here briefly review the considerations and notations in II necessary for discussions in this paper. For detailed accounts the reader is referred to II.

The problem is to calculate the partition function

$$Z = \mathrm{Trace}[\exp(-\beta H)], \tag{1}$$

where

$$H = -B \sum_{\langle i,j \rangle} a_i^* a_j, \quad B = \hbar^2/2md^2. \tag{2}$$

a_i^* and a_i are operators which respectively creates and annihilates an atom at the i-th lattice point. We have seen that expanding Z as

$$Z = \mathrm{Trace}[\sum_n (-\beta H)^n/n!], \tag{3}$$

there corresponds to each term of the right-hand side of (3) certain transfer of atoms forming as their trajectories a set of closed polygons in the lattice space. We have classified these sets of closed polygons defining a fundamental graph (F. G.). We have asked, then, in how many ways we can transfer atoms on a F. G. and return to the configuration identical with the initial one after $2n$ transfers.

In order to settle the problem we have introduced three assumptions in II. According to Assumption I there are approximately $[2\bar{m}\rho_0\rho_1]^{2n}$ ways of arranging the $2n$ transfers, if we disregard the fact that the final configuration of particles should be identical with the initial one. Here $\bar{m}$ is the number of lines connecting nearest neighbour lattice points in a skeleton, and $\rho_0=(M-N)/M$ and $\rho_1=N/M$. M is the total number of lattice points and N is the total number of particles. By Assumption II we have to multiply $[2\bar{m}\rho_0\rho_1]^{2n}$ by $[s!/\{(s\rho_0)!\,(s\rho_1)!\}]^{-1}$ in order to correct the number so that the final configuration is identical with the initial one. Here, s is the number of lattice points forming the skeleton. This corresponds to the assumption that complete randamization of particles has occurred in the F. G. after $2n$ transfers independent of the initial configuration.

In order that a graph drawn by arranging $2n$ transfers according to Assumptions I and II belongs to the original F. G. with side length $2m$, we have to put a further correction factor $f(m, n)$, which is approximated according to Assumption III by

$$f(m,\ n)\simeq[(2n)!/2^{2(n-m)}]\binom{2m}{2(n-m)}(2m)^{-2n} \tag{4}$$

for $n>m$ and $n-m\ll m$.

Thus we have obtained

$$Z=\frac{M!}{(M-N)!\,N!}S\sum_{n=0}^{\infty}f(m,\ n)[2\bar{m}\rho_0\rho_1]^{2n}\frac{(\beta B)^{2n}}{(2n)!}\left[\frac{s!}{(s\rho_0)!\,(s\rho_1)!}\right]^{-1}, \tag{5}$$

where S means the summation over all possible F. G.'s.

Each F. G. has been classified by a set of parameters $\{x_i\}$ and $\{y_i\}$ specifying a type of F. G. Denoting by $g(\{x_i\},\ \{y_i\})$ the number of F. G.'s belonging to the set of parameters $\{x_i\}$ and $\{y_i\}$, we have

$$\begin{aligned}Z=&\frac{M!}{(M-N)!\,N!}\sum_{\{x_i\},\{y_i\}}g(\{x_i\},\ \{y_i\})\sum_{n=0}^{\infty}f(m,\ n)[2\bar{m}\rho_0\rho_1]^{2n}\\&\times\frac{(\beta B)^{2n}}{(2n)!}\left[\frac{s!}{(s\rho_0)!\,(s\rho_1)!}\right]^{-1}.\end{aligned} \tag{6}$$

We have seen in II that in the vicinity of the λ-point we can approximate as

$$s\simeq 2m\simeq 2n\simeq 2\bar{m}. \tag{7}$$

Admitting (7), we obtain from (6) for the logarithm of the partition function per lattice point without making direct use of Kikuchi's approximation

$$\begin{aligned}\bar{P}&=\mathrm{Max}\left[\frac{\ln g(\{x_i\},\ \{y_i\})}{M}+\frac{2n(\Gamma+A)}{M}-A\right]\\&=\mathrm{Max}\left[\frac{P(\{x_i\},\ \{y_i\},\ n)}{M}\right],\end{aligned} \tag{8}$$

where

$$\Gamma=\ln(\rho_0\rho_1\beta B),\quad A=\rho_0\ln\rho_0+\rho_1\ln\rho_1. \tag{9}$$

The values of $\{x_i\}$, $\{y_i\}$ and n are to be determined so as to make $P(\{x_i\},\ \{y_i\},\ n)$

maximum for given values of Γ and A. Therefore, $\bar{P}$ can be written as

$$\begin{aligned} \bar{P} &= \varphi(Y) - A, \\ Y &= \exp[A+\Gamma], \end{aligned} \tag{10}$$

where $\varphi(Y)$ is a *universal function* dependent solely through Y on the variables specifying the thermodynamical state of the system.

When we deal with the mixtures of He^4 and He^3, we note that (i) if the concentration of He^3 is low the possibility that He^3 atoms exchange their positions is very small compared with that of He^4 atoms. Therefore, the most part of the contributions to the partition function comes from the graphs in which most He^3 atoms return after $2n$ transfers to their own initial points*; (ii) even when they happen to exchange their places, due to the Fermi statistics the contribution is negative if the permutation is odd. Both of these two facts will reduce the contribution from transfers of He^3 atoms to a considerable extent.

Thus, it will be safe to regard He^3 atoms as fixed, so far as the concentration of He^3 is not too high and the temperature is not too low. The $2n\Gamma$ in (8) should now be written as

$$\begin{aligned} 2n\Gamma &= 2n \ln(\rho_0 \rho_4 \beta B), \\ \rho_4 &= N_4/M, \end{aligned} \tag{11}$$

where N_4 is the number of He^4 atoms. Corresponding to Assumption II of II, the probability of returning to the initial configuration will here be approximated by the reciprocal number of ways that $s\rho_4$ He^4 atoms and $s\rho_0$ holes can be distributed among $s(\rho_0+\rho_4)$ lattice points, i. e. $[\{s(\rho_0+\rho_4)\}!/\{(s\rho_0)!\ (s\rho_4)!\}]^{-1}$.**

This requires that the $2nA$ appeared in (8) should be

$$2nA = 2n\{\rho_0 \ln \rho_0 + \rho_4 \ln \rho_4 - (\rho_0+\rho_4)\ln(\rho_0+\rho_4)\}, \tag{12}$$

and the last term A in (8) should be replaced by

$$\bar{A} = \rho_0 \ln \rho_0 + \rho_4 \ln \rho_4 + \rho_3 \ln \rho_3, \quad \rho_3 = N_3/M \tag{13}$$

respectively, where N_3 is the number of He^3 atoms.

Summarizing the above we have for the logarithm of the partition function per lattice point

$$\bar{P} = \varphi(Y) - \bar{A}, \tag{14}$$

where

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* This consideration applies to the system of liquid He^4 containing as dilute solute any foreign atoms. It is then expected that these systems will show similar thermodynamical properties.

** Strictly speaking the immobile He^3 atoms will act as an obstacle to the transfer of He^4 atoms, contradictory to the fact that the s lattice points belong to a F. G. defined in II. Here, we may think we have roughly taken into account the possibility that He^3 atoms evade the He^4 atom and return afterwards to their initial positions.

$$Y=\exp(A+\Gamma), \quad \Gamma=\ln(\rho_0\rho_1\beta B),$$

$$A=\rho_0 \ln \rho_0+\rho_4 \ln \rho_4-(\rho_0+\rho_4) \ln(\rho_0+\rho_4),$$

$$\bar{A}=\rho_0 \ln \rho_0+\rho_4 \ln \rho_4+\rho_3 \ln \rho_3,$$

$$B=\hbar^2/2m_4 d^2.$$

Here m_4 is the mass of a He^4 atom.

Free energy F, internal energy E, entropy S, heat capacity at constant volume C, and pressure of the system p can easily be written down as

$$F=-MkT\bar{P}, \tag{15}$$

$$E=-M(\partial\bar{P}/\partial Y)(\partial Y/\partial\beta)=-M(\partial\varphi/\partial Y)\,e^A\rho_0\rho_4 B, \tag{16}$$

$$S=(E-F)/T=kM(\bar{P}-Y\cdot\partial\varphi/\partial Y), \tag{17}$$

$$C=kMY^2\cdot\partial^2\varphi/\partial Y^2, \tag{18}$$

$$\begin{aligned} p&=kT/d^3\cdot(\bar{P}+M\cdot\partial\bar{P}/\partial M)\\ &=kT/d^3\cdot[\bar{P}+\rho_4 \ln \rho_4+\rho_3 \ln \rho_3-\rho_1 \ln \rho_0\\ &\quad+\partial\varphi/\partial Y\cdot Y\{\rho_1 \ln \rho_0-\rho_4 \ln \rho_4-\rho_3 \ln(1-\rho_3)+\rho_1/\rho_0-1\}]. \end{aligned} \tag{19}$$

At the λ-point the heat capacity expressed in (18) should have a discontinuity. Since $\varphi(Y)$ is a universal function, the singularity should always occur at the same value of Y. The λ-point is, therefore, determined by

$$\exp[A+\Gamma]=Y_\lambda, \tag{20}$$

where Y_λ is a constant.

So far we have left the effect of attractive forces out of consideration. As the range of attractive force is larger than that of repulsive force we may take as a first approximation that every atom experiences the constant field of attraction so long as it is in the liquid. The potential energy per atom due to attractive force being thus assumed constant, its inclusion does not change the matter in the process where the number of atoms is kept constant as above. In order to obtain the expression for chemical potential, let this mean attractive potential energy per atom be ε.

The chemical potential for He^4 in the liquid is then

$$\begin{aligned} \mu_4&=(\partial F/\partial N_4)_{N_3}=1/M\cdot(\partial F/\partial\rho_4)_{\rho_3}\\ &=-kT\left(\frac{\partial\varphi}{\partial Y}\frac{\partial Y}{\partial\rho_4}-\frac{\partial\bar{A}}{\partial\rho_4}\right)+\varepsilon\\ &=-kT\left[\frac{\partial\varphi}{\partial Y}Y\left\{\left(\frac{1}{\rho_4}-\frac{1}{\rho_0}\right)+\ln\left(\frac{\rho_4}{\rho_0}\right)\right\}-\ln\left(\frac{\rho_4}{\rho_0}\right)\right]+\varepsilon, \end{aligned} \tag{21}$$

and for He^3

$$\mu_3 = (\partial F/\partial N_3)_{N_4'} = 1/M \cdot (\partial F/\partial \rho_3)_{\rho_4}$$
$$= -kT\left[\frac{\partial\varphi}{\partial Y} Y\left\{-\frac{1}{\rho_0} + \ln\left(\frac{1-\rho_3}{\rho_0}\right)\right\} - \ln\left(\frac{\rho_3}{\rho_0}\right)\right] + \varepsilon. \qquad (22)$$

If we assume the vapour of helium as an ideal gas, the chemical potential in the vapour for He^4 and for He^3 are

$$\tilde{\mu}_4 = kT(\ln\tilde{\rho}_4 - \tfrac{3}{2}\ln\lambda_4), \qquad (23)$$

$$\tilde{\mu}_3 = kT(\ln\tilde{\rho}_3 - \tfrac{3}{2}\ln\lambda_3), \qquad (24)$$

respectively, where

$$\lambda_4 = (2\pi m_4 kT)/h^2, \quad \lambda_3 = (2\pi m_3 kT)/h^2.$$

$\tilde{\rho}_4$ and $\tilde{\rho}_3$ are the number density of He^4 and He^3 in the vapour, and m_4, m_3 are the mass of a He^4 and He^3 atom respectively.

From the equation

$$\mu_4 = \tilde{\mu}_4 \quad \text{and} \quad \mu_3 = \tilde{\mu}_3, \qquad (25)$$

we obtain

$$\tilde{\rho}_4 = \lambda_4^{3/2}\frac{\rho_4}{\rho_0}\exp\left[-\frac{\partial\varphi}{\partial Y} Y\left\{\frac{1}{\rho_4} - \frac{1}{\rho_0} + \ln\left(\frac{\rho_4}{\rho_0}\right)\right\}\right], \qquad (26)$$

$$\tilde{\rho}_3 = \lambda_3^{3/2}\frac{\rho_3}{\rho_0}\exp\left[-\frac{\partial\varphi}{\partial Y} Y\left\{-\frac{1}{\rho_0} + \ln\left(\frac{1-\rho_3}{\rho_0}\right)\right\}\right]. \qquad (27)$$

Hence,

$$C_V/C_L \equiv \frac{\tilde{\rho}_3/\tilde{\rho}_4}{\rho_3/\rho_4} = (m_3/m_4)^{3/2}\exp\left[\frac{\partial\varphi}{\partial Y} Y\left\{\frac{1}{\rho_4} + \ln\left(\frac{\rho_4}{1-\rho_3}\right)\right\}\right]. \qquad (28)$$

§ 3. Determination of the universal function $\varphi(Y)$

In II we have seen that if we set the lattice spacing $d = 3.1$ Å, the correct density dependence of T_λ is obtained. Here, we use the same value for d as before, but use the true mass 6.69×10^{-24} gr for m_4. From the fact that when $\rho_4 = 0.65$, $\rho_3 = 0$, the density is about 0.147 gr/c.c. and the corresponding experimental λ-temperature is about 2.16°K, we obtain

$$Y_\lambda = (e^A \rho_0 \rho_4 \beta B)_\lambda \simeq 0.0344.$$

We can get the explicit form of $\varphi(Y)$ by numerical integration of eq. (18) making use of the experimental value of C at a fixed density. We employ for simplicity the data of C under saturated vapour pressure, the density of which approximately corresponds to $\rho_4 = 0.65$, $\rho_3 = 0$. As eq. (18) is of the second order in the derivative of $\varphi(Y)$, we have to fix two integration constants from experiments. Using eqs. (17) and (28) and experimental values $C_V/C_L \simeq 10$ and $S \simeq 0.36$ cal/gr for $\rho_4 = 0.65$, $\rho_3 = 0$, we obtain at the λ-point

$$(\partial\varphi/\partial Y)_{\lambda} \simeq 64.9$$

$$\varphi(Y_{\lambda}) \simeq 2.06.$$

The universal function $\varphi(Y)$ thus determined is illustrated in Fig. 1, together with its derivative $d\varphi/dY$.

§ 4. Comparison with experiments

We can now obtain values for entropy and C_V/C_L for various temperatures and concentrations using eqs. (17) and (28) and the universal function determined in § 3. These curves are shown in Fig. 2 and Fig. 3. To compare the values of pressure with experiments we have to correct (19) for the effect of attractive force. As the anomaly in isopycnals near the λ-line is considered to be due to that of the kinetic energy, we have assumed for simplicity that the contribution to the pressure from the attractive potential is independent of temperature.

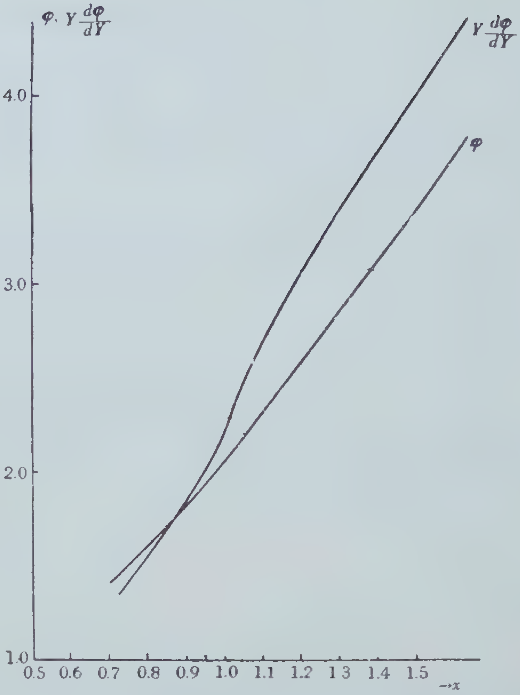

Fig. 1. Universal Function $\varphi(Y)$ and $Y\dfrac{d\varphi}{dY}$ versus $x \equiv Y/Y_{\lambda}$, $Y_{\lambda}=0.0344$.

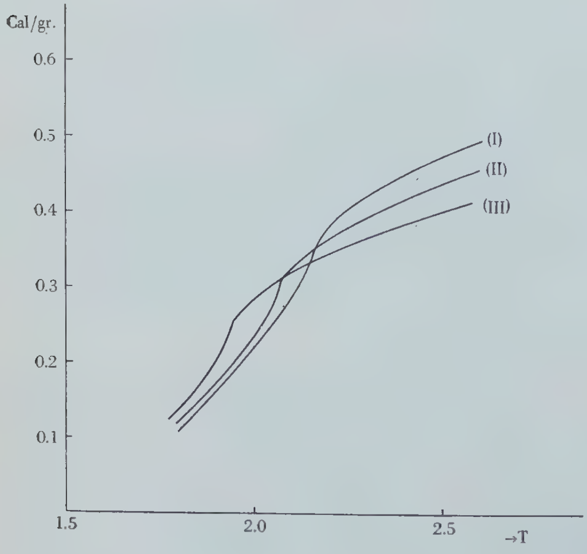

Fig. 2. Entropy versus Temperature of Liquid He^4 (Theoretical)
(I) : $\rho_1=0.65$ (II) : $\rho_1=0.70$ (III) : $\rho_1=0.75$

The isopycnals drawn in Fig. 4 are obtained by adjusting the additive constant due to the attractive potential which may depend on density but is assumed independent of temperature so that the correct value of pressure may be obtained at the λ-line. These curves are generally in good agreement with experiments. The too slow rise of isopycnals with temperature in the region of He I will be due to ignoring the temperature dependence of the effect of attractive potential.

From eq. (20) the concentration dependence of the

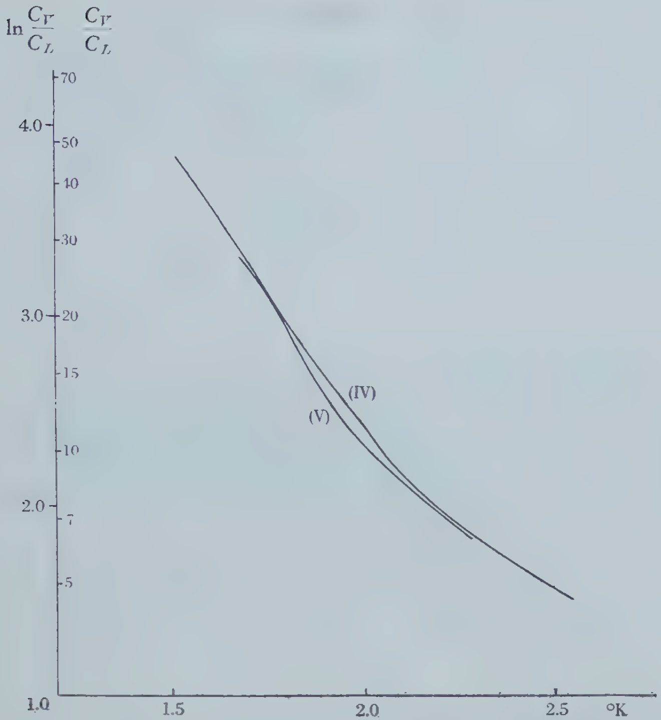

Fig. 3. The Distribution Function versus Temperature (Theoretical)
(IV) $\rho_4=0.60$ $\rho_3=0.05$ (V) $\rho_4=0.55$ $\rho_3=0.10$

λ-temperature is obtained. If we put

$$X \equiv \rho_3/\rho_1$$

we have

$$-dT_\lambda/dX = Y_\lambda B \rho_0 \rho_4 e^A \rho_1 \{1/\rho_4 + \ln \rho_4 + \ln(1-\rho_3)\}. \tag{29}$$

In the limit of zero concentration of He^3, we obtain

$$-dT_\lambda/dX = T_\lambda (1+\rho_1 \ln \rho_1). \tag{30}$$

The result is tabulated in Table I and is compared with experiments.

Thus we have obtained by use of the lattice model a consistent explanation of peculiar properties of liquid He^4 and mixtures of He^4-He^3 with small concentration of He^3, although we are not free from using several assumptions made in II.

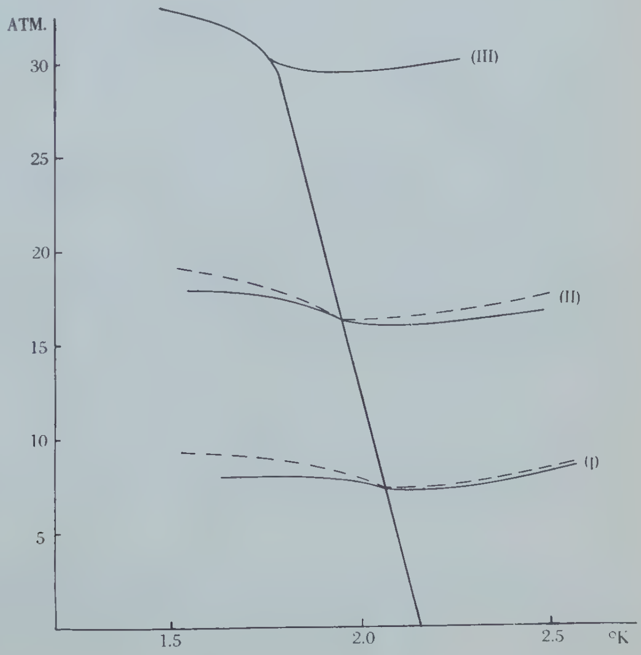

Fig. 4. Isopycnals of liquid He^4
—— Theoretical curve
.... Experimental curve
(I) : $\rho_1=0.70$
(II) : $\rho_1=0.75$
(III) : $\rho_1=0.80$

Table I. Density ρ of He^4, theoretical and experimental T_λ, and initial slope of the curve of T_λ versus X.

ρ_1	ρ (gr/c.c.)	T_λ (theor)	T_λ (exp)	$\left(\frac{\partial T_\lambda}{\partial X}\right)_{\text{(theor)}}$	$\left(\frac{\partial T_\lambda}{\partial X}\right)_{\text{(exp)}}$ 9)
0.65	0.147	2.16	2.16	-1.56	-1.5
0.70	0.158	2.07	2.10	-1.55	—
0.75	0.169	1.94	1.97	-1.52	—
0.80	0.180	1.76	1.76	-1.44	—

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On the Vibration of Disordered Linear Lattice, II

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The exact eigenfrequency-distribution of the Poisson lattice containing few isotopic impurities at random positions was calculated by the matrix method developed by the author in a preceding paper (I) and by Kerner. Within the original band, this comes out to coincide with that obtained in I from the average eigenvalue equation $\langle \mathrm{Trace}\, \boldsymbol{H} \rangle_{AV}=2$. This justifies the use of the average eigenvalue equation for the Poisson lattice, at least within the band. The distribution function obtained shows in addition that outside the band there appears a narrow "impurity band."

Secondly the exact average eigenvalue equation for non-Poisson lattice was calculated, which shows that, on the average, the eigenfrequency-distribution is the same as that of the regular lattice except that the limiting frequency is shifted corresponding to the average increase or decrease of the atomic mass. Finally the validity of the ergodic assumption was examined, with a negative answer. The actual lattices may, therefore, have the eigenfrequency-distribution which is markedly different from the average distribution.

§ 1. Introduction

In a preceding paper[1], which hereafter will be referred to as I, we treated the problem of the vibration of linear disordered lattice by a method of transfer matrix and showed that it is particularly effective for investigating the problem of random lattice.* The argument given there as to the random lattice was however incomplete inasmuch as it was based on a rough argument about the ergodicity of the quantity Trace $\boldsymbol{H}$, on the one hand, and it was restricted to a limiting case of few impurities (Poisson lattice), on the other. In this paper it is attempted to remove this incompleteness to some extent.

In the first place, after giving some remarks about the moments of Trace $\boldsymbol{H}$ and the validity of the ergodic assumption, the exact probability distribution function of Trace $\boldsymbol{H}$ for the Poisson case is calculated by the method of Kerner[2], from which the limiting eigenfrequency-distribution for a long lattice is obtained. Within the frequency-band of the regular lattice this proves to coincide with that obtained in I from the average eigenvalue equation, which shows that we may use here, at least within the band, the average eigenvalue equation to obtain the true distribution, though in this case the ergodic assumption is not valid for Trace $\boldsymbol{H}$. In addition, our eigenfrequency-distribution tells us that there should appear an "impurity band" whose center lies at the impurity frequency of a long lattice containing only one impurity.

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* Recently we have become aware of that H. Schmidt treated the problem of the impurity band by starting from a matrix formulation which is very similar to ours. His procedure of calculation is, however, mathematically different form and much more intricate than that used here and in I.

In the second place we calculate the average eigenvalue equation for non-Poisson lattice, which shows that, on the average, the eigenfrequency-distribution is the same as that of the regular lattice except that we must replace the mass of the regular atoms by the average mass of the atoms in the random lattice, in the formula giving the eigenfrequency-distribution. Finally we examine the validity of the ergodic assumption for Trace $\boldsymbol{H}$, which unfortunately leads to a negative answer. Though it does not necessarily mean that the assumption fails to be valid also for the eigenfrequency-distribution itself, we cannot prove the validity as in the case of Poisson lattice, and cannot therefore exclude the possibility that the actual lattice may have the eigenfrequency-distribution which is markedly different from the average distribution just obtained.

We shall use throughout the paper the same notations as used in I.

§ 2. Eigenfrequency-distribution of Poisson lattice

Let us first consider the case in which q becomes infinitesimal as $N\to\infty$ but $Nq=\alpha$, the average number of impurities, remains constant, i.e., the case of the Poisson distribution treated already in I. We follow the procedure of Kerner[2] to calculate the moments of Trace $\boldsymbol{H}$: In the formula

$$\langle \mathrm{Trace}\,\boldsymbol{H}\rangle_{\mathrm{AV}}=\mathrm{Trace}\,[\,(\boldsymbol{I}+\boldsymbol{\Lambda})+q\,\{\boldsymbol{Q}'-(\boldsymbol{I}+\boldsymbol{\Lambda})\}\,]^{N}\equiv\mathrm{Trace}\,\boldsymbol{J}^{N},$$

we regard $q\{\boldsymbol{Q}'-(\boldsymbol{I}+\boldsymbol{\Lambda})\}$ as the perturbation and neglect the non-diagonal elements of the matrix $\boldsymbol{Q}'-(\boldsymbol{I}+\boldsymbol{\Lambda})$. (For simplicity, we restrict ourselves to the Case M, i.e. the case of isotopic impurities.) Then

$$\boldsymbol{J}^{N}\cong\begin{pmatrix}\exp(2Ni\beta) & 0\\ 0 & \exp(-2Ni\beta)\end{pmatrix}\exp\alpha\begin{pmatrix}i\mathcal{E}\tan\mathcal{E} & 0\\ 0 & -i\mathcal{E}\tan\beta\end{pmatrix},$$

and we have

$$\langle \mathrm{Trace}\,\boldsymbol{H}\rangle_{\mathrm{AV}}=2\cos(2N\beta+\alpha\mathcal{E}\tan\beta)\,. \tag{2·1}$$

This coincides with the result obtained in I, and justifies the argument in I about the nature of our approximation. In the limit $N\to\infty$, the second term of the argument of the cosine function can be omitted, and the average eigenvalue equation becomes the same as that of the regular lattice.

The second moment can be calculated by the same perturbation procedure: Starting from

$$\langle(\mathrm{Trace}\,\boldsymbol{H})^{2}\rangle_{\mathrm{AV}}=\mathrm{Trace}\,[\,(\boldsymbol{I}+\boldsymbol{\Lambda})\cdot(\boldsymbol{I}+\boldsymbol{\Lambda})+q\,\{\boldsymbol{Q}'\cdot\boldsymbol{Q}'-(\boldsymbol{I}+\boldsymbol{\Lambda})\cdot(\boldsymbol{I}+\boldsymbol{\Lambda})\}\,]^{N},$$

where $\boldsymbol{A}\cdot\boldsymbol{B}$ means the direct product of $\boldsymbol{A}$ and $\boldsymbol{B}$,[2) we get

$$\langle(\mathrm{Trace}\,\boldsymbol{H})^{2}\rangle_{\mathrm{AV}}=2\exp(\alpha\mathcal{E}^{2}\tan^{2}\beta)+\exp(-\alpha\mathcal{E}^{2}\tan^{2}\beta)[\langle\mathrm{Trace}\,\boldsymbol{H}\rangle_{\mathrm{AV}}^{2}-2], \tag{2·2}$$

which means that there is a finite (central) second moment, which increases rapidly with β.

Since in the present case (finite average number of impurities) the ergodicity is evidently invalid, it is a matter of course that the finite second moment exists. This means that the eigenfrequency distribution of the actual lattice exhibits fortuitous fluctuations from the one obtained from the average eigenvalue equation $\langle \text{Trace}\, \boldsymbol{H} \rangle_{\text{AV}}=2$. It does not, however, necessarily mean that the actual lattice has usually an eigenfrequency distribution which is markedly different from that obtained from the average eigenvalue equation, e.g., the one in which exist a considerable number of impurity frequencies. It is possible that, even if the Trace $\boldsymbol{H}$ for some β-value (say β') which satisfies $\langle \text{Trace}\, \boldsymbol{H} \rangle_{\text{AV}}=2$ is very different from 2, there is always a β-value which is very close to β' and for which Trace $\boldsymbol{H}=2$. Then the overall eigenfrequency distribution will still be given approximately by $\langle \text{Trace}\, \boldsymbol{H} \rangle_{\text{AV}}=2$. This is highly probable, since Trace $\boldsymbol{H}$ will oscillate very rapidly with β, as might be expected from the formula for Trace $\boldsymbol{H}$ obtained in I for the case of a definite number of impurities (e.g. eq. (6·3) of I).

To investigate whether this is true or not, we directly calculate the probability distribution function of Trace $\boldsymbol{H}$ without using the moments. According to Kerner, the characteristic function of the distribution of Trace $\boldsymbol{H}$ is given by

$$f(u)=\exp(-\alpha)\sum_{\nu}\exp(iuQ_{\nu})\,\alpha^{\nu}/\nu!, \tag{2·3}$$

where $Q_{\nu}=\text{Trace}\, \boldsymbol{D}\boldsymbol{E}^{\nu}$, and

$$\boldsymbol{D}=\begin{pmatrix} \exp(2Ni\beta) & 0 \\ 0 & \exp(-2Ni\beta) \end{pmatrix}, \quad \boldsymbol{E}=\begin{pmatrix} 1+i\varepsilon\tan\beta & 0 \\ 0 & 1-i\varepsilon\tan\beta \end{pmatrix},$$

in our case. Then the probability density for the event Trace $\boldsymbol{H}=2$ is given by

$$\begin{aligned} F(2)&=\sum_{\nu}\exp(-\alpha)\,\alpha^{\nu}/\nu!\left[(2\pi)^{-1}\int \exp iu(Q_{\nu}-2)\,du\right] \\ &=\sum_{\nu}\exp(-\alpha)\,\alpha^{\nu}/\nu!\left[\delta(Q_{\nu}-2)\right], \end{aligned} \tag{2·4}$$

where $\delta(x)$ is Dirac's delta function. Q_{ν} can easily be calculated as

$$\begin{aligned} Q_{\nu}&=\exp(2Ni\beta)\,(1+i\varepsilon\tan\beta)^{\nu}+\exp(-2Ni\beta)\,(1-i\varepsilon\tan\beta)^{\nu} \\ &=2\,(1+\varepsilon^{2}\tan^{2}\beta)^{\nu/2}\cos(2N\beta+\nu\tan^{-1}(\varepsilon\tan\beta)). \end{aligned} \tag{2·5}$$

It is seen that the β-value which satisfies $Q_{\nu}=2$ depends on ν in a complicated way, and it is difficult to visualize $F(2)$ as a function of β. It is, however, easy to obtain its limiting form for $N\rightarrow\infty$. Then the function $\cos(2N\beta+\nu\tan^{-1}(\varepsilon\tan\beta))$ oscillates infinitely within the interval $(0\leq\beta\leq\pi/2)$, and fills up the region almost continuously, as is shown in Fig. 1. Therefore the line representing the constant $(1+\varepsilon^{2}\tan^{2}\beta)^{-\nu/2}$ with any arbitrary value of β, which is less than unity for all β's, always intersects the curve at that value of β. This means that for all β's Trace $\boldsymbol{H}=2$ with probability unity, i.e., every β necessarily becomes the eigenfrequency. This eigenfrequency distribution coincides with that given by $\langle \text{Trace}\, \boldsymbol{H} \rangle_{\text{AV}}=2$ in the limit $N\rightarrow\infty$, as is seen from

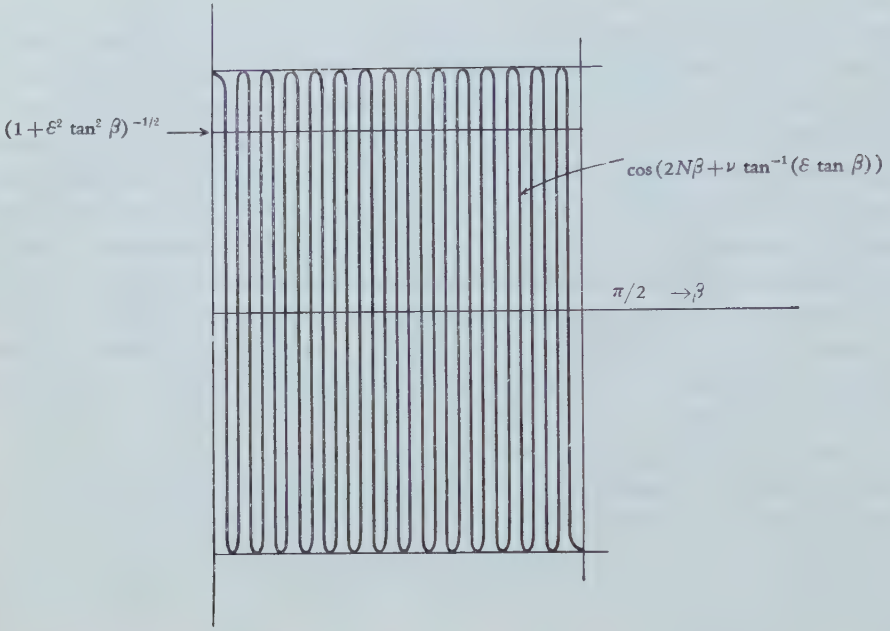
$(1+\mathcal{E}^2\tan^2\beta)^{-1/2}$
$\cos(2N\beta+\nu\tan^{-1}(\mathcal{E}\tan\beta))$
$\pi/2$ $\rightarrow\beta$

Fig. 1.

(2·1), showing that in the present case there is a certain kind of ergodicity for the eigenfrequency distribution and we have only to calculate the average eigenvalue equation to obtain the limiting distribution of eigenfrequencies within the band $(0, \pi/2)$.

The above result does not, however, exclude the possibility of appearance of impurity frequencies. In fact, if we put $\beta=\pi/2+i\gamma$, Q_ν becomes

$$Q_\nu=\pm[\exp(-2N\gamma)(1-\mathcal{E}\coth\gamma)^\nu+\exp(2N\gamma)(1+\mathcal{E}\coth\gamma)^\nu]. \qquad (2\cdot6)$$

For $\nu=1$, for example, this gives

$$Q_1=\pm[2\cosh(2N\gamma)+2\mathcal{E}\coth\gamma\sinh(2N\gamma)]. \qquad (2\cdot7)$$

The equation $Q_1=2$ is thus identical with the equation (I, (4·8)) for the impurity frequency of the lattice containing only one impurity. Hence the impurity frequency of the lattice containing one impurity becomes that of the random Poisson lattice with the probability $\alpha\exp(-\alpha)$, which is just the probability that there is one impurity. It is easily seen that also for other values of ν, Q_ν coincides with the expression of Trace $\boldsymbol{H}$ for the lattice containing the respective number of impurities, averaged over their relative configurations (see I, (7·3)). Thus we can generally conclude that the average impurity frequency of the lattice containing ν impurities becomes that of random lattice just with the probability that there are ν impurities. This is a natural result inasmuch as for $N\rightarrow\infty$ the value of Trace $\boldsymbol{H}$ for complex argument $\pi/2+i\gamma$ does not depend sensitively on the relative configuration of the impurities, as can be seen, for example, from (I, (6·3)).

It can be seen that for $N\rightarrow\infty$ the equation $Q_\nu=2$ becomes

$$(1+\mathcal{E}\coth\gamma)^{\nu}=0 \text{ for every } \nu. \tag{2·8}$$

The root of this equation just gives the impurity frequency γ_1 of the infinitely long lattice containing only one impurity (see I, (4·9)). It is thus seen that for sufficiently long lattices the impurity frequencies crowd around γ_1 almost certainly, and form a narrow "impurity band." This result is just the same as that obtained by Schmidt[3].*

Such an information on the impurity frequencies cannot be obtained, as is easily seen, from the average eigenvalue equation. This is because of the essentially discrete character of the impurity frequency-distribution, and can be anticipated from the expression of Trace $\boldsymbol{H}$ for definite number of impurities, since it does not exhibit such a rapid oscillation as mentioned above with the variation of γ.

§ 3. Non-Poisson lattice with isotopic impurities

Next we consider the non-Poisson case, i.e., the case in which q remains constant even if $N\to\infty$, so that the average number of impurities Nq is infinite. $\langle \text{Trace } \boldsymbol{H}\rangle_{\text{AV}}$ is then given by Trace $[p(\boldsymbol{I}+\boldsymbol{\Lambda})+q\boldsymbol{Q}']^N=\lambda_1^N+\lambda_2^N$, where λ_1 and λ_2 are the eigenvalues of the matrix $p(\boldsymbol{I}+\boldsymbol{\Lambda})+q\boldsymbol{Q}'$. Putting $\cos\alpha=\cos(2\beta)-q\mathcal{E}\tan\beta\sin(2\beta)$, they turn out to be $\exp(\pm i\alpha)$. Hence we have

$$\langle \text{Trace } \boldsymbol{H}\rangle_{\text{AV}}=2\cos[N\{\cos^{-1}(\cos(2\beta)-q\mathcal{E}\tan\beta\sin(2\beta))\}]. \tag{3·1}$$

The average eigenvalue equation thus becomes

$$\cos^{-1}\{\cos(2\beta)-q\mathcal{E}\tan\beta\sin(2\beta)\}=2n\pi/N,$$
$$n=1,\ 2,\cdots\cdots,\ N/2 \text{ or } (N-1)/2. \tag{3·2}$$

The roots of this equation can be obtained graphically, as shown in Fig. 2. If the ergodic assumption be valid for Trace $\boldsymbol{H}$, these roots would give the eigenfrequencies for the actual lattice: all the eigenfrequencies decrease when $\mathcal{E}>0$, and increase when $\mathcal{E}<0$, and in the latter case there appear several impurity frequencies. At first sight it seems that the density of eigenfrequencies is minimum at the top of the band $(0,\pi/2)$, and there appears an impurity band. That this is not the case can easily be seen if we transform the variable from β to ω: From the relations

$$\begin{aligned} d\cos^{-1}&\{\cos(2\beta)-q\mathcal{E}\tan\beta\sin(2\beta)\}=\\ &=-1/\{1-(\cos(2\beta)-q\mathcal{E}\tan\beta\sin(2\beta))^2\}^{1/2}\\ &\times[-2\sin(2\beta)-q\mathcal{E}\{2\tan\beta\cos(2\beta)+\sin(2\beta)(1+\tan^2\beta)\}]\,d\beta \end{aligned}$$

and

$$d\beta=d\omega/(\omega_0^{\,2}-\omega^2)^{1/2},$$

---

* The fact that γ_1 is the ν-fold root of (2·8) shows that there must appear in general ν impurity frequencies if the lattice contains ν impurities. For $N\to\infty$ it is almost certain that there are Nq impurities and consequently Nq impurity frequencies, crowding around γ_1.

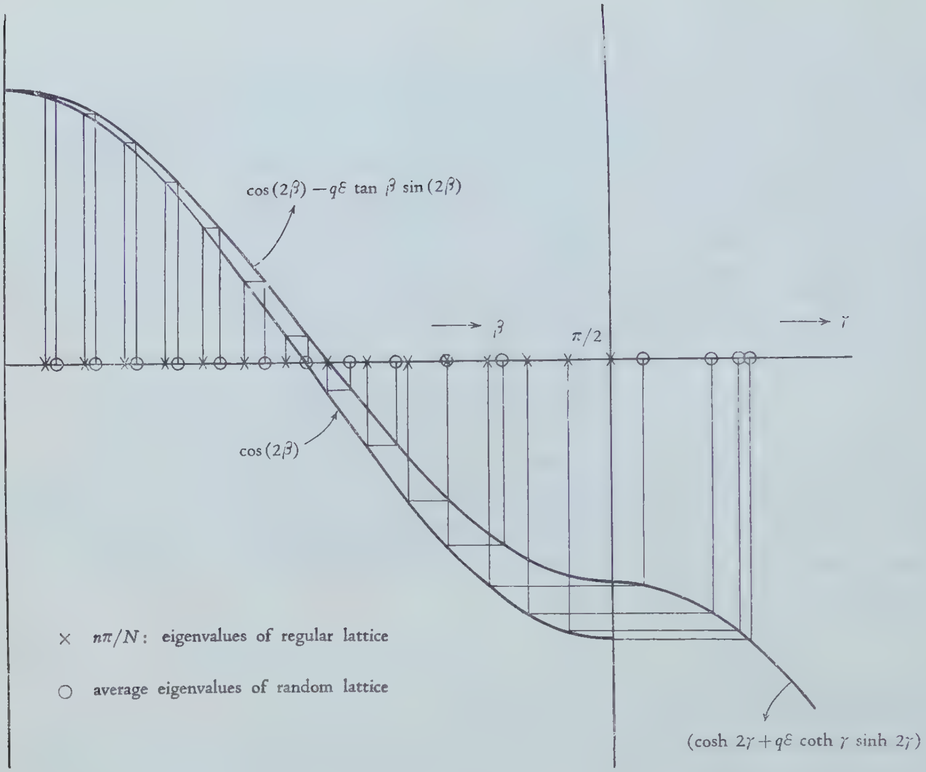
$\cos(2\beta) - q\mathcal{E} \tan\beta \sin(2\beta)$
$\longrightarrow \beta$
$\pi/2$
$\longrightarrow \gamma$
$\cos(2\beta)$
× $n\pi/N$: eigenvalues of regular lattice
○ average eigenvalues of random lattice
$(\cosh 2\gamma + q\mathcal{E} \coth\gamma \sinh 2\gamma)$

Fig. 2.

where $\omega_0 = (4\lambda/m)^{1/2}$, we obtain the density distribution on ω-axis:

$$\begin{aligned} f(\omega)\,d\omega &= 2d\omega/\pi(1-\omega^2/\omega'^2)^{1/2}\omega', \quad \omega < \omega', \\ &= 0 \qquad\qquad\qquad\qquad\quad , \quad \omega > \omega', \end{aligned} \tag{3·3}$$

where $\omega' = \omega_0/(1+q\mathcal{E})^{1/2}$. Thus we obtain the physically reasonable result that the eigenfrequency distribution is just the same as that of regular lattice, except that we have to replace the limiting frequency ω_0 by a new limiting frequency $\omega' = \{4\lambda/m(1+q\mathcal{E})\}^{1/2}$, where $m(1+q\mathcal{E})$ is the average mass of the atoms.

When $\mathcal{E} < 0$ all the eigenfrequencies increase, and there appear impurity eigenfrequencies outside the original band $(0, \omega_0)$. It is however open to question whether we may consider them to be impurity frequencies in the true sense: In the present case we may freely interchange the roles of the regular and impurity atoms, i.e. interchange p and q. Then after interchange the " impurities " become heavier than the " regular " atoms, and the eigenfrequencies in question now lie *within* the original band! This shows that, although it is evident that the modes corresponding to these frequencies have more or less localized character (the complex values of β's), the essential difference between the non-localized mode and the localized one now disappears and the transition between them becomes a continuous one.

In order to examine the validity of the ergodic assumption, we next calculate the second moment of Trace $\boldsymbol{H}$. This is given by, after Kerner[2],

$$\langle (\text{Trace } \boldsymbol{H})^2 \rangle_{\text{AV}} = [p(\boldsymbol{I}+\boldsymbol{\Lambda}) \cdot (\boldsymbol{I}+\boldsymbol{\Lambda}) + q\boldsymbol{Q}' \cdot \boldsymbol{Q}']^N.$$

The characteristic equation of the matrix in the square bracket comes out to be

$$\begin{aligned}\lambda^4 - \lambda^3 [4\cos^2(2\beta) &+ q\{4G^2\sin^2(2\beta) - 8G\sin(2\beta)\cos(2\beta)\}] \\ &+ \lambda^2[8\cos^2(2\beta) + 8qG\sin(4\beta) + 8q^2G^2\sin^2(2\beta) - 2] \\ &- \lambda[4\cos^2(2\beta) - q\{4G^2\sin^2(2\beta) + 8G\sin(2\beta)\cos(2\beta)\} \\ &\qquad + 8q^2G^2\sin^2(2\beta)] + 1 = 0,\end{aligned}$$

where $G = \varepsilon \tan\beta$. Let the roots of this equation be λ_i's ($i=1$, 2, 3 and 4). For $\lim_{N\to\infty} \langle (\text{Trace } \boldsymbol{H})^2 \rangle_{\text{AV}} = \lim_{N\to\infty} \sum \lambda_i^N$ to be finite, all the four roots must be of modulus unity. The above equation must, therefore, have the form

$$\begin{aligned}&(\lambda^2 - A\lambda + 1)(\lambda^2 - B\lambda + 1) \\ &\quad = \lambda^4 - \lambda^3(A+B) + \lambda^2(2+AB) - \lambda(A+B) + 1 = 0,\end{aligned}$$

which evidently is not the case. Whence the result that there must always exist the roots whose moduli are not unity. Since, however, $\prod_i \lambda_i = 1$, there must be at least one root with modulus greater than unity; that is, $\lim_{N\to\infty} \langle (\text{Trace } \boldsymbol{H})^2 \rangle_{\text{AV}}$ always diverges.

This is an unpleasant result and shows that either the ergodic assumption is not valid or there are lattices, though few, for which Trace $\boldsymbol{H}$ becomes infinite. To investigate which alternative corresponds to the truth, we next calculate $\langle \text{Trace } \boldsymbol{H} \rangle_{\text{AV}}$ for the lattice containing non-isotopic impurities. In this case the exact calculation proves to be difficult, because at the position of impurity three successive transfer matrices have to be replaced by the impurity ones. As an approximation we assume that the impurity atoms can occupy, with a probability q, only every third position. Then

$$\langle \text{Trace } \boldsymbol{H} \rangle_{\text{AV}} = \text{Trace}[p(\boldsymbol{I}+\boldsymbol{\Lambda})^3 + q\boldsymbol{Q}]^{N/3}.$$

The characteristic equation of the matrix in the square bracket comes out to be

$$\begin{aligned}\lambda^2 - 2\lambda[\cos(6\beta) &- q\{2\delta\tan\beta\sin(6\beta) - 4\delta^2\tan\beta\sin^2\beta\sin(4\beta)\}] \\ &+ 1 - 8pq\delta^2\sin^2\beta\tan\beta\sin(2\beta) = 0, \qquad (3\cdot4)\end{aligned}$$

if for simplicity we put $\varepsilon = 0$. Since the constant term is not unity, the moduli of the roots cannot be both unity, which means that in this case $\langle \text{Trace } \boldsymbol{H} \rangle_{\text{AV}}$ becomes zero or infinity for $N \to \infty$, and the eigenvalue equation has no roots.

This difficulty originates from the fact that $\|[p(\boldsymbol{I}+\boldsymbol{\Lambda})^3 + q\boldsymbol{Q}]\|$ is not unity, i.e., the fact that the "average lattice" whose transfer matrix is given by $p(\boldsymbol{I}+\boldsymbol{\Lambda})^3 + q\boldsymbol{Q}$ has no real significance. This clearly indicates that the ergodic assumption fails to be valid for Trace $\boldsymbol{H}$, since otherwise $\langle \text{Trace } \boldsymbol{H} \rangle_{\text{AV}}$ should be equal to Trace $\boldsymbol{H}$ of the individual lattices and should give a significant eigenvalue equation.

It has thus been shown that the ergodic assumption does not hold good for Trace $\boldsymbol{H}$, contrary to the anticipation made in I. This does not necessarily mean, as is seen from the result obtained above for the Poisson lattice, that it fails to be valid also for the eigenfrequency-distribution itself. Considering the essentially continuous character of the average eigenfrequency-distribution obtained above, it is likely that the assumption is true for the eigenfrequency-distribution in the whole frequency domain. Inasmuch as we are not able to prove it, however, we cannot at present exclude the possibility that the contrary is true and the actual lattice may have the eigenfrequency-distribution markedly different from the average one, for example a distribution in which appears an impurity band well separated from the main band.

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Anomalous Magnetic Moments of Hyperons and Mirror Theorem

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According to the theory of d'Espagnat and Prentki the anomalous magnetic moments of hyperons are calculated in the lowest order perturbation approximation. It is further shown that their charge-independent interaction Hamiltonian leads to the mirror theorem for the anomalous magnetic moments of baryons (and of K mesons) without use of perturbation approximation. Finally the anomalous magnetic moments of Σ hyperons determined from the mass differences of Σ and the mirror theorem are compared with the perturbation results.

§ 1. The lowest order perturbation calculation

To estimate the unknown anomalous magnetic moments of Λ, Σ and Ξ hyperons using the d'Espagnat and Prentki interaction,[1] seems to be interesting. So far, however, there is no reliable method to carry this out. Any method which neglects the recoil of baryons would give a poor approximation in view of the relatively large rest mass of a K meson. Although the perturbation treatment takes account of the recoil effect, it would not be considered as an acceptable method for this problem, when the interactions are fairly strong. Now one could pay attention to the facts that (i) even the lowest order perturbation contribution from the $(NN\pi)$ interaction has given the correct signs for the anomalous magnetic moments of a proton and a neutron (but with the wrong proportion), (ii) the recent analyses of the $K-N$ scattering experiment and of the binding energy of hyperfragments seem to give somewhat smaller coupling constants of (ΣNK) and $(\Lambda N\bar{K})$ interactions, that is $g^2 \lesssim 1$,[2,3] and moreover (iii) Schwinger's theory[4] suggests that $g^2(\Sigma NK)$, $g^2(\Lambda NK)$, $g^2(\Sigma\Xi K)$, and $g^2(\Lambda\Xi K)$ are $\sim 1/10$ to 1, and $g^2(\Lambda\Sigma\pi)$, $g^2(\Sigma\Sigma\pi)$ and $g^2(\Xi\Xi\pi)$ either vanish or are much smaller than $g^2(NN\pi)$ in order to explain the observed mass levels of baryons. Thus the lowest order perturbation calculation for the anomalous magnetic moments of the hyperons is carried out in the present report in the hope that at least some qualitative features of them might be obtained. The validity of the perturbation approximation for this problem could be examined in the comparison of the calculated values with those of future experiments.

With omission of the ordinary spin operators the d'Espagnat and Prentki interaction Hamiltonians are written as

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$$\left.\begin{aligned}
H_1(NN\pi) &= f(NN\pi)\bar{\psi}_N^{*T}\tau_i\psi_N\phi_\pi,\\
H_2(\Lambda\Sigma\pi) &= f(\Lambda\Sigma\pi)\bar{\psi}_\Lambda{}^*\psi_{\Sigma i}\phi_{\pi i}+\text{H.C.},\\
H_3(\Sigma\Sigma\pi) &= f(\Sigma\Sigma\pi)\bar{\psi}_\Sigma^{*T}I_i\psi_\Sigma\phi_{\pi i},\\
H_1(\Xi\Xi\pi) &= f(\Xi\Xi\pi)\bar{\psi}_\Xi^{*T}\tau_i\psi_\Xi\phi_{\pi i},
\end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned}
H_5(\Lambda NK) &= f(\Lambda NK)\bar{\psi}_\Lambda{}^*\phi_K^{*T}\psi_N+\text{H.C.},\\
H_6(\Sigma NK) &= f(\Sigma NK)\bar{\psi}_{\Sigma i}^*\phi_K^{*T}\tau_i\psi_N+\text{H.C.},\\
H_7(\Lambda\Xi K) &= f(\Lambda\Xi K)\bar{\psi}_\Lambda{}^*\psi_\Xi{}^T\tau_2\phi_K+\text{H.C.},\\
H_8(\Sigma\Xi K) &= f(\Sigma\Xi K)\bar{\psi}_{\Sigma i}^*\psi_\Xi{}^T\tau_2\tau_i\phi_K+\text{H.C.}
\end{aligned}\right\} \qquad (2)$$

where τ's and I's are 2×2 and 3×3 matrices in the isobaric spin space respectively, and the other notations are the same as in reference 1.

According to these interaction Hamiltonians, the anomalous magnetic moments of baryons in the g^2-approximation are, in units of nuclear magnetons,

$$\left.\begin{aligned}
\mu(p^+) &= \frac{1}{\pi}\{g^2(NN\pi)[B_1(NN\pi)+2B_2(NN\pi)]\\
&\quad +g^2(N\Sigma K)[2B_1(N\Sigma K)+B_2(N\Sigma K)]+g^2(N\Lambda K)B_2(N\Lambda K)\},\\
\mu(n^0) &= \frac{1}{\pi}\{g^2(NN\pi)[2B_1(NN\pi)-2B_2(NN\pi)]\\
&\quad +g^2(N\Sigma K)[-2B_1(N\Sigma K)+2B_2(N\Sigma K)]\},
\end{aligned}\right\} \qquad (3)$$

$$\begin{aligned}
\mu(\Lambda^0) &= \frac{M_N}{M_\Lambda}\frac{1}{\pi}\{-g^2(\Lambda\Xi K)[B_1(\Lambda\Xi K)-B_2(\Lambda\Xi K)]\\
&\quad +g^2(\Lambda N\bar{K})[B_1(\Lambda N\bar{K})-B_2(\Lambda N\bar{K})]\},
\end{aligned} \qquad (4)$$

$$\left.\begin{aligned}
\mu(\Sigma^+) &= \frac{M_N}{M_\Sigma}\frac{1}{\pi}\{g^2(\Sigma N\bar{K})2B_1(\Sigma N\bar{K})+g^2(\Sigma\Xi K)2B_2(\Sigma\Xi K)\\
&\quad +g^2(\Sigma\Sigma\pi)[B_1(\Sigma\Sigma\pi)+B_2(\Sigma\Sigma\pi)]+g^2(\Sigma\Lambda\pi)B_2(\Sigma\Lambda\pi)\},\\
\mu(\Sigma^-) &= \frac{M_N}{M_\Sigma}\frac{1}{\pi}\{-g^2(\Sigma N\bar{K})2B_2(\Sigma N\bar{K})-g^2(\Sigma\Xi K)2B_1(\Sigma\Xi K)\\
&\quad -g^2(\Sigma\Sigma\pi)[B_1(\Sigma\Sigma\pi)+B_2(\Sigma\Sigma\pi)]-g^2(\Sigma\Lambda\pi)B_2(\Sigma\Lambda\pi)\},\\
\mu(\Sigma^0) &= \frac{M_N}{M_\Sigma}\frac{1}{\pi}\{g^2(\Sigma N\bar{K})[B_1(\Sigma N\bar{K})-B_2(\Sigma N\bar{K})]\\
&\quad -g^2(\Sigma\Xi K)[B_1(\Sigma\Xi K)-B_2(\Sigma\Xi K)]\},
\end{aligned}\right\} \qquad (5)$$

$$\left.\begin{aligned}
\mu(\Xi^-) &= \frac{M_N}{M_\Xi}\frac{1}{\pi}\{-g^2(\Xi\Xi\pi)[B_1(\Xi\Xi\pi)+2B_2(\Xi\Xi\pi)]\\
&\quad -g^2(\Xi\Sigma\bar{K})[2B_1(\Xi\Sigma\bar{K})+B_2(\Xi\Sigma\bar{K})]-g^2(\Xi\Lambda\bar{K})B_2(\Xi\Lambda\bar{K})\},
\end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned}\mu(\Xi^0)=\frac{M_N}{M_\Xi}\frac{1}{\pi}\{&-g^2(\Xi\Xi\pi)[2B_1(\Xi\Xi\pi)-2B_2(\Xi\Xi\pi)]\\&+g^2(\Xi\Sigma\overline{K})[2B_1(\Xi\Sigma\overline{K})-2B_2(\Xi\Sigma\overline{K})]\}\end{aligned}\right\}$$

where $g^2=f^2/4\pi\hbar c$, and M_N, M_Λ, M_Σ, and M_Ξ are the rest masses of the N, Λ, Σ and Ξ respectively. B_1 and B_2 denote the contributions from the baryon and boson currents. Assuming all the baryons are Dirac particles and the K meson is a spin 0 particle, one has the expressions for B_1 and B_2 in cases of *ps* and *s* couplings as follows.[5] (Here *ps* or *s* coupling means that in $(\Xi\Sigma K)$, for example, the Ξ and Σ have the same (opposite) parities and the K is a pseudoscalar (scalar) or a scalar (pseudoscalar) particle.)

ps coupling :

$$\left.\begin{aligned}B_1=-\frac{1}{4}\{&1+2(\lambda+\kappa-\kappa^2)+[(\lambda+\kappa-\kappa^2)(1+\lambda-\kappa^2)-\lambda]\log(\kappa^2/\lambda)\\&+[(\lambda+\kappa-\kappa^2)[(1+\lambda-\kappa^2)^2-2\lambda]-\lambda(1+\lambda-\kappa^2)]/\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}\\&\times\left(\tan^{-1}\frac{1-\lambda+\kappa^2}{\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}}+\tan^{-1}\frac{1+\lambda-\kappa^2}{\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}}\right)\Big\},\\B_2=\frac{1}{4}\Big\{&1-2(\lambda+\kappa-\kappa^2)+[(\lambda+\kappa-\kappa^2)(\kappa^2-\lambda)+\lambda]\log(\kappa^2/\lambda)\\&+[(\lambda+\kappa-\kappa^2)\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}\\&+(1-\lambda+\kappa^2)[(1+\lambda-\kappa^2)(\lambda+\kappa-\kappa^2)-2\lambda]/\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}]\\&\times\left(\tan^{-1}\frac{1-\lambda+\kappa^2}{\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}}+\tan^{-1}\frac{1+\lambda-\kappa^2}{\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}}\right)\Big\}.\end{aligned}\right\}\quad(7)$$

s coupling :

($\kappa\to-\kappa$ in eq. (7), since the matrix elements for the two couplings are obtained from each other by "mass reversal" about the mass of a baryon in the intermediate state.) (8)

Here $\lambda=(m/M)^2$ and $\kappa=M'/M$, where M' and m denote the masses of a baryon and a boson in the intermediate state and M is the baryon mass in question. It should be remembered that $\tan^{-1}\frac{1-\lambda+\kappa^2}{\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}}+\tan^{-1}\frac{1+\lambda-\kappa^2}{\sqrt{4\lambda-(1+\lambda-\kappa^2)^2}}<\pi/2$ for $\lambda+\kappa^2-1>0$ (i. e., $(N\Sigma K)$, $(N\Lambda K)$, $(NN\pi)$, $(\Lambda\Xi K)$, $(\Sigma\Xi K)$, $(\Sigma\Sigma\pi)$, $(\Xi\Xi\pi)$) and $>\pi/2$ for $\lambda+\kappa^2-1<0$ (i. e., $(\Lambda N\overline{K})$, $(\Sigma N\overline{K})$, $(\Sigma\Lambda\pi)$, $(\Xi\Sigma\overline{K})$, $(\Xi\Lambda\overline{K})$).

Using the observed masses, $M_\Xi=2580\,m_e$, $M_\Sigma=2335\,m_e$, $M_\Lambda=2182\,m_e$, $M_N=1837\,m_e$, $m_K=967\,m_e$, $m_\pi=270\,m_e$, one has the numerical values of B_1 and B_2 for every case as shown in Table I.[6),7)]

Table 1

Baryon	Interaction	*ps* coupling		*s* coupling	
		B_1	B_2	B_1	B_2
N	$(N\Sigma K)$	−0.157	0.090	0.264	−0.247
	$(N\Lambda K)$	−0.165	0.092	0.291	−0.269
	$(NN\pi)$	−0.237	0.174	—	—
Λ	$(\Lambda\Xi K)$	−0.172	0.101	0.308	−0.310
	$(\Lambda N\bar{K})$	−0.235	0.105	0.571	−0.573
Σ	$(\Sigma N\bar{K})$	−0.260	0.110	0.716	−0.761
	$(\Sigma\Xi K)$	−0.186	0.107	0.356	−0.368
	$(\Sigma\Sigma\pi)$	−0.240	0.187	—	—
	$(\Sigma\Lambda\pi)$	−0.271	0.216	0.862	−2.125
Ξ	$(\Xi\Sigma\bar{K})$	−0.232	0.119	0.548	−0.610
	$(\Xi\Lambda\bar{K})$	−0.250	0.120	0.648	−0.733
	$(\Xi\Xi\pi)$	−0.242	0.191	—	—

Tables 2 to 4 list the anomalous magnetic moments of Λ, Σ and Ξ hyperons given from eqs. (3) to (6) and Table 1, when all the g^2's are assumed to be unity.

Table 2. $\mu(\Lambda)$

Case	I	II	III	IV
$(\Lambda\Xi K)$	*ps*	*ps*	*s*	*s*
$(\Lambda N\bar{K})$	*ps*	*s*	*ps*	*s*
$\mu(\Lambda^\circ)$	−0.018	0.379	−0.257	0.141

Table 3. $\mu(\Sigma)$

Case	I	II	III	IV	V	VI	VII	VIII
$(\Sigma N\bar{K})$	*ps*	*ps*	*ps*	*ps*	*s*	*s*	*s*	*s*
$(\Sigma\Xi K)$	*ps*	*ps*	*s*	*s*	*s*	*s*	*ps*	*ps*
$(\Sigma\Sigma\pi)$	*ps*	*ps*	*ps*	*ps*	*ps*	*ps*	*ps*	*ps*
$(\Sigma\Lambda\pi)$	*ps*	*s*	*ps*	*s*	*ps*	*s*	*ps*	*s*
$\mu(\Sigma^+)$	−0.036	−0.622	−0.274	−0.860	0.215	−0.371	0.453	−0.133
$\mu(\Sigma^-)$	−0.003	0.583	−0.274	0.312	0.162	0.748	0.433	1.020
$\mu(\Sigma^\circ)$	−0.019	−0.019	−0.274	−0.274	0.189	0.189	0.443	0.443

Table 4. $\mu(\Xi)$

Case	I	II	III	IV
$(\Xi\Sigma\bar{K})$	*ps*	*ps*	*s*	*s*
$(\Xi\Lambda\bar{K})$	*ps*	*s*	*ps*	*s*
$(\Xi\Xi\pi)$	*ps*	*ps*	*ps*	*ps*
$\mu(\Xi^-)$	0.019	0.212	−0.169	0.024
$\mu(\Xi^\circ)$	0.037	0.037	0.719	0.719

§ 2. The perturbation-independent mirror theorem

Eqs. (3) to (6) show that $\mu(p^+)+\mu(n^0)$, $\mu(\Lambda^0)$, $\mu(\Sigma^+)+\mu(\Sigma^-)=2\mu(\Sigma^0)$, and $\mu(\Xi^-)+\mu(\Xi^0)$ have no contribution from the π, Λ, and Σ currents. The fact that this is true without regard to the perturbation approximation is discussed in the following.

The current operators for mesons and baryons are

$$j_\mu^{(\Lambda)}=0, \tag{9}$$

$$\left.\begin{aligned} j_\mu^{(\pi)}&=F(\partial_\mu)\,(\phi_\pi^{*T}I_3\phi_\pi),\\ j_\mu^{(\Sigma)}&=\bar{\psi}_\Sigma^{*T}\gamma_\mu I_3\psi_\Sigma, \end{aligned}\right\}\ \text{(Third components of isovectors)} \tag{10}$$

$$\left.\begin{aligned} j_\mu^{(K)}&=G(\partial_\mu)\left(\phi_K^{*T}\frac{1+\tau_3}{2}\phi_K\right),\\ j_\mu^{(N)}&=\bar{\psi}_N^{*T}\gamma_\mu\frac{1+\tau_3}{2}\psi_N,\\ j_\mu^{(\Xi)}&=-\bar{\psi}_\Xi^{*T}\gamma_\mu\frac{1-\tau_3}{2}\psi_\Xi, \end{aligned}\right\}\ \begin{aligned}&\text{(Linear combinations of isoscalars}\\ &\text{and third components of isovectors)}\end{aligned} \tag{11}$$

where $F(\partial_\mu)$ and $G(\partial_\mu)$ denote certain linear functionals of the differential operator ∂_μ in the ordinary space, the explicit forms of which are not needed here, and $\tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $I_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$.

The expectation value of the current operator in the state which contains one baryon (or one meson) and no other particles is

$$\langle j_\mu(x)\rangle_{1,0}=e\sum_{n=0}^{\infty}\sum_{(\sum_j n_j=n)}\frac{(-i)^n}{\prod_j n_j!}\int_{-\infty}^{\infty}\cdots\int dx_1\cdots dx_n\langle P[j_\mu(x),\ H_1(x_1),\ \cdots,$$
$$H_1(x_{n_1}),\ H_2(x_{n_1+1}),\ \cdots,\ H_8(x_n)]\rangle_{1,0}, \tag{12}$$

where H_1, H_2, $\cdots$, H_8 are the charge-independent Hamiltonians shown in (1) and (2). According to an argument similar to that of Takahashi,8) who treated the pion current around nucleons, the expectation value of an isoboson or isofermion (anti-isofermion for the Ξ) current around any baryon or meson is proportional to the matrix element of the isospin operators given in Table 5.

Table 5

Current around	Isoscalar particle (Λ)	Isospinor particles (K, N, Ξ)	Isovector particles (π, Σ)
$\langle$Isoboson (π, Σ) current$\rangle_{1,0} \propto$	0	τ_3	I_3
$\langle$Isofermion (K, N, Ξ) current$\rangle_{1,0} \propto$	1	$1+a\tau_3$	$1+bI_3$

Here a, b are certain (positive or negative) scalar constants independent of isospin operators. These results could be expected from the consideration of transformation properties of the current operators (9) to (11) in the isospin space.

From the results in Table 5, *the mirror theorem* for the anomalous magnetic moments of the baryons (and of the K meson, if the K meson is not a spin 0 particle) is readily obtained :

$$\mu(\Lambda^\circ),\ \mu(p^+)+\mu(n^\circ),\ \mu(\Xi^-)+\mu(\Xi^\circ),\ (\mu(K^+)+\mu(K^\circ))\ \text{and}$$
$$\mu(\Sigma^+)+\mu(\Sigma^-)=2\mu(\Sigma^\circ) \tag{13}^*$$

get a zero contribution from the isoboson (Λ, π, Σ) *currents and depend on the isofermion and anti-isofermion* (K, N, *and* Ξ) *currents only.*

It should be noted that this theorem comes from the charge-independent interactions which satisfy the Nishijima and Gell-Mann rule, and no use is made of any approximation like weak coupling or any specification of the parities of either mesons or baryons.

As to the anomalous magnetic moments of nucleons, the above mirror theorem shows that, even if the recoil of baryons in the intermediate states is neglected, a nonvanishing value of $\mu(p^+)+\mu(n^\circ)$ can be expected from the K current.

The relation (13) on $\mu(\Sigma^+)$, $\mu(\Sigma^-)$ and $\mu(\Sigma^\circ)$ is useful for the argument on the mass differences of the Σ hyperons.[9] This will be discussed in the next section.

§ 3. The anomalous magnetic moments of Σ hyperons

Combining the perturbation-independent relation (13) for $\mu(\Sigma)$'s and their hyperbolic relations[9] obtained from the observed mass differences of Σ's, one may determine a set of values of $\mu(\Sigma^+)$, $\mu(\Sigma^-)$ and $\mu(\Sigma^\circ)$. Table 6 shows how these values depend upon the cutoff momentum k and the observed $\Sigma^\circ-\Sigma^+$ mass difference. (The three different cases, i.e., $M(\Sigma^\circ)-M(\Sigma^+)=$ (i) $5\,m_e$, (ii) 0, and (iii) $-5\,m_e$, were examined in ref. 9, because of an inaccuracy of the experimental mass of Σ°.[10])

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* After this note was prepared, the author noticed that Marshak and others have independently obtained the same relation as (13) on the $\mu(\Sigma)$'s. (Phys. Rev. **106** (1957), 599).

Table 6

Cutoff k/M_Σ		1.0*		1.5			2.0		
Choice of $M(\Sigma^0)$		(i)	(ii)	(i)	(ii)	(iii)	(i)	(ii)	(iii)
$\mu(\Sigma^+)+\mu(\Sigma^-)>0$	$\mu(\Sigma^+)$	1.36	1.86	0.68	0.51	0.44	0.42	0.31	0.19
	$\mu(\Sigma^-)$	0.36	2.51	0.03	0.39	1.06	-0.03	0.13	0.33
	$\mu(\Sigma^\circ)$	0.86	2.19	0.35	0.45	0.75	0.19	0.22	0.26
$\mu(\Sigma^+)+\mu(\Sigma^-)<0$	$\mu(\Sigma^+)$	-3.08	-3.58	-2.43	-2.27	-2.19	-2.19	-2.08	-1.96
	$\mu(\Sigma^-)$	1.36	-0.79	1.73	1.37	0.70	1.80	1.64	1.44
	$\mu(\Sigma^\circ)$	-0.86	-2.19	-0.35	-0.45	-0.75	-0.19	-0.22	-0.26

* The case (iii) for $k/M_\Sigma=1.0$ has no real solution.

Comparing the $\mu(\Sigma)$'s in Table 6 with those in Table 3, one may clearly observe that both the orders of magnitude and the signs agree for the cases of II, IV (solution $\mu(\Sigma^+)+\mu(\Sigma^-)<0$) and V, VII (solution $\mu(\Sigma^+)+\mu(\Sigma^-)>0$). For example, if one takes values of g^2 shown in Table 7 in accordance with the cases II, IV, V, and VII in Table 3, one obtains the $\mu(\Sigma)$'s for $k/M_\Sigma=1.5$ and (ii) in Table 6.

Table 7

Solution	$\mu(\Sigma^+)+\mu(\Sigma^-)<0$		$\mu(\Sigma^+)+\mu(\Sigma^-)>0$	
Case	II	IV	V	VII
$g^2(\Sigma N\bar{K})$	5.5	1.6	2.4	1.0
$g^2(\Sigma\Xi K)$	0.8	1.6	2.4	1.0
$g^2(\Sigma\Sigma\pi)$	2.9	3.2	2.4	2.0
$g^2(\Sigma\Lambda\pi)$	2.9	3.2	2.4	2.0

A similar agreement can be attained for the other choices of k and $M(\Sigma^\circ)$. The other combinations of coupling types in Table 3, i. e., I, III, VI, and VIII, however, require an unreasonably large coupling constant or an imaginary coupling constant. As is seen from the above result, one cannot exclude the solution $\mu(\Sigma^+)+\mu(\Sigma^-)<0$, which was omitted in Sudershan and Marshak's discussion.$^{7)}$

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6) The values of B_1 and B_2 for nucleons are listed for the sake of comparison. The contributions from the $(NN\pi)$ interaction were early calculated by K. M. Case (Phys. Rev. **76** (1949), 1) and others. Although the cotributions from the $(N\Sigma K)$ and $(N\Lambda K)$ interactions have been calculated by Iso and Iwao (ref. 5), the present numerical value of $B_1(N\Sigma K)$ for *ps* coupling does not agree with theirs.

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Static Model in the Meson Theory

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A non-linear static model is constructed from the relativistic γ_5-theory, by approaching the static limit after a canonical transformation is applied to the original γ_5-Hamiltonian, with no perturbation expansion used. It is then proven that the same analysis as due to Chew and Drell et al. of *P*- and *S*-wave pion-nucleon scattering can be applied even for the general non-linear static model obtained, as far as the one-meson approximation is done. In our model, however, various renormalized coupling constants are uniquely defined in terms of the unrenormalized $ps-ps$ coupling constant. Furthermore, their numerical values depend in general on the types of the process concerned; the renormalized *P*-wave coupling constant effective in pion-nucleon scattering is different from one determining threshold gamma-pion production. Rough numerical estimations of these renormalized coupling constants are done for two transformations found by Dyson and by Foldy. The results seem qualitatively satisfactory, including a strong meson-pair term damping, though a definite statement is not permitted yet.

§ 1. Introduction

A static model seems to have been established successfully regarding low energy pion-nucleon phenomena, employing the Chew-Low-Wick formalism[1], including the *S*-wave pion-nucleon interaction[2]. This success suggests that the static approximation might be a valid approximation for the low energy phenomena in the relativistic γ_5-theory, as far as it can describe these phenomena correctly. However, the inter-relation between the γ_5-theory and the static model proposed has not yet been clarified; especially the *S*-wave static model is a purely phenomenological assumption.

The purpose of the present investigation is to show how we can construct a static model from the relativistic γ_5-theory, without recourse to the perturbation expansion and furthermore how the resulting non-linear model is equivalent to the static model proposed previously[1,2]. The recipe is the following: Firstly we apply a suitable canonical transformation to the original γ_5-Hamiltonian and then approach the static limit for nucleons, without employing the perturbation expansion, thns leading in general to a non-linear interaction Hamiltonian.

We can, however, show that the Chew-Low-Wick[1] formalism is applied to such a general type of Hamiltonian and, as far as the one-meson approximation is made, exactly the same analysis is permitted of pion-nucleon scattering as was done by Chew-Low[1] and Drell-Friedman-Zachariasen[2], in terms of various renormalized coupling constants. While they are in these models adjustable independent parameters to be determined to fit data, they are, according to our model, uniquely related to the unrenormalized $ps-ps$ coupling constant. It is, therefore, possible in principle to evaluate the numerical values of these renormalized parameters in terms of the unrenormalized $ps-ps$ coupling constant. In section 2, the general nature of the requisite canonical transformation is discussed; the Dyson and the Foldy transformations belong to our category. The Chew-Low-Wick[1] formalism is then applied to the transformed (non-linear) Hamiltonian in section 3.

We apply the same procedure to the case where the electromagnetic field exists (section 4). The electromagnetic interaction is introduced before or after the canonical transformation is performed, both giving the same result. It is then shown that the renormalized coupling constant g_γ which determines threshold γ-pion production is different in general from the corresponding coupling constant g which is effective in pion-nucleon scattering.

No approximation has thus far been made except the static approximation when constructing our static model. In order to estimate the numerical values of various renormalized coupling constants (section 5), the intermediate coupling meson theory is used. Rough estimations are made for two transformations by Dyson and by Foldy. The result seems qualitatively satisfactory; especially a strong damping of the meson-pair term is reproduced for the Dyson transformation and g_γ is somewhat smaller than g. However, the definite conclusion is certainly premature, since the interrelation between renormalized and unrenormalized coupling constants concerns essentially the self-meson cloud around the physical nucleon, which we solved here only crudely. As a general conclusion, it is suggested that a successful static model may be constructed by choosing a suitable canonical transformation, which proves that the relativistic γ_5-theory may well be consistent with the low energy pion-nucleon data, including both P- and S-wave pion-nucleon interactions. Numerical results are summarized at the end of section 5. In the final section the physical contents of our non-linear model are discussed.

§ 2. Canonical transformation

The relativistic γ_5-Hamiltonian is given in the Schrödinger representation by

$$H=\int\bar{\psi}(i\gamma\boldsymbol{p}+m)\psi d\boldsymbol{x}+\frac{1}{2}\int(\boldsymbol{\pi}^2+\boldsymbol{\phi}(\mu^2-\Delta)\boldsymbol{\phi})\,d\boldsymbol{x}+if\int\bar{\psi}\gamma_5\boldsymbol{\tau\phi}\psi d\boldsymbol{x}, \tag{1}$$

where the notations are those commonly used. To construct a static model that reproduces particular features of the above Hamiltonian in the low energy region, we apply to (1) a canonical transformation exp (iS), for which we demand the following requirements:

(i) It is an invariant scalar in the charge space, because the charge independence is known to be valid especially in the low energy region.

(ii) It is invariant against the space rotation and also the space inversion, since the angular momentum and parity are conserved in strong interactions.

(iii) It is not always an invariant quantity against the Lorentz transformation changing the time axis, because the static approximation for nucleons necessarily violates the invariance against the full Lorentz group.

(iv) It creates or annihilates odd number of S-wave mesons, since static interaction can create or annihilate only even number of S-wave mesons at the same time.

In addition to these requirements which are definitely necessary, we impose the following requirement which seems to be reasonable :

(v) It is a function of ϕ only, excluding its derivatives and its canonical momentum π. Our motivation for this is the following : Because of the characteristic γ_5-interaction in (1), one should take due accounts of the higher order corrections with respect to S-wave pions accompanying virtual nucleon pair processes, before we approach the static limit, while the interaction may well be linear as regards P-wave mesons in the static limit. We furthermore presume that ϕ plays a much more important role than π as far as a static model is concerned. Since S must be dimensionless, it is finally a function of dimensionless combination $(f/2m)\boldsymbol{\tau}\phi$.

It is evident that the most general unitary transformotion satisfying above requirements is given by

$$e^{iS},\ S=\frac{f}{2m}\int\overline{\psi}[iv(\chi)+\gamma_4 w(\chi)]\gamma_5\boldsymbol{\tau}\phi\psi d\boldsymbol{x}, \tag{2}$$

where $v(x)$ and $w(x)$ are real even functions of x, being $(f/2m)\sqrt{\phi^2}$.

After having applied to (1) the canonical transformation (2), we make the static approximation for nucleons, neglecting all terms containing nucleon momenta and odd Dirac matrices. Thus we get our static model with the following interaction Hamiltonian :

$$\begin{aligned}H'=&\frac{f}{2m}((\sigma\nabla)\boldsymbol{\tau}\phi)f(x)+(f/2m)^3(\boldsymbol{\tau}\boldsymbol{\phi})((\sigma\nabla)\boldsymbol{\tau}\boldsymbol{\phi})(\boldsymbol{\tau}\boldsymbol{\phi})g(x)+\\&+2mh(x)+(f/2m)^2\{\tfrac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\pi k(x)+\tfrac{1}{2}k(x)\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\pi\}.\end{aligned} \tag{3}$$

The functions $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are all related to $v(x)$ and $w(x)$ and are also real even functions of x. We easily see that the interaction (3) is the most general one under the five requirements mentioned above. Some features of (3) are being linear with respect to P-wave mesons, while highly nonlinear regarding S-wave mesons and invariant against rotations in spin and charge space including space inversion.

We add a remark here that the part containing $v(x)$ in (2) commutes with the γ_5-interaction term in (1) and, therefore, $w(x)$ would be much more important than $v(x)$ in reproducing a successful static model. Furthermore, it is shown that $v(x)$ gives rise to non-Hermitian terms after the static approximation is made. This difficulty could naturally be avoided by treating the nucleon motion more properly when approaching the static limit. However, we shall in the following neglect $v(x)$ for reasons mentioned above. Thus, using a menthod by Berger, Foldy and Osborn[4], we get

$$\begin{aligned}f(x)&=\frac{\sin(2xw(x))}{2x}+\frac{1}{2}\left\{x\frac{dw(x)}{dx}+w(x)-\frac{\sin(2xw(x))}{2x}\right\},\\g(x)&=\frac{1}{2x^2}\left\{x\frac{dw(x)}{dx}+w(x)-\frac{\sin(2xw(x))}{2x}\right\},\\h(x)&=\frac{1}{2}\{\cos(2xw(x))-1+2x\sin(2xw(x))\},\end{aligned} \tag{4}$$

$$k(x)=\frac{1}{2x^2}\{1-\cos(2xw(x))\},$$

which also agree with the results obtained by Akiba and Sawada[5]. We have simply dropped in (4) those contributions which are due to the terms biquadratic in ψ^* and ψ, since they contain $\delta(0)$ and therefore may be eliminated by mass renormalization.

If we finally require that the canonical transformation (2) assures the equivalence of *ps*—*ps* and *ps*—*pv* couplings for sufficiently small *f*, it follows that

$$v(x)\to 0,\qquad w(x)\to 1, \tag{5}$$

in the weak coupling limit or as x goes to zero. This is another reason why we put $v(x)$ equal to zero. The transformations proposed by Dyson[6] and Foldy[4] are two special cases of our transformation, satisfying all the requirements mentioned above, including (5). Indeed, the former is characterized by

$$v(x)=0,\qquad w(x)=1, \tag{6}$$

and the latter by

$$v(x)=0,\quad w(x)=(\tan^{-1}2x)/2x\,. \tag{7}$$

In section 5, we treat these two cases in detail in discussing some numerical results. We get from (4) and (6), for the Dyson transformation,

$$\begin{aligned} &f(x)=(2x+\sin 2x)/4x,\quad g(x)=(2x-\sin 2x)/4x^3,\\ &h(x)=\tfrac{1}{2}[\cos 2x-1+2x\sin 2x],\\ &k(x)=\frac{1}{2x^2}[1-\cos 2x]. \end{aligned} \tag{8}$$

The Foldy transformation (7) leads to

$$\begin{aligned} &f(x)=(1+\xi)/2\xi^2,\quad g(x)=-2/\xi^2(1+\xi),\\ &h(x)=\tfrac{1}{2}(\xi-1),\qquad k(x)=2/\xi(\xi+1), \end{aligned} \tag{9}$$

where $\xi=\sqrt{1+4x^2}$.

§ 3. Pion-nucleon scattering

According to Drell, Friedman and Zachariasen,[2] *S*-matrix element $\tilde{S}_{fi}$ for pion-nucleon scattering is given by

$$\tilde{S}_{fi}=\delta_{fi}-2\pi i\delta(\omega_f-\omega_i)T_{fi}\,, \tag{10}$$

where

$$T_{fi}=\langle\psi_0|[a_f[H',a_i^+]]|\psi_0\rangle+\frac{1}{\omega_f}(T^*_{of}T_{oi}-S^*_{oi}S_{of})-\sum_{n\geq 1}\left\{\frac{T^*_{nf}T_{ni}}{E_n-\omega_f-i\epsilon}+\frac{S^*_{ni}S_{nf}}{E_n+\omega_f}\right\},$$

$$S_{fi}=\langle\psi_0|[a_f[H', a_i]]|\psi_0\rangle+\frac{1}{\omega_f}(T^*_{of}S_{oi}-T^*_{oi}S_{of})-\sum_{n\geq1}\left\{\frac{T^*_{nf}S_{ni}}{E_n-\omega_f-i\epsilon}+\frac{T^*_{ni}S_{nf}}{E_n+\omega_f}\right\}, \quad (11)$$

and

$$T_{oi}=\langle\psi_0|[H', a_i^+]|\psi_0\rangle, \quad S_{oi}=\langle\psi_0|[H', a_i]|\psi_0\rangle, \quad (12)$$

ψ_0 describing a physical nucleon whose energy eigenvalue is now chosen to be zero, a_i^+ and a_i being creation and annihilation operators of a meson specified by i with energy ω_i and n representing all possible intermediate states with energy E_n except the physical nucleon states.

As far as we assume the one-meson approximation by dropping all but the one-meson states in the sum over n in (11), equations (11) constitute closed integral equations by themselves. We shall now prove that these integral equations with our non-linear Hamiltonian (3) become identical with those obtained by Chew-Low[1] and Drell et al.[2] The difference is that various renormalized coupling constants are all related to the unrenormalized $ps-ps$ coupling constant f, thus permitting us to evaluate them in terms of f and the cut-off energy $\omega_{\max}$ which should inevitably be introduced so that the static approximation may be consistent.

In order to prove the above statement, we need only to evaluate the inhomogeneous terms in (11). First we note the following expansions valid for meson fields in (3):

$$\begin{gathered}\phi_\alpha=\sum_K\frac{v_K}{\sqrt{\omega_K}}A_{K\alpha}, \quad \pi_\alpha=\sum_K v_K\sqrt{\omega_K}\,B_{K\alpha}, \\ (\sigma\nabla)\boldsymbol{\tau}\boldsymbol{\phi}=-\sum_{K\alpha}\frac{v_K}{\sqrt{\omega_K}}(\sigma\boldsymbol{K})\tau_\alpha B_{K\alpha},\end{gathered} \quad (13)$$

with

$$\begin{gathered}A_{K\alpha}=\frac{1}{\sqrt{2}}(a^+_{K\alpha}+a_{K\alpha}), \quad B_{K\alpha}=\frac{i}{\sqrt{2}}(a^+_{K\alpha}-a_{K\alpha}), \\ [A_{K\alpha}, B_{K'\alpha'}]=i\delta_{KK'}\delta_{\alpha\alpha'},\end{gathered} \quad (14)$$

where the cut-off function v_K is explicitly introduced. Then, employing following commutation relations

$$\begin{gathered}[\boldsymbol{\phi}^2, B_i]=2i\frac{v_i}{\sqrt{\omega_i}}\phi_i, \\ [(\sigma\nabla)\boldsymbol{\tau}\boldsymbol{\phi}, A_i]=i\frac{v_i}{\sqrt{\omega_i}}(\sigma\boldsymbol{K}_i)\tau_i, \\ [\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}, B_i]=-i(\boldsymbol{\tau}\times\boldsymbol{\pi})_i\frac{v_i}{\sqrt{\omega_i}}, \\ [\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}, A_i]=-i(\boldsymbol{\tau}\times\boldsymbol{\phi})_i v_i\sqrt{\omega_i},\end{gathered} \quad (15)$$

with abbreviations $A_i\equiv A_{K_i i}$ and $B_i\equiv B_{K_i i}$, we can easily calculate the commutators of

H' with A_i and B_i. For example, one gets

$$[H',\, A_i]=i\frac{v_i}{\sqrt{\omega_i}}\Big[\,(\sigma\boldsymbol{K}_i)\,\Big\{\tau_i\frac{f}{2m}f(x)+ \\ +(\boldsymbol{\tau}\boldsymbol{\phi})\,\tau_i(\boldsymbol{\tau}\boldsymbol{\phi})\,(f/2m)^3 g(x)\Big\}-\omega_i(\boldsymbol{\tau}\times\boldsymbol{\phi})_i(f/2m)^2 k(x)\Big]. \tag{16}$$

The last term on the right-hand side of (16) is odd with respect to S-wave mesons and therefore does not contribute to the physical nucleon expectation value, as was shown by Drell et al.[2)] in Appendix 1 of their original paper. We easily see that all terms in $[H',\, B_i]$ are also odd as regards the S-wave mesons. Thus we get

$$\langle\psi_0|[H',\, A_i]|\psi_0\rangle=i\frac{v_i}{\sqrt{\omega_i}}(f/2m)\langle\psi_0|\,(\sigma\boldsymbol{K}_i)\,\{\tau_i f(x)+ \\ +(\boldsymbol{\tau}\boldsymbol{\phi})\,\tau_i(\boldsymbol{\tau}\boldsymbol{\phi})\,(f/2m)^2 g(x)\}\,|\psi_0\rangle, \tag{17}$$

$$\langle\psi_0|[H',\, B_i]|\psi_0\rangle=0.$$

We can now define a real scalar g with the same dimension as f by

$$\frac{g}{\mu}\langle u_0|\sigma_j\tau_i|u_0\rangle=\frac{f}{2m}\langle\psi_0|\sigma_j\{\tau_i f(x)+(\boldsymbol{\tau}\boldsymbol{\phi})\,\tau_i(\boldsymbol{\tau}\boldsymbol{\phi})\,(f/2m)^2 g(x)\}\,|\psi_0\rangle, \tag{18}$$

where u_0 is a bare spinor corresponding to ψ_0. This definition is consistent with the transformation properties of the quantities involved and is nothing but the relation which connects the renormalized P-wave coupling constant g effective in pion-nucleon scattering to the unrenormalized *ps-ps* coupling constant f. Thus we may rewrite (17) as, dropping bare spinors,

$$\begin{aligned}&\langle\psi_0|[H',\, A_i]|\psi_0\rangle=i\frac{g}{\mu}\frac{v_i}{\sqrt{\omega_i}}(\sigma\boldsymbol{K}_i)\,\tau_i\,,\\ &\langle\psi_0|[H',\, B_i]|\psi_0\rangle=0,\end{aligned} \tag{19}$$

or equivalently from (12)

$$T_{\alpha i}=S_{\alpha i}=i\frac{g}{\mu}\frac{v_i}{\sqrt{2\omega_i}}(\sigma\boldsymbol{K}_i)\,\tau_i\,, \tag{20}$$

which is exactly the same as was obtained in the linear P-wave static model[1)].

The first terms in (11) can be evaluated in a similar way. In constructing the physical nucleon expectation value we use the following theorem, besides one mentioned above. Let $\Phi_{\alpha\beta}$ be an arbitrary Hermitian function of σ, τ, ϕ and π, which is a symmetric tensor (with respect to α and β) in the charge space and a scalar in the ordinary spin space. Then it follows

$$\langle\psi_0|\,\Phi_{\alpha\beta}|\psi_0\rangle=\delta_{\alpha\beta}\tfrac{1}{3}\sum_{\alpha}\langle\psi_0|\,\Phi_{\alpha\alpha}|\psi_0\rangle, \tag{21}$$

since the only Hermitian symmetric tensor in the charge space is $\tau_\alpha\tau_\beta+\tau_\beta\tau_\alpha$ or $\delta_{\alpha\beta}$.

Using these two theorems, one gets after a straightforward calculation, dropping bare spinors again,

$$\begin{aligned}
&\langle\psi_0|[A_f[H', A_i]]|\psi_0\rangle=0,\\
&\langle\psi_0|[A_f[H', B_i]]|\psi_0\rangle=-\langle\psi_0|[B_f[H', A_i]]|\psi_0\rangle=\lambda\frac{v_iv_f}{\sqrt{\omega_i\omega_f}}\omega_f\epsilon_{i\alpha f}\tau_\alpha,\\
&\langle\psi_0|[B_f[H', B_i]]|\psi_0\rangle=2\lambda_0\frac{v_iv_f}{\sqrt{\omega_i\omega_f}}\delta_{if},
\end{aligned}\tag{22}$$

where $\epsilon_{\alpha\beta\gamma}$ is equal to $+1$ or -1 for even or odd permutations of its indices, respectively, and we have defined two real scalars λ and λ_0 by

$$\begin{aligned}
\lambda\langle u_0|\tau_\alpha|u_0\rangle&=\left(\frac{f}{2m}\right)^2\left\langle\psi_0\middle|\tau_\alpha\left\{k(x)+\frac{x}{3}\frac{dk(x)}{dx}\right\}\middle|\psi_0\right\rangle,\\
\lambda_0\langle u_0|u_0\rangle&=\frac{f^2}{2m}\left\langle\psi_0\middle|\frac{1}{3x}\frac{dh(x)}{dx}+\frac{1}{6}\frac{d^2h(x)}{dx^2}+\right.\\
&\quad+\frac{f}{(2m)^2}((\sigma\nabla)\tau\phi)\left\{\frac{1}{3x}\frac{df(x)}{dx}+\frac{1}{6}\frac{d^2f(x)}{dx^2}-\frac{1}{3}g(x)\right\}+\\
&\quad+\frac{f^3}{(2m)^4}(\tau\phi)((\sigma\nabla)\tau\phi)(\tau\phi)\left\{\frac{1}{x}\frac{dg(x)}{dx}+\frac{1}{6}\frac{d^2g(x)}{dx^2}\right\}+\\
&\quad+\frac{f^2}{(2m)^3}\frac{1}{2}\tau\cdot\phi\times\pi\left\{\frac{1}{3x}\frac{dk(x)}{dx}+\frac{1}{6}\frac{d^2k(x)}{dx^2}\right\}+\\
&\quad\left.+\frac{f^2}{(2m)^3}\left\{\frac{1}{3x}\frac{dk(x)}{dx}+\frac{1}{6}\frac{d^2k(x)}{dx^2}\right\}\frac{1}{2}\tau\cdot\phi\times\pi\middle|\psi_0\right\rangle.
\end{aligned}\tag{23}$$

These definitions are also consistent with the transformation properties of various factors in (23) and specify the relations between the renormalized *S*-wave coupling constants λ and λ_0 effective in pion-nucleon scattering and the unrenormalized *ps-ps* coupling constant *f*. The equations (22) show that the first terms in (11) contribute only to *S*-wave pion-nucleon scattering and are just what comes out in an *S*-wave static model by Drell et al.[2] We have, therefore, shown that, as far as the one-meson approximation is made, exactly the same analyses as were developed by Chew and others[1)2)] of *P*- and *S*-wave pion-nucleon scattering in terms of three renormalized coupling constants g, λ and λ_0 are allowed also in our non-linear static model (3). In our model, however, three coupling constants g, λ and λ_0, which are determined by scattering data, are now uniquely related to the unrenormalized *ps-ps* coupling constant *f* by relations (18) and (23). It is therefore possible in principle to evaluate the theoretical values of these coupling constants and to compare them with the empirical values determined by Chew and others[1)2)]. Although ψ_0 has not yet been solved satisfactorily, some estimation of g, λ and λ_0 wiil be given in section 5 for the transformations (6) and (7).

§ 4. Gamma-pion production

We would like to stress another particular feature of our non-linear model (3): The renormalized coupling constant depends in general on the process concerned. For example, the renormalized P-wave coupling constant g defined by (18) is effective only in pion-nucleon scattering and may be different from renormalized P-wave coupling constant g_γ which determines the threshold γ-pion production cross section. In the linear P-wave model by Chew[1], on the other hand, g is rigorously equal to g_γ[1]. It is easily seen that g and g_γ cannot be the same in our model because of the non-linearity of the interaction (3). It is the effective interaction and not the true or basic interaction that is concerned in the static model. It is, therefore, physically reasonable to expect that a non-linear static interaction thus permit various renormalized coupling constants to depend on the type of the process concerned.

The introduction of the electromagnetic field can be accomplished in two alternative ways; by introducing it into (1) before the canonical transformation is applied, followed by the same procedure as was explained before or by simply replacing $\boldsymbol{\nabla}$ by $\boldsymbol{\nabla}-ie\boldsymbol{A}$ in our static Hamiltonian, $\boldsymbol{A}$ being the vector potential of the electromagnetic field. It is shown that the results are the same and the additional interaction Hamiltonian is

$$H'_A=-\frac{ef}{2m}(\sigma\boldsymbol{A})(\boldsymbol{\tau}\times\phi)_3\{f(x)-x^2g(x)\}-e\int\boldsymbol{A}(\phi_2\boldsymbol{\nabla}\phi_1-\phi_1\boldsymbol{\nabla}\phi_2)\,d\boldsymbol{x}. \tag{24}$$

It is known[2] that (24) is not gauge-invariant. However, we will not concern ourselves with this point, since the above two procedures give the same interaction Hamiltonian (24) and the static model is just an approximation and not the rigorous theory. We recall here that Drell et al.[2] have shown that the correction terms to (24) which make their model gauge-invariant give only small contributions to the low energy γ-pion production.

The transition amplitude $m_k(p)$ is now defined as the amplitude, to the first order in $\boldsymbol{A}$, from a state of a free photon k, specifying its momentum $\boldsymbol{k}$ and polarization $\boldsymbol{e}$, together with a physical nucleon, to a one-meson final state specified by p. Using the same notations as in (11) and (12), $m_k(p)$ satisfies the integral equation[2]

$$m_k(p)=\langle\psi_0|[a_p,H'_A]|\psi_0\rangle+\frac{1}{\omega_p}(T^*_{op}m_k(0)+m^*_k(0)S_{op})-$$
$$-\sum_{n\geq 1}\left\{\frac{T^*_{np}m_k(n)}{E_n-\omega_p-i\epsilon}-\frac{m_k{}^*(n)S_{np}}{E_n+\omega_p}\right\}, \tag{25}$$

with

$$m_k(0)=\langle\psi_0|H'_A|\psi_0\rangle, \tag{26}$$

where the vector potential $\boldsymbol{A}$ in H'_A is now to be replaced by $\boldsymbol{e}/\sqrt{2k}$.

Using the procedure similar to the one in the previous section, especially the theorem proven there, one gets, as the contribution of the first term in (24) to the first term in (25),

$$\left\langle \psi_0 \middle| \left[a_p, \frac{-ef}{2m} \frac{(\sigma e)}{\sqrt{2k}} (\tau \times \phi)_3 \{ f(x) - x^2 g(x) \} \right] \middle| \psi_0 \right\rangle$$

$$= \frac{ef}{2m} \frac{v_p}{\sqrt{4k\omega_p}} \left\langle \psi_0 \middle| (\sigma e) \left\{ \tau_{3p} \left(1 + \frac{x}{4} \frac{d}{dx} \right) (f(x) - x^2 g(x)) - \right.\right.$$

$$\left.\left. - (f/2m)^2 (\tau\phi) \tau_{3p} (\tau\phi) \frac{1}{4x} \frac{d}{dx} (f(x) - x^2 g(x)) \right\} \middle| \psi_0 \right\rangle, \tag{27}$$

where $\tau_{33} = 0$, $\tau_{31} = \tau_2$ and $\tau_{32} = -\tau_1$. We can now define another real scalar g_Υ with the same dimension as g or f by

$$\frac{g_\Upsilon}{\mu} \langle u_0 | \sigma_j \tau_i | u_0 \rangle = \frac{f}{2m} \left\langle \psi_0 \middle| \sigma_j \left\{ \tau_i \left(1 + \frac{x}{4} \frac{d}{dx} \right) (f(x) - x^2 g(x)) - \right.\right.$$

$$\left.\left. - (f/2m)^2 (\tau\phi) \tau_i (\tau\phi) \frac{1}{4x} \frac{d}{dx} (f(x) - x^2 g(x)) \right\} \middle| \psi_0 \right\rangle. \tag{28}$$

Thus the first term in (25) takes the same expression as in the previous model[1)2)], with the renormalized *P*-wave coupling constant replaced by g_Υ, which is different from g defined by (18). Since the second term in (25) contributes only to *P*-wave production, as is seen from (20), the analysis due to Drell et al.[2)] of threshold γ-pion production need not be modified at all.

Thus we see that the renormalized *P*-wave coupling constant determined by Drell et al.[2)] to fit threshold γ-pion production is not g defined by (18), which is effective only in pion-nucleon scattering, but g_Υ defined by (28), according to our non-linear model. The theoretical estimates of g_Υ will be given in the next section together with its comparison with the empirical value.

§ 5. Estimation of renormalized coupling constants and comparison with empirical values

Let us now turn over to the theoretical estimation of four renormalized coupling constants g, λ, λ_0 and g_Υ defined by (18), (23) and (28), respectively. We shall, however, be satisfied with rather rough estimates, since we have no powerful technique to evaluate the physical nucleon expectation values of highly non-linear quantities.

Our main interest is how much would be the difference between g and g_Υ and whether an extremely strong pair damping (small λ_0) would be realized in our static model which contains only one coupling constant f. For this purpose, we expand $|\psi_0\rangle$, according to the intermediate coupling theory, as

$$|\psi_0\rangle = c_0 |u_0\rangle + c_1 |\text{one meson}\rangle + \cdots^{7)}. \tag{29}$$

Since $|\text{one meson}\rangle$ consists only of creation operators of *P*-wave mesons, which are commutable with any functions of x, we easily get, as the contributions up to the one-meson amplitude,

$$\frac{g}{\mu}=\frac{f}{2m}\left(c_0^2+\frac{1}{9}c_1^2\right)\left\langle f(x)-\frac{1}{3}x^2g(x)\right\rangle_0+\cdots,$$

$$\frac{g_\Upsilon}{\mu}=\frac{f}{2m}\left(c_0^2+\frac{1}{9}c_1^2\right)\left\langle\left(1+\frac{x}{3}\frac{d}{dx}\right)(f(x)-x^2g(x))\right\rangle_0+\cdots,$$

$$\lambda=(f/2m)^2\left(c_0^2-\frac{1}{3}c_1^2\right)\left\langle\left(1+\frac{x}{3}\frac{d}{dx}\right)k(x)\right\rangle_0+\cdots,$$

$$\lambda_0=\frac{f^2}{2m}\left[(c_0^2+c_1^2)\left\langle\left(\frac{1}{3x}\frac{d}{dx}+\frac{1}{6}\frac{d^2}{dx^2}\right)h(x)\right\rangle_0+\right. \tag{30}$$

$$\left.+2c_0c_1\frac{fV}{\sqrt{4\pi}}\left\langle\left(\frac{1}{3x}\frac{d}{dx}+\frac{1}{6}\frac{d^2}{dx^2}\right)\left(f(x)-\frac{1}{3}x^2g(x)\right)\right\rangle_0\right]+\cdots,$$

where $\langle\ \rangle_0$ means the meson vacuum expectation value and

$$V=\frac{1}{(2m)^2}\sqrt{\frac{3}{\pi}}\int\frac{k^4}{\omega^2}dk\Big/\sqrt{\int\frac{k^4}{\omega^3}dk}\,, \tag{31}$$

for the bound P-wave configuration proportional to $k^2/(\omega\sqrt{\omega})$. The values of $fV/\sqrt{4\pi}$ are given in Table 1 for various values of $f^2/4\pi$ and $\omega_{\max}$.

Table 1. The values of $fV/\sqrt{4\pi}$

$\omega_{\max}$	4μ	5μ	6μ
$f^2/4\pi=20$	0.171	0.275	0.411
30	0.209	0.337	0.504
40	0.242	0.389	0.582
50	0.270	0.435	0.651

According to (30), the renormalized coupling constants depend on the amplitudes c_0 and c_1 and the meson vacuum expectation values of five functions of x, which we shall denote R_g, R_{g_Υ}, R_λ, R_{λ_0} and R'_{λ_0}, the latter two being those appearing in λ_0, respectively. They are given, for the Dyson transformation (6), by

$$R_g=\left\langle\frac{1}{3}\left(1+\frac{\sin 2x}{x}\right)\right\rangle_0,\quad R_{g_\Upsilon}=\left\langle\frac{1}{3}\left(\cos 2x+\frac{\sin 2x}{x}\right)\right\rangle_0,$$

$$R_\lambda=\left\langle\frac{1}{3}\left(\frac{\sin^2 x}{x^2}+\frac{\sin 2x}{x}\right)\right\rangle_0,\quad R_{\lambda_0}=\left\langle\cos 2x-\frac{2x}{3}\sin 2x\right\rangle_0, \tag{32}$$

$$R'_{\lambda_0}=-\left\langle\frac{2\sin 2x}{9x}\right\rangle_0,$$

while they are, for the Foldy transformation (7),

$$R_g=\left\langle\frac{1+2\xi}{3\xi^2}\right\rangle_0,\quad R_{g_\Upsilon}=R_{\lambda_0}=\left\langle\frac{1+2\xi^2}{3\xi^3}\right\rangle_0,$$

$$R_\lambda = \left\langle \frac{2(1+\xi+\xi^2)}{3\xi^3(1+\xi)} \right\rangle_0, \quad R'_{\lambda_0} = -\left\langle \frac{4(4-\xi)(1+\xi)}{9\xi^6} \right\rangle_0. \tag{33}$$

These vacuum expectation values are evaluated according to a method due to Glauber.[8] For an arbitrary function $V(x)$ of x, we have

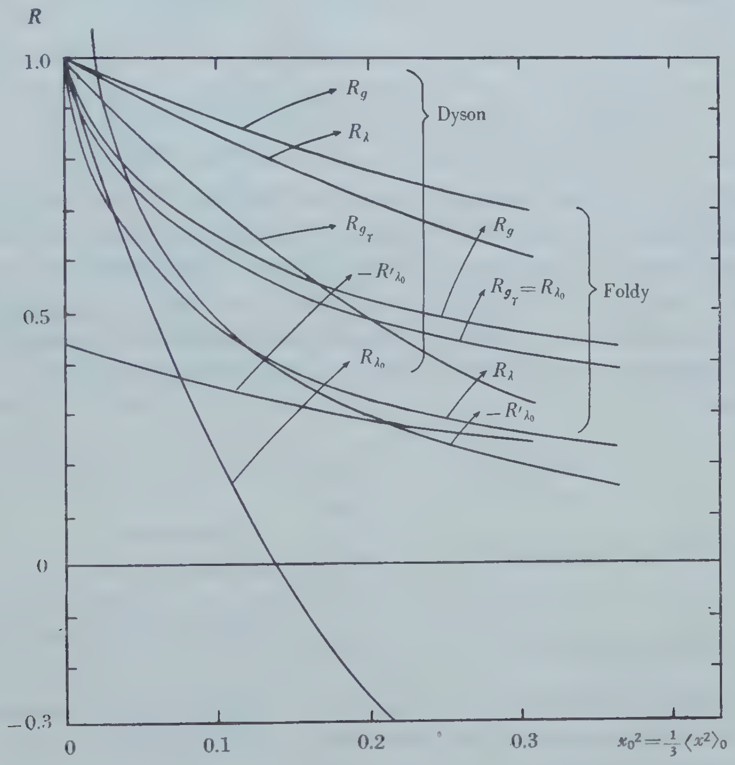
$$\langle V(x) \rangle_0 = (2/\pi x_0^6)^{1/2} \int_0^\infty x^2 V(x) e^{-x^2/2x_0^2} dx, \tag{34}$$

with

$$x_0^2 = \tfrac{1}{3} \langle x^2 \rangle_0. \tag{35}$$

For the functions in (32), analytic integrations were done, but the numerical method was used for (33). All R's as functions of x_0^2 which is in turn a function of f and $\omega_{\max}$ are shown in Fig. 1.

Fig. 1. Plot of R-functions defined by (32) and (33) against x_0^2 given by (35).



Let us now summarize the empirical values of these renormalized coupling constants :

$$\begin{aligned} &g^2/4\pi = 0.087^{9)}, \quad 0.082 \pm 0.015^{10)} \quad \text{or} \quad 0.107 \pm 0.015^{11)}, \\ &g_T^2/4\pi = 0.064 \pm 0.006^{2)}, \quad 0.067 \pm 0.003^{12)}, \\ &\lambda = 0.4/\mu^{2\ 2)}, \quad \lambda_0 = 0.4/\mu^{\ 2)}. \end{aligned} \tag{36}$$

The simple perturbation expansion or putting $c_0 = 1$, $c_1 = 0$ and all R's equal to one in

(30) gives

$$g^2/4\pi : g_\Upsilon^2/4\pi : \lambda\mu^2 : \lambda_0\mu = 1 : 1 : 4\pi : 4\pi(2m/\mu) = 1 : 1 : 12.6 : 169. \qquad (37)$$

It is, therefore, interesting to examine first whether R_{λ_0} and R'_{λ_0} or its combination in (30) would really be vanishingly small, thus explaining the strong meson-pair term damping. From Table 1, the coefficient of R'_{λ_0} in (30) or $2c_0c_1fV/\sqrt{4\pi}$ would be at most about 0.5. We thus see that the strong damping is actually the case for the Dyson transformation, while it seems unlikely for the Foldy transformation, as far as the higher meson amplitudes neglected in (30) would not dominate. It seems that $x_0^2=0.1$ is roughly the right place where the correct λ_0 is reproduced. It would, however, be premature to discuss the right value of x_0^2 and/or to conclude that the correct damping of λ_0 would hardly be attained for the Foldy transformation, since the approximation in (30) is crude.

Let us then observe how much would be the difference between g and g_Υ. The ratio g^2/g_Υ^2 depends only on R_g and R_{g_Υ} and is given as follows ;

$$\begin{array}{lll} x_0^2=0.1 & x_0^2=0.2 & \\ g^2/g_\Upsilon^2=1.54, & 2.52, & \text{(Dyson trans.)} \\ g^2/g_\Upsilon^2=1.08, & 1.18. & \text{(Foldy trans.)} \end{array} \qquad (38)$$

These figures are not inconsistent with the present data (36). It is interesting that both transformations predict g_Υ smaller than g, though the difference seems somewhat too large for the Dyson, while somewhat too small for the Foldy transformation.

Finally, to estimate λ, we need to fix c_0 and c_1. Let us, for example, put $c_0^2=0.7$ and $c_1^2=0.3$ arbitrarily, since any other reasonable assignments do not modify the following figures appreciably. Let us also fix f so that g_Υ takes its value in (36). Thus we get the values :

$$\begin{array}{rll} x_0^2=0.1 & x_0^2=0.2 & \\ \lambda\mu^2=1.51, & 2.69, & \left.\right\} \text{(Dyson trans.)} \\ 1/4\pi\cdot(f\mu/2m)^2=0.237, & 0.493, & \\ \lambda\mu^2=1.13, & 1.24, & \left.\right\} \text{(Foldy trans.)} \\ 1/4\pi\cdot(f\mu/2m)^2=0.317, & 0.498. & \end{array} \qquad (39)$$

They are in general too large by factors 3 or 4 compared with the empirical value in (36), for both transformations. The discrepancy may perhaps consist in too small R_g and R_{g_Υ} compared with R_λ.

We now summarize these numerical results : $g^2/4\pi$ is somewhat larger than $g_\Upsilon^2/4\pi$ for both transformations as given by (38), which seems consistent with the available data (36) ; λ is, however, too large compared with the empirical one (36), as is seen from (39), again for both transformations : The strong meson-pair term damping is well reproduced in the case of the Dyson transformation, while it seems unlikely for the Foldy transformation. However, we should stress that our numerical estimates are rough. Thus a definite conclusion is not given, while the qualitative aspect of our non-linear model seems to be satisfactory.

§ 6. Concluding remarks

As was summarized at the end of the previous section, our non-linear static model seems qualitatively consistent with the available data concerning low energy pion-nucleon scattering and threshold γ-pion production. However, the definite statement about the quantitative agreement is not given, since we cannot evaluate exactly the physical nucleon expectation values and also because we have no unique criterion which suffices to determine the form of the canonical transformation in the very beginning. The important point is that the present investigation suggests that a suitable canonical transformation exists which connects the relativistic γ_5-meson theory to a static model that describes low-energy pion-nucleon scattering and threshold γ-pion production correctly.

If this expectation turns out to be true, the static meson-nucleon interaction should be regarded as highly non-linear with respect to S-wave mesons, while may well be linear regarding P-wave mesons. Therefore, the renormalized coupling constants depend in general on the type of the process concerned. For example, the renormalized P-wave coupling constant is different in pion-nucleon scattering and in threshold γ-pion production.

Our non-linear model would also explain why S-wave mesons interact only weakly with nucleons in low energy pion-nucleon scattering, though virtual nucleon pair creation and annihilation accompanied by an emission or absorption of an S-wave meson takes place strongly. However, it is not known from the beginning whether this is also the case in other processes than low energy pion-nucleon scattering. It is, therefore, very interesting to see what would be the predictions of our non-linear model concerning, for example, meson production in pion-nucleon collision.

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A Remark on the Moments of Inertia of Rotating Nuclei

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The quantum mechanical description of a two dimensional rotating system of particles is studied for the purpose to elucidate the meaning of the effective moments of inertia of nuclei. We employ the method of canonical transformations and obtain the rotational kinetic energy of a "canonical form" by introducing an internal angular momentum L_{in}. The Coriolis force arising from the coupling of L_{in} with the rotational angular velocity is found to be responsible for the deviation of the moment of inertia from its hydrodynamical value. The explicit form of L_{in} is given as a function of the positions and momenta of individual particles. This form of L_{in} should be used for deriving the effective moment of inertia. Comparison with the cranking model of Inglis is discussed.

§ 1. Introduction

The existence of the rotational states in heavy nuclei was considered as a strong support for the collective model of the nucleus.[1] Empirically the rotational states are identified by a sequence of energy levels represented by

$$E=\epsilon_0+\hbar^2 I(I+1)/2\mathcal{J} \tag{1·1}$$

where ϵ_0 is a constant energy for a particular band concerned and I the angular momenta of such states. This relation gives one the empirical values of the moments of inertia $\mathcal{J}$ for various nuclei. These values are, however, found to deviate systematically from those which are predicted on the basis of the irrotational collective flow,[2] a starting point of the collective model in their original work of Bohr and Mottelson.[1]

A number of attempts have been put forward to elucidate the above situation.[2-9] Among them the "cranking model" of Inglis[3] is often referred to understand the qualitative reason why the moment of inertia is larger than the irrotational value. However, the validity of this model is questioned by Lipkin, de-Shalit and Talmi,[10] since the rotation of the nucleus in Inglis' model is externally forced rotation imposed by "cranking" the external potential, while the collective rotation in the actual nucleus is the free rotation resulting from the mutual interaction of the nucleons. As one sees from their critical work,[10] it is necessary to reconsider this problem on the basis of an orthodox approach to the many body problem.

The main purpose of our paper is to make clear the theoretical foundation of the cranking model of Inglis by using the method of the quantum mechanical description of the collective motion. The outcome of our present work throws light on the method of calculating the moment of inertia. It has not been clear what is responsible for the deviation of the moment of inertia from its irrotational value, as critically investigated by Lipkin et al.[10] In our method the responsible interaction can be introduced rather unambiguously; hence the relation between the moment of inertia and the internal structure becomes clear.

In course of performing our work, it has come to our notice that Villars[11] has published his work, with the same purpose as ours. Using the method of the coordinate transformation, he has succeeded in obtaining a canonical form of the kinetic energy of a many-particle system, in which the dependences on the total and the intrinsic angular momenta are explicitly given; the latter is a function only of those intrinsic variables which are complicated functions of particle coordinates. Although our method is different from that of Villars, the essential idea is very similar. In this paper, we shall compare our results with those of Villars.

Our method is based on that used by Miyazima and Tamura[12] and by the present authors[13] in which some auxiliary variables describing the collective motions are introduced together with appropriate subsidiary conditions, and canonical transformations are employed. This method seems superior to that of the coordinate transformation in the sense that the derivation is simple and the particle coordinates can be explicitly expressed in the transformed state without knowing the functional form of the intrinsic variables.

The application of this method to the collective rotational mode was already made by Tamura.[14] Keeping a close contact with his work, we shall develop our method.

In the present paper, however, we do not discuss quantitative problems but emphasize the physical implication of our method. In order to avoid unnecessary complication, we restrict ourselves to the rotational motion of a two dimensional system of particles that is subject to elliptic deformation.

The extension to the three dimensional case does not involve any fundamental difficulty. Such extension as well as the application to actual nuclei are left over in a subsequent paper.

In § 2 the method of the quantum mechanical description of the rotational motion is discussed along the line of Tamura's.[14] The recapitulation is made for completeness.

In § 3 is performed a reduction of the Hamiltonian obtained in § 2 to that similar to the usual canonical form, and the meaning of the "internal angular momentum" is discussed. The comparison with the cranking model of Inglis is made in § 4. The prescription to obtain the effective moment of inertia is also given.

§ 2. Quantum mechanical description of the rotational motion

Let us consider a two dimensional system of particles, which we may call the nucleus. The positions of the particles are designated by (x_i, y_i).* Following Tamura[14] we introduce a couple of functions of positions,

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* Here we are not concerned with the separation of the center-of-mass motion; hence (x_i, y_i) may be regarded as the coordinates relative to the center of mass.

$$\xi_1=\sum_i (x_i^2-y_i^2)/2,\ \xi_2=\sum_i x_i y_i. \tag{2·1}$$

By use of ξ_1 and ξ_2 an angular coordinate

$$\varphi=(1/2)\tan^{-1}(\xi_2/\xi_1) \tag{2·2}$$

is defined. The total angular momentum of the system,

$$L_\varphi=\sum_i (x_i p_y^{(i)}-y_i p_x^{(i)}), \tag{2·3}$$

where $(p_x^{(i)},\ p_y^{(i)})$ are the momenta of the particles, is proved to satisfy the commutation relation

$$[\varphi,\ L_\varphi]=i\hbar. \tag{2·4}$$

Of course, L_φ commutes with the original Hamiltonian

$$H_0=(1/2m)\sum_i (p_x^{(i)2}+p_y^{(i)2})+V(x_1,\ y_1,\cdots x_A,\ y_A). \tag{2·5}$$

Here we introduce an auxiliary variable θ, and regard the original eigenfunction Φ_0 as a function of θ, imposing on Φ_0 the subsidiary condition

$$\theta\Phi_0=0. \tag{2·6}$$

Otherwise Φ_0 behaves as usual ; for example, it satisfies

$$(H_0-E)\Phi_0=0. \tag{2·7}$$

The first canonical transformation

$$\Phi_1=U_1^{-1}\Phi_0;\ U_1=\exp[iL_\theta\varphi/\hbar], \tag{2·8}$$

where L_θ is the canonical conjugate momentum to θ satisfying

$$[\theta,\ L_\theta]=i\hbar, \tag{2·9}$$

changes the subsidiary condition (2·6) into

$$(\theta-\varphi)\Phi_1=0. \tag{2·10}$$

The second transformation

$$\Psi=U_2^{-1}\Phi_1;\ U_2=\exp(-iL_\varphi\theta/\hbar) \tag{2·11}$$

changes the subsidiary condition further into

$$\varphi\Psi=0. \tag{2·12}$$

Taking account of

$$[\sum_{i,\nu}(\nabla_\nu^{(i)}\varphi)^2,\ L_\varphi]=0,\ [\sum_{i,\nu}(\nabla_\nu^{(i)}\varphi)\cdot p_\nu^{(i)},\ L_\varphi]=0, \tag{2·13}$$

where ν stands for x and y, and also of

$$\sum_i \Delta^{(i)} \varphi = 0,$$

the Hamiltonian in this representation is expressed as

$$\begin{aligned} H &= U_2^{-1} U_1^{-1} H_0 U_1 U_2 \\ &= H_0 + (1/2m) \sum_{i,\nu} (\nabla_\nu^{(i)} \varphi)^2 (L_\theta - L_\varphi)^2 + (1/m) \sum_{i,\nu} (\nabla_\nu^{(i)} \varphi) p_\nu^{(i)} (L_\theta - L_\varphi). \qquad (2\cdot14) \end{aligned}$$

As discussed by Tamura,[14)] the physical meaning of the canonical transformation,

$$\Psi = U_2^{-1} U_1^{-1} \Phi_0,$$

can be interpreted as follows. The introduction of a pair of the auxiliary variables θ and L_θ means that of an imaginary rotor. The canonical transformation U_1 superposes the imaginary rotor on the elliptic nucleus with a constant density particle distribution, as one sees from (2·10).

The second transformation U_2 means that in the representation Ψ the rotational motion of the elliptic nucleus is described by that of the imaginary rotor, and the particle motions are confined in a body fixed frame. This body fixed frame is characterized by (2·12) which may be of the same meaning as

$$\sum_i x_i y_i \Psi = 0, \qquad (2\cdot12')$$

as one easily sees from the definition of φ in (2·2). The condition (2·12′) indicates that the body fixed system has its axis on the principal axis of the nucleus. The above picture is also supported by the fact that L_θ is nothing but the total angular momentum in this representation, as readily seen from

$$L_\theta = U_2^{-1} U_1^{-1} L_\varphi U_1 U_2. \qquad (2\cdot15)$$

It should be noted that the Hamiltonian (2·14) contains neither φ nor L_φ; that is, it satisfies

$$[H, \varphi] = [H, L_\varphi] = 0, \qquad (2\cdot16)$$

as easily verified by taking account both of the subsidiary condition (2·12) and of the relation (2·13). This implies that we have displayed the dependence of the Hamiltonian (2·5) on the total angular momentum without making explicit use of the intrinsic coordinates which should be complicated functions of particle coordinates. On the other hand, using the method of the coordinate transformation, Villars[11)] has derived the nuclear Hamiltonian written in a canonical form which displays the dependence on the total angular momentum. According to the above discussion, therefore, it is clear that our Hamiltonian (2·14) is equivalent to that of Villars. Detailed discussion on this point will be given in the next section.

§ 3. Reduction of Hamiltonian and "internal angular momentum"

Now we try to rewrite the Hamiltonian (2·14) to that similar to the usual canonical form. Following Tamura,[14)] we introduce quantities β and τ through

$$\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)^2=1/\beta^2\tau, \qquad (3\cdot1)$$

where

$$\beta=2\sqrt{\xi_1^2+\xi_2^2}/\tau,\quad \tau=\sum_i(x_i^2+y_i^2).$$

$1/\beta^2\tau$ is the coefficient of $(L_\theta-L_\varphi)^2$ in $(2\cdot14)$ and satisfies the commutation relations

$$[L_\varphi,\ \beta]=[L_\varphi,\ \tau]=0. \qquad (3\cdot2)$$

Since both β and τ can be proved to commute with the coefficient of $(L_\theta-L_\varphi)$ in $(2\cdot14)$, $\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)\cdot p_\nu^{(i)}$, with the aid of

$$\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)(\Gamma_\nu^{(i)}\beta)=\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)(\Gamma_\nu^{(i)}\tau)=0, \qquad (3\cdot3)$$

the following relation is found to hold:

$$[\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)^2,\ \sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)\cdot p_\nu^{(i)}]=0. \qquad (3\cdot4)$$

Thus we are able to rewrite the Hamiltonian $(2\cdot14)$ to that similar to the usual canonical form:

$$\begin{cases} H=H_{\text{in}}+H_{\text{rot}}, \\ H_{\text{rot}}=(1/2m\beta^2\tau)(L_\theta-L_{\text{in}})^2 \\ H_{\text{in}}=(1/2m)\sum_i(p_x^{(i)2}+p_y^{(i)2})+V(x_1y_1\cdots x_Ay_A)-(\beta^2\tau/2m)\{\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi\cdot p_\nu^{(i)})\}^2, \end{cases} \qquad (3\cdot5)$$

where

$$L_{\text{in}}=L_\varphi-\beta^2\tau(\sum_{i,\nu}(\Gamma_\nu^{(i)}\varphi)\cdot p_\nu^{(i)}). \qquad (3\cdot6)$$

H_{in} is the Hamiltonian for the internal motion, while H_{rot} expresses the rotational part written as the "canonical form". L_θ is the total angular momentum and L_{in} is an "internal angular momentum". This form completes the comparision with the formalism of Villars; Hamiltonian (13) with (22) in his paper[11] is identical with our $(3\cdot5)$.

L_{in} must be a function only of the intrinsic coordinates and their canonical conjugate momenta. This fact is guaranteed by

$$[L_{\text{in}},\ \varphi]=[L_{\text{in}},\ L_\varphi]=0.$$

L_{in} in $(3\cdot6)$ is written explicitly as

$$L_{\text{in}}=\sum_i x_i\{(1-2\xi_1/\tau)p_y^{(i)}+(2\xi_2/\tau)p_x^{(i)}\}$$

$$-\sum_i y_i\{(1+2\xi_1/\tau)p_x^{(i)}+(2\xi_2/\tau)p_y^{(i)}\}.$$

Remarking the subsidiary condition $(2\cdot12')$ and remembering the definition of ξ_1 and τ, we have

$$\begin{aligned}[x_i,\ L_{\text{in}}]\Psi &= -i\hbar\frac{2\sum_k (x_k^2)}{\sum_k (x_k^2+y_k^2)}y_i\Psi,\\ [y_i,\ L_{\text{in}}]\Psi &= i\hbar\frac{2\sum_k (y_k^2)}{\sum_k (x_k^2+y_k^2)}x_i\Psi,\end{aligned} \tag{3·7}$$

which are identical with eq. (31) of Villars.[11)]

Here the following should be noted. If the approximation based on the large number theorem is applied, as was employed by Tomonaga[15)] in his theory of the collective motion,* L_{in} has the usual property of the angular momentum with respect to the particle coordinates.

§ 4. The cranking model of Inglis and the effective moment of inertia

It is apparent from H_{rot} in (3·5) that the zeroth order moment of inertia arises from $m\beta^2\tau$. Explicitly this is defined as

$$\mathcal{J}_0=\langle m\beta^2\tau\rangle. \tag{4·1}$$

Here the expectation value is taken in the eigenstate Ψ. Remembering the subsidiary condition (2·12′) and the definition of β and τ in (3·1), we have

$$\mathcal{J}_0=m\langle\{\sum_i (x_i^2-y_i^2)\}^2/\sum_i (x_i^2+y_i^2)\rangle. \tag{4·1′}$$

This is nothing but the hydrodynamical value of the moment of inertia for the irrotational flow.

By introducing $\mathcal{J}_0$, the Hamiltonian for the rotational motion, H_{rot}, in (3·5) is decomposed as

$$\left.\begin{aligned}H_{\text{rot}}&=H_{\text{rot}}^{(0)}+H'_{\text{rot}}\,;\\ H_{\text{rot}}^{(0)}&=(1/2\mathcal{J}_0)\,(L_\theta-L_{\text{in}})^2,\\ H'_{\text{rot}}&=(1/2m\beta^2\tau-1/\mathcal{J}_0)\,(L_\theta-L_{\text{in}})^2.\end{aligned}\right\} \tag{4·2}$$

Hereafter we neglect H'_{rot}, although its effect needs examination. Then the angular velocity of the rotational motion is given by

$$\omega=(i/\hbar)[H,\ \theta]=(L_\theta-L_{\text{in}})/\mathcal{J}_0.$$

In order to show the relation of the Hamiltonian (3·5) with the cranking model of Inglis,[3)] we define the corresponding classical angular velocity as

---

* Tomonaga employed the approximation in which the quantity $\sum_i (x_i^2+y_i^2)$ is replaced by its mean value $\sum_i (\langle x_i^2\rangle+\langle y_i^2\rangle)$. This approximation is good if the number of particles, A, is so large that $1/\sqrt{A}$ can be neglected against unity. See reference 15, Appendix I.

$$\omega=(1/\mathscr{J}_0)\,(\hbar I-\langle L_{\rm in}\rangle), \tag{4·3}$$

where

$$L_\theta \Psi=\hbar I\Psi \tag{4·4}$$

is taken into account. $\langle L_{\rm in}\rangle$ will be determined in (4·15) and (4·16).

With this definition of (4·3), we can decompose the Hamiltonian (3·5) in the form:

$$\left.\begin{aligned} &H=H_{\rm part}+T_{\rm rot}\,;\\ &T_{\rm rot}=(1/2\mathscr{J}_0)\,\{(\hbar I-\langle L_{\rm in}\rangle)^2+2(\hbar I-\langle L_{\rm in}\rangle)\langle L_{\rm in}\rangle+(L_{\rm in}-\langle L_{\rm in}\rangle)^2\},\\ &H_{\rm part}=H_{\rm in}-\omega L_{\rm in}, \end{aligned}\right\} \tag{4·5}$$

where $H_{\rm in}$ is given in (3·5). In this expression we have replaced L_θ by $\hbar I$ on account of (4·4).

It is to be noted that the expectation value of $L_{\rm in}$, used in (4·3) and (4·5), should be taken by employing the eigenfunction of $H_{\rm part}$ which is solved self-consistently.

The Schrödinger equation,

$$H_{\rm part}\,\psi\equiv(H_{\rm in}-\omega L_{\rm in})\,\psi=\epsilon\psi, \tag{4·6}$$

with the subsidiary condition $\varphi\psi=0$, represents the particle motion in the body fixed frame, and the term $-\omega L_{\rm in}$ expresses the interaction arising from the Coriolis force.

Now it is clear that eq. (4·6) is just the basic equation in the cranking model of Inglis. The only fundamental difference lies in that the rotational motion in Inglis' model is externally forced rotation, while in our case this rotation is a free rotation resulting from the particle system itself. Further, in Inglis' model, $H_{\rm in}$ in (4·6) is approximated by the Hamiltonian for the particle motion in an averaged, deformed potential, and also $L_{\rm in}$ is replaced by the apparent total angular momentum of particles, L_φ, in the body fixed system.

If ω is small compared with the internal frequencies arising from $H_{\rm in}$, the Coriolis term, $-\omega L_{\rm in}$, may be treated as a perturbation. In this case the Schrödinger equation can be solved by using the states of $H_{\rm in}$, unperturbed by rotation and the corresponding unperturbed energies; namely

$$H_{\rm in}|m\rangle=\epsilon_m|m\rangle.$$

Then the eigenfunction of eq. (4·6) is represented as

$$\psi=|0\rangle-\omega\sum_m(\epsilon_0-\epsilon_m)^{-1}|m\rangle\langle m|L_{\rm in}|0\rangle, \tag{4·7}$$

and the eigenvalue of $H_{\rm part}$ is given by

$$\epsilon=\epsilon_0-\omega\langle 0|L_{\rm in}|0\rangle-\mathscr{J}'\omega^2/2, \tag{4·8}$$

where

$$\mathscr{J}'=2\sum_m(\epsilon_m-\epsilon_0)^{-1}|\langle 0|L_{\rm in}|m\rangle|^2. \tag{4·9}$$

$\mathcal{J}'$ has a similar form to the moment of inertia obtained by the cranking model of Inglis,[3)] except that L_φ is replaced by L_{in}. If only the particles outside a core contribute to the matrix elements $\langle 0|L_{\text{in}}|m\rangle$, our result is reduced to the one obtained by Villars.[11)] The expectation value of T_{rot} in ψ is

$$\langle T_{\text{rot}}\rangle=(1/2\mathcal{J}_0)\{(\hbar I-\langle L_{\text{in}}\rangle)^2+2(\hbar I-\langle L_{\text{in}}\rangle)\langle L_{\text{in}}\rangle\}$$
$$+(1/2\mathcal{J}_0)\langle(L_{\text{in}}-\langle L_{\text{in}}\rangle)^2\rangle. \tag{4·10}$$

Therefore, if we can neglect the last term of (4·10), the eigenvalue of the total Hamiltonian (4·5) is given by

$$E=\epsilon_0-\omega\langle 0|L_{\text{in}}|0\rangle-\tfrac{1}{2}\mathcal{J}'\omega^2+\tfrac{1}{2}\mathcal{J}_0\omega^2+\omega\langle L_{\text{in}}\rangle. \tag{4·11}$$

Here we note that, by using (4·7), the expectation value of L_{in} is represented as

$$\langle L_{\text{in}}\rangle=\langle 0|L_{\text{in}}|0\rangle+2\omega\sum_m(\epsilon_m-\epsilon_0)^{-1}|\langle 0|L_{\text{in}}|m\rangle|^2$$
$$=\langle 0|L_{\text{in}}|0\rangle+\mathcal{J}'\omega. \tag{4·12}$$

Substituting (4·12) in (4·11), we have

$$E=\epsilon_0+\tfrac{1}{2}(\mathcal{J}_0+\mathcal{J}')\omega^2. \tag{4·13}$$

From this form we can presume the effective moment of inertia is given by

$$\mathcal{J}_{\text{eff}}=(\mathcal{J}_0+\mathcal{J}'). \tag{4·14}$$

A slight difference of the expression for $\mathcal{J}_{\text{eff}}$ obtained by Lüders[9)] is due merely to the difference in the choice of $\langle L_{\text{in}}\rangle$.

With the definition of ω in (4·3) and eq. (4·12) the self-consistent solution of $\langle L_{\text{in}}\rangle$ is obtained from the equation,

$$\langle L_{\text{in}}\rangle=\langle 0|L_{\text{in}}|0\rangle+(\mathcal{J}'/\mathcal{J}_0)(\hbar I-\langle L_{\text{in}}\rangle). \tag{4·15}$$

If we adopt the relation*

$$\langle 0|L_{\text{in}}|0\rangle=0,$$

as in the cranking model of Inglis, the solution of (4·15) is given by

$$\langle L_{\text{in}}\rangle=\hbar I\mathcal{J}'/(\mathcal{J}_0+\mathcal{J}'), \tag{4·16}$$

and ω is expressed as

$$\omega=\hbar I/(\mathcal{J}_0+\mathcal{J}'). \tag{4·17}$$

Substituting (4·17) in (4·13), we have

$$E=\epsilon_0+\hbar^2I^2/2(\mathcal{J}_0+\mathcal{J}'). \tag{4·18}$$

By comparing (4·18) with the expression (1·1), the effective moment of inertia given by (4·14) is just the one deduced from the experimental rotational spectrum.

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* On this point, see references 3, 8 and 9.

The effective moment of inertia given by (4·14) with the definition of $\mathcal{J}'$ in (4·9) allows us to anticipate how the moments of inertia deduced from the rotational spectra deviate from the hydrodynamical values. The deviation is greater, according as the excited single particle levels are more closely spaced and the matrix elements of L_{in} are larger. Therefore, considerable deviation is expected for the nuclei far from the closed shell and those of large deformation. A quantitative study of this problem will be made in a subsequent paper.

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On the Ground Level Splitting of Mn^{++} and Fe^{+++} in Nearly Cubic Crystalline Field

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The ground level splitting and g-factor of Mn^{++} and Fe^{+++} in nearly cubic crystalline field, with a small axially symmetric field superposed on it, are studied by higher-order perturbation calculations. It is shown that the electronic energy levels arising from the configuration half-filled with electrons are unsplit under the linear effect of crystalline potentials, but can be split by the effects of even powers of them. Various higher-order processes involving crystalline potential, spin-orbit coupling and magnetic spin-spin interaction within an ion that are supposed to be dominant in the ground level splitting and the g-factor deviation, are formulated and order of magnitude discussions are given for them.

§ 1. Introduction

During the past few years the paramagnetic resonance method has been applied to study the splitting of the ground energy level of Mn^{++} and of Fe^{+++} in paramagnetic crystals, such as manganese Tutton salts and iron alums,[1] in MgO[2] and in zincblende crystals.[3] These energy levels can be described in a simple way by a 'spin Hamiltonian' which is expressed in terms of the spin operators. The coefficients of a spin Hamiltonian depend in a complicated way upon the symmetry and strength of the crystalline field, the spin-orbit coupling (W_{LS}), the magnetic spin-spin interaction between pairs of electrons within the ion (W_{SS}) and so on.[1] The order of magnitude of their splittings is smaller than that of the other iron-group ions, since both Mn^{++} and Fe^{+++} are in $3d^5\ {}^6S$ state which has no orbital angular momentum.

On the other hand, these ground level splittings correspond to the part of the magnetic anisotropy energy of these ions which is produced by a combined effect of the crystalline field, W_{LS} and W_{SS}[4]. And this anisotropy energy is expected to be dominant in paramagnetic crystals and in anti- and ferrimagnetics in which the spin arrangement is of cubic symmetry, since in this case the ordinary magnetic dipole-dipole interaction between ions contributes nothing to the anisotropy energy. Recently, Yosida and Tachiki[5] suggested that the origin of the cubic anisotropy of Mn- and Ni-ferrites might be attributed to the magnetic anisotropy of Fe^{+++} ions situated on the $16c$ and $8f$ sites of these ferrites, and somewhat to a less extent to Mn^{++} ions on the $8f$ site in the case of Mn-ferrite. The reason is that the observed anisotropy energies per ion are comparable with those of Fe^{+++} in ferric alums and that their temperature dependence appears to be similar to those of one-ion cubic anisotropy. In Ni-ferrite, the anisotropic exchange interaction is small and the combined effect of the spin-orbit coupling and crystalline potential for Ni^{++} is isotropic.

It seems, therefore, desirable to clarify the electronic processes which contribute mostly to the anisotropy energies of Mn^{++} and Fe^{+++} in a cubic or nearly cubic crystalline field.

There appeared two papers which devoted themselves to theoretical discussion of these ground level splittings. One is due to Van Vleck and Penney,6) who proposed several mechanisms and estimated the corresponding order of magnitude to explain the anisotropic magnetic susceptibility of paramagnetic crystals containing manganese ions, such as $Mn(NH_4)_2(SO_4)_2 6H_2O$. And the other is due to Pryce,7) who showed that Van Vleck and Penney's mechanisms are inadequate to give the observed magnitude of the magnetic anisotropy, or of the splitting, and proposed other mechanisms. He attributed the quadratic part (DS_z^2) of the spin Hamiltonian to a process which involves the first powers of W_{SS} and the axially symmetric crystalline potential via an intermediate state $3d^4 4s\ {}^6D$, and calculated the coefficient D. This process may be dominant but comparable with another process which will be proposed in this paper. The process he proposed for the quartic part of the spin Hamiltonian $(a/6)(S_x^4+S_y^4+S_z^4)$ is a fifth-order one, quartic in W_{LS} and *linear* in cubic crystalline potential, through intermediate states formed from $3d^5$, but this process gives nothing since all the matrix elements of any crystalline potential within the same multiplet arising from $3d^5$ identically vanish, as will be shown in section 2.

The purpose of this paper is to discuss the mechanisms of the ground level splitting, or of the magnetic anisotropy, of Mn^{++} and Fe^{+++} ions, placed in a cubic crystalline field superimposed by a less amount of an axial field. Section 2 is devoted to the discussion of the effect of cubic crystalline potential on the excited quartets formed from $3d^5$. In section 3 the ground level splitting and the g-value deviation are studied by various higher-order processes through excited multiplets and configurations, involving crystalline potential, W_{LS} and W_{SS}. These are then followed in section 4 by discussion of the order of magnitude of the coefficients of the spin Hamiltonian and of the g-value deviation.

§ 2. The energy level splitting of excited quartet multiplets in cubic crystalline electric field

The ground multiplet of Mn^{++} and Fe^{+++} is $3d^5\ {}^6S$. Therefore, any crystalline field does not directly affect this state. To explain the ground level splitting, it is necessary to consider an admixing to ${}^6S$ of higher multiplets and configurations which are split under the action of the crystalline field, as was pointed out by Van Vleck and Penney, and Pryce. Any multiplet formed from $3d^5$ or, more generally, from a configuration with half a number of electrons of a closed shell, is unsplit and unshifted by any crystalline field, provided that the matrix elements of the crystalline potential among different multiplets are neglected and that the potential satisfies Laplace's equation, since we know that the matrix elements of the crystalline potential within the same multiplet identically vanish. This was already pointed out by Kynch8) but can easily be understood from the following consideration. Two ions are said to be complementary when their incomplete shells add up to a closed shell when they are united to a single atom. For example, Ti^{+++} and Cu^{++} are complementary, the former having one electron and the latter one hole in the $3d$ shell. As is well known, such two complementary ions show splitting patterns inverted with respect to each other when placed in the same crystalline field. On the other hand, an ion with $3d^5$ is complementary to itself, and if its multiplet were split into sublevels of different symmetry properties by a crystalline potential, two inverted splitting patterns would have to be obtained simultaneously. This is obviously contradictory to the fact that one multiplet gives only one pattern. Thus, any multiplet formed from $3d^5$ is unsplit under the linear action of the crystalline potential, though it can be split by the even-order effects through non-diagonal matrix elements of the potential among different multiplets.

In this section, the splitting patterns of the quartets arising from $3d^5$ (^4G, ^4F, ^4D, ^4P and ^4S) are treated when the ion is subject to a cubic crystalline field. The energy diagram will be given only for Mn^{++}, since in this ion the energy levels of quartets except ^4S are spectroscopically known.[9] Those for Fe^{+++} are less accurately known.[9] The energy levels for the doublets in Mn^{++} are unknown and will not be taken into consideration, but the matrix elements of the crystalline potential between quartets and doublets vanish. Moreover, we shall treat in this paper the case in which the crystalline field is so weak as to conserve the spin quantum number of $5/2$ for the ground state of the ion, not as in the case of iron cyanides in which Fe^{+++} ions have spin $1/2$.

The cubic electrostatic potential around a Mn^{++} ion can be developed in spherical harmonics as

$$V(\boldsymbol{r})=c_{40}r^4[Y_4^0(\theta\varphi)+(5/14)^{1/2}(Y_4^4(\theta\varphi)+Y_4^{-4}(\theta\varphi))]. \tag{1}$$

The energy matrix elements of this potential with respect to the quartets under consideration are:

$$\begin{aligned}
&\langle{}^4G\Gamma_1|V|{}^4S\Gamma_1\rangle=0,\\
&\langle{}^4G\Gamma_{3i}|V|{}^4D\Gamma_{3j}\rangle=0, \quad i,\ j=1,\ 2\\
&\langle{}^4G\Gamma_{41}|V|{}^4P\Gamma_{41}\rangle=-\langle{}^4G\Gamma_{42}|V|{}^4P\Gamma_{42}\rangle=\langle{}^4G\Gamma_{43}|V|{}^4P\Gamma_{43}\rangle=-4(5)^{1/2}Dq,\\
&\langle{}^4G\Gamma_{41}|V|{}^4F\Gamma_{41}\rangle=\langle{}^4G\Gamma_{42}|V|{}^4F\Gamma_{42}\rangle=-\langle{}^4G\Gamma_{43}|V|{}^4F\Gamma_{43}\rangle=-2(5)^{1/2}Dq,\\
&\langle{}^4G\Gamma_{5i}|V|{}^4D\Gamma_{5j}\rangle=0, \quad i,\ j=1,\ 2\ 3,\\
&\langle{}^4G\Gamma_{51}|V|{}^4F\Gamma_{51}\rangle=\langle{}^4G\Gamma_{52}|V|{}^4F\Gamma_{52}\rangle=-\langle{}^4G\Gamma_{53}|V|{}^4F\Gamma_{53}\rangle=-10(3/7)^{1/2}Dq,\\
&\langle{}^4G\Gamma_{4i}|V|{}^4P\Gamma_{4j}\rangle=0, \quad i,\ j=1,\ 2,\ 3\\
&\langle{}^4F\Gamma_{51}|V|{}^4D\Gamma_{51}\rangle=-\langle{}^4F\Gamma_{52}|V|{}^4D\Gamma_{52}\rangle=\langle{}^4F\Gamma_{53}|V|{}^4D\Gamma_{53}\rangle=-20/(7)^{1/2}Dq,
\end{aligned} \tag{2}$$

where $Dq=c_{40}\langle r^4\rangle/14(\pi)^{1/2}$, $\langle r^4\rangle$ being the average of r^4 with respect to the radial part of the $3d$-orbital. The sign of these matrix elements is necessary for the calculation of the splitting parameter in the next section. $|\Gamma_{ij}\rangle$ are wave functions for the irreducible representations of the octahedral group. The first index i is that of Bethe's notation[10] and the second j is introduced to indicate the degenerate wave functions of the same representation. The wave functions used to calculate these matrix elements (2) are eigenfunctions in the cubic field, i. e.,

$$
\begin{aligned}
&|{}^4G\Gamma_1\rangle=(7/12)^{1/2}\varphi_0+(5/24)^{1/2}[\varphi_4+\varphi_{-4}],\\
&|{}^4G\Gamma_{31}\rangle=(5/12)^{1/2}\varphi_0-(7/24)^{1/2}[\varphi_4+\varphi_{-4}],\\
&|{}^4G\Gamma_{32}\rangle=(1/2)^{1/2}[\varphi_2+\varphi_{-2}],\\
&|{}^4G\Gamma_{41}\rangle=(1/2)^{1/2}[\varphi_4+\varphi_{-4}],\\
&|{}^4G\Gamma_{42}\rangle=(1/8)^{1/2}[\varphi_{-3}+(7)^{1/2}\varphi_1],\\
&|{}^4G\Gamma_{43}\rangle=(1/8)^{1/2}[\varphi_3+(7)^{1/2}\varphi_{-1}],\\
&|{}^4G\Gamma_{51}\rangle=(1/2)^{1/2}[\varphi_2-\varphi_{-2}],\\
&|{}^4G\Gamma_{52}\rangle=(1/8)^{1/2}[(7)^{1/2}\varphi_{-3}-\varphi_1],\\
&|{}^4G\Gamma_{53}\rangle=(1/8)^{1/2}[(7)^{1/2}\varphi_3-\varphi_{-1}],\\
&|{}^4F\Gamma_2\rangle=(1/2)^{1/2}[\varphi_2-\varphi_{-2}],\\
&|{}^4F\Gamma_{41}\rangle=\varphi_0,\\
&|{}^4F\Gamma_{42}\rangle=(1/8)^{1/2}[(5)^{1/2}\varphi_{-3}+(3)^{1/2}\varphi_1],\\
&|{}^4F\Gamma_{43}\rangle=(1/8)^{1/2}[(5)^{1/2}\varphi_3+(3)^{1/2}\varphi_{-1}],\\
&|{}^4F\Gamma_{51}\rangle=(1/2)^{1/2}[\varphi_2+\varphi_{-2}],\\
&|{}^4F\Gamma_{52}\rangle=(1/8)^{1/2}[(5)^{1/2}\varphi_1-(3)^{1/2}\varphi_{-3}],\\
&|{}^4F\Gamma_{53}\rangle=(1/8)^{1/2}[(5)^{1/2}\varphi_{-1}-(3)^{1/2}\varphi_3],\\
&|{}^4D\Gamma_{31}\rangle=\varphi_0,\\
&|{}^4D\Gamma_{32}\rangle=(1/2)^{1/2}[\varphi_2+\varphi_{-2}],\\
&|{}^4D\Gamma_{51}\rangle=(1/2)^{1/2}[\varphi_2-\varphi_{-2}],\\
&|{}^4D\Gamma_{52}\rangle=\varphi_1,\\
&|{}^4D\Gamma_{53}\rangle=\varphi_{-1},\\
&|{}^4P\Gamma_{41}\rangle=\varphi_0,\\
&|{}^4P\Gamma_{42}\rangle=\varphi_1,\\
&|{}^4P\Gamma_{43}\rangle=\varphi_{-1},\\
&|{}^4S\Gamma_1\rangle=\varphi_0.
\end{aligned}
\tag{3}
$$

In these wave functions (3), φ's for ^4G multiplet, for example, are normalized spherical harmonics of degree four, and so on. Since the electrostatic potential V is given by a function of the coordinates of a single electron, straightforward calculation of the matrix elements (2) are worked out with the use of the spherical harmonics of (3) rewritten

	$^4F\Gamma_4$	$^4P\Gamma_4$	$^4G\Gamma_4$
$^4F\Gamma_4$	δ_1	0	$2(5)^{1/2}$Dq
$^4P\Gamma_4$	0	δ_2	$4(5)^{1/2}$Dq
$^4G\Gamma_4$	$2(5)^{1/2}$Dq	$4(5)^{1/2}$Dq	0

(4)

	$^4F\Gamma_5$	$^4D\Gamma_5$	$^4G\Gamma_5$
$^4F\Gamma_5$	δ_1	$20/(7)^{1/2}$Dq	$10(3/7)^{1/2}$Dq
$^4D\Gamma_5$	$20/(7)^{1/2}$Dq	δ_3	0
$^4G\Gamma_5$	$10(3/7)^{1/2}$Dq	0	0

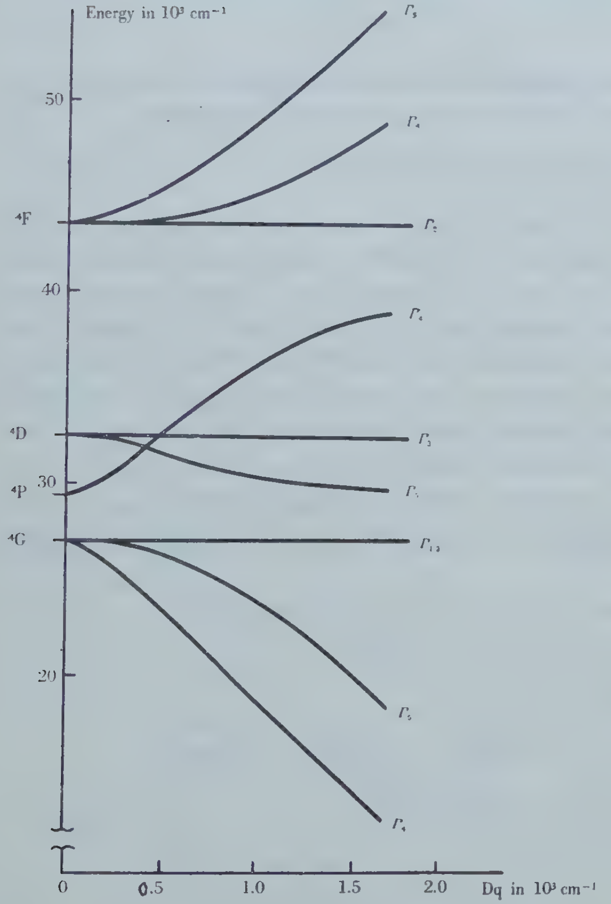

Fig. 1. Energy diagram for quartet multiplets from 5d^5 of Mn^{++}

by the antisymmetrized product of one-electron functions. For the brevity, these detailed functions are omitted.

The energy levels for these quartets are calculated by solving the secular matrices (4), where plus and minus signs to be attached to the off-diagonal matrix elements are omitted since only squared quantities are needed, and δ_1, δ_2 and δ_3 are multiplet separations between ^4F and ^4G, ^4P and ^4G, and ^4D and ^4G, respectively. As can easily be seen, energy levels for $^4G\Gamma_1$, $^4G\Gamma_3$, $^4F\Gamma_2$, $^4D\Gamma_3$ and $^4S\Gamma_1$ are unaffected, since they have no matrix element within themselves and between other multiplets and themselves. The level of $^4P\Gamma_4$ is unsplit but shifted. The three multiplets ^4G, ^4F and ^4D are split by the cubic crystalline potential into three, three and two levels, respectively. The results are shown in Fig. 1, where the ^4S-level is not plotted, being unknown from spectroscopic data.[9]

A similar energy diagram was reported by Tanabe and Sugano,[11] with the use of another method of calculation, but they did not show explicitly the behaviors of ^4F, ^4D and ^4P levels. The absence of the linear effect of the crystalline field is seen from the vanishing slope of these curves at the vanishing field strength.*

§ 3. Combined effects of V, W_{LS} and W_{SS}; mechanisms for the g-value deviation

Bethe's theoretical study shows that the state $^6S_{5/2}$ is split by a cubic field into a twofold (Γ_7) and a fourfold level (Γ_8). Important mechanisms for this splitting are to be attributed to an admixture of higher multiplets arising from $3d^5$ and from other configurations, due to combined actions of V, W_{LS} and W_{SS}. Let us introduce the perturbing Hamiltonian:

$$\boldsymbol{H}=V_{\text{cryst.}}+\sum_i \xi(i)\,\boldsymbol{l}_i\boldsymbol{s}_i$$
$$+\sum_{i>j}(g^2\beta^2/r_{ij}^5)\left[(\boldsymbol{s}_i\boldsymbol{s}_j)r_{ij}^2-3(\boldsymbol{s}_i\boldsymbol{r}_{ij})(\boldsymbol{s}_j\boldsymbol{r}_{ij})\right].$$

As is well known, the ground level splitting can be written in the form of a spin Hamiltonian when $V_{\text{cryst.}}$ is a cubic potential superimposed by an axial potential:

$$\boldsymbol{H}_S=(a/6)\,(S_x^4+S_y^4+S_z^4)+DS_1^2, \tag{5}$$

where the first term corresponds to the cubic deformation of the charge cloud of the ground state and the second one to the axial deformation whose axis we suppose to lie generally not along one of the principal axes of the cubic field. This is indicated by a suffix 1. W_{LS} and W_{SS} have matrix elements between different multiplets formed from $3d^5$ configuration, and these are already calculated by Racah[12] and Trees,[13] respectively. There are, on the other hand, matrix elements of W_{LS} and W_{SS} between the multiplets from $3d^5$ and those of other configurations whose contributions to the ground level splitting may be expected to be less than those within $3d^5$ because of their higher excitation energies. Many processes are responsible for the ground level splitting, but the followings may give dominant contributions to the quartic part of the spin Hamiltonian (5).

(I) Four sixth-order processes quartic in W_{LS} and quadratic in V_{cub}:

(1) ($^6S-{}^4P-{}^4D-{}^4F-{}^4G-{}^4P-{}^6S$), being doubled
$W_{LS}W_{LS}W_{LS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

since there is another process reverted with respect to the order of sequences,

(2) ($^6S-{}^4P-{}^4D-{}^4F-{}^4G-{}^4P-{}^6S$), being doubled
$W_{LS}W_{LS}V_{\text{cub}}W_{LS}V_{\text{cub}}W_{LS}$

similarly as above,

---

* This was pointed out by Tanabe and Sugano in a privated communication to the author.

(3) $({}^6S-{}^4P-{}^4G-{}^4P-{}^4G-{}^4P-{}^6S)$,

$W_{LS}V_{\text{cub}}W_{LS}W_{LS}V_{\text{cub}}W_{LS}$

(4) $({}^6S-{}^4P-{}^4D-{}^4F-{}^4D-{}^4P-{}^6S)$.

$W_{LS}W_{LS}V_{\text{cub}}V_{\text{cub}}W_{LS}W_{LS}$

There are two other sixth-order processes :

$({}^6S-{}^4P-{}^4D-{}^4P-{}^4G-{}^4P-{}^6S)$ and

$W_{LS}W_{LS}W_{LS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

$({}^6S-{}^4P-{}^6S-{}^4P-{}^4G-{}^4P-{}^6S)$.

$W_{LS}W_{LS}W_{LS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

These two processes contribute nothing in splitting but only in shifting the ground level.

(II) Five fifth-order processes involving the second powers of W_{LS} and V_{cub} and first power of W_{SS} :

(1) $({}^6S-{}^4D-{}^4F-{}^4G-{}^4P-{}^6S)$, being doubled,

$W_{SS}W_{LS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

(2) $({}^6S-{}^4P-{}^4F-{}^4G-{}^4P-{}^6S)$, being doubled,

$W_{LS}W_{SS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

(3) $({}^6S-{}^4D-{}^4F-{}^4G-{}^4P-{}^6S)$, being doubled,

$W_{SS}V_{\text{cub}}W_{LS}V_{\text{cub}}W_{LS}$

(4) $({}^6S-{}^4P-{}^4G-{}^4G-{}^4P-{}^6S)$,

$W_{LS}V_{\text{cub}}W_{SS}V_{\text{cub}}W_{LS}$

(5) $({}^6S-{}^4D-{}^4F-{}^4D-{}^4P-{}^6S)$, being doubled.

$W_{SS}V_{\text{cub}}V_{\text{cub}}W_{LS}W_{LS}$

Two other processes, similar to those above, contribute nothing to the splitting of the ground level, i. e.,

$({}^6S-{}^4D-{}^4P-{}^4G-{}^4P-{}^6S)$ and

$W_{SS}W_{LS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

$({}^6S-{}^4P-{}^4P-{}^4G-{}^4P-{}^6S)$.

$W_{LS}W_{SS}V_{\text{cub}}V_{\text{cub}}W_{LS}$

(III) A fourth-order process involving the second powers of W_{SS} and V_{cub} :

$({}^6S-{}^4D-{}^4F-{}^4D-{}^6S)$.

$W_{SS}V_{\text{cub}}V_{\text{cub}}W_{SS}$

(IV) Two sixth-order processes quadratic in W_{SS} and quartic in V_{cub} :

$$
\begin{array}{ll}
(1) & ({}^6S-{}^4D-{}^4F-{}^4D-{}^4F-{}^4D-{}^6S) \text{ and} \\
 & W_{SS}V_{\mathrm{cub}}V_{\mathrm{cub}}V_{\mathrm{cub}}V_{\mathrm{cub}}W_{SS} \\
(2) & ({}^6S-{}^4D-{}^4F-{}^4G-{}^4F-{}^4D-{}^6S). \\
 & W_{SS}V_{\mathrm{cub}}V_{\mathrm{cub}}V_{\mathrm{cub}}V_{\mathrm{cub}}W_{SS}
\end{array}
$$

Each of these processes contributes to the splitting parameter a, which is equal to one third of the energy difference between Γ_8 and Γ_7. The contribution from (I) (1), for example, is formulated as follows:

$$
\begin{aligned}
a_{(\mathrm{I},1)} = (2/3) & \sum_{J_z=-5/2}^{5/2} \sum_{i=1}^{3} \sum_{J'_z=-5/2}^{5/2} \\
& [\langle \Gamma_{8m} | W_{LS} | {}^4P\, 5/2\, J_z \rangle \langle {}^4P\, 5/2\, J_z | W_{LS} | {}^4D\, 5/2\, J_z \rangle \\
& \langle {}^4D\, 5/2\, J_z | W_{LS} | {}^4F\, 5/2\, J_z \rangle \langle {}^4F\, 5/2\, J_z | V_{\mathrm{cub}} | {}^4G\, \Gamma_{4i}\, S_z \rangle \\
& \langle {}^4G\, \Gamma_{4i}\, S_z | V_{\mathrm{cub}} | {}^4P\, 5/2\, J_z' \rangle \langle {}^4P\, 5/2\, J_z' | W_{LS} | \Gamma_{8m} \rangle \\
& \qquad - (\text{the same for } \Gamma_{7n})] / P^2DFG,
\end{aligned}
\tag{6}
$$

where P, D, F and G in the denominator are energy separations between the excited quartets and ${}^6S$ state in the case of free ion. The wave functions $|{}^{2S+1}L\Gamma S_z\rangle$ are those of Eqs. (3), multiplied with spin functions. The wave functions $|\Gamma_{7n}\rangle$ and $|\Gamma_{8m}\rangle$ are eigenfunctions of ${}^6S_{5/2}$ in the cubic field:

$$
\begin{aligned}
|\Gamma_{71}\rangle &= (1/6)^{1/2}[\varphi_{5/2} - (5)^{1/2}\varphi_{-3/2}], \\
|\Gamma_{72}\rangle &= (1/6)^{1/2}[\varphi_{-5/2} - (5)^{1/2}\varphi_{3/2}], \\
|\Gamma_{81}\rangle &= \varphi_{1/2}, \\
|\Gamma_{82}\rangle &= \varphi_{-1/2}, \\
|\Gamma_{83}\rangle &= (1/6)^{1/2}[(5)^{1/2}\varphi_{5/2} + \varphi_{-3/2}], \\
|\Gamma_{84}\rangle &= (1/6)^{1/2}[(5)^{1/2}\varphi_{-5/2} + \varphi_{3/2}].
\end{aligned}
\tag{7}
$$

The formulation of the other processes is worked out in the same way and, for the brevity, omitted. A detailed calculation shows that the processes give contributions to the splitting parameter a as follows:

$$
\begin{array}{llll}
(\mathrm{I}) & & (\mathrm{II}) & \\
(1) & 320\,(\mathrm{Dq})^2\zeta^4/P^2DFG & (1) & 12\cdot 320\,(\mathrm{Dq})^2\zeta^2(M_0-8M_2)/PDFG \\
 & & (2) & 12\cdot 320\,(\mathrm{Dq})^2\zeta^2(M_0-(11/2)M_2)/P^2FG \\
(2) & 80\,(\mathrm{Dq})^2\zeta^4/P^2DFG & (3) & 12\cdot 80\,(\mathrm{Dq})^2\zeta^2(M_0-8M_2)/PDFG \\
(3) & -10\,(\mathrm{Dq})^2\zeta^4/P^2FG^2 & (4) & 2880\,(\mathrm{Dq})^2\zeta^2(M_0-13M_2)/P^2G^2 \\
(4) & -160\,(\mathrm{Dq})^2\zeta^4/P^2D^2F & (5) & -24\cdot 160\,(\mathrm{Dq})^2\zeta^2(M_0-8M_2)/PD^2F \\
(\mathrm{III}) & -144\cdot 160\,(\mathrm{Dq})^2(M_0-8M_2)^2/D^2F & &
\end{array}
\tag{8}
$$

(IV)

(1) $-(8/35)\cdot 576\cdot 10^4(\mathrm{Dq})^4(M_0-8M_2)^2/D^3F^2$

(2) $-(8/35)\cdot 432\cdot 10^4(\mathrm{Dq})^4(M_0-8M_2)^2/D^2F^2G$

where ζ is the one-electron parameter of the spin-orbit coupling in $3d^5$ configuration, M_0 and M_2 are the radial parameters relating the spin-spin interaction within an ion:

$$M_0=(\beta^2/2)\,(1/7)\int_0^\infty\!\!\int_0^\infty(1/r_<^3)\,\psi_{3d}^2(r_1)\,\psi_{3d}^2(r_2)\,r_1^{\,2}r_2^{\,2}\,dr_1\,dr_2,$$

$$M_2=(\beta^2/2)\,(1/49)\int_0^\infty\!\!\int_0^\infty(r_>^2/r_<^5)\,\psi_{3d}^2(r_1)\,\psi_{3d}^2(r_2)\,r_1^{\,2}r_2^{\,2}\,dr_1\,dr_2. \tag{9}$$

There may be other processes which contribute to the quartic part of the spin Hamiltonian (5), such as:

$$({}^6S-{}^4P-{}^2D-{}^2F-{}^2D-{}^4P-{}^6S),$$
$$W_{LS}W_{LS}V_{\mathrm{cub}}V_{\mathrm{cub}}W_{LS}W_{LS}$$

$$({}^6S-{}^4P-{}^2D-{}^2F-{}^2D-{}^6S),$$
$$W_{LS}W_{LS}V_{\mathrm{cub}}V_{\mathrm{cub}}W_{SS}$$

$$({}^6S-{}^2D-{}^2F-{}^2D-{}^6S).$$
$$W_{SS}V_{\mathrm{cub}}V_{\mathrm{cub}}W_{SS}$$

These processes are restricted to $3d^5$, but there may be other processes which involve excited configurations, e.g.,

$$({}^6S-{}^6P(3d^4\,4d)-{}^6D(3d^4\,4d)-{}^6D(3d^4\,4d)-{}^6P(3d^4\,4d)-{}^6S),$$
$$W_{LS}\qquad W_{LS}\qquad V_{\mathrm{cub}}\qquad W_{LS}\qquad W_{LS}$$

$$({}^6S-{}^6D(3d^4\,4s)-{}^6D(3d^4\,4s)-{}^6S).$$
$$W_{SS}\qquad V_{\mathrm{cub}}\qquad W_{SS}$$

These processes might contribute to the splitting parameter a less amounts than one tenth of those from the processes discussed above because of their larger excitation energies to the intermediate multiplets even though the energy levels of ${}^2D$ and ${}^2F$ are not known from spectroscopic data in case of Mn^{++}. The spin-other-orbit couplings are not taken into consideration assuming the smallness of their coupling parameters. If not so, their contributions to the splitting parameter a must be taken into account. It is known that the configuration interactions mix excited ${}^6S'$ to the ground ${}^6S$ state and contribute nothing to the anisotropic deformation of the charge cloud of the ground state.

Now, let us turn to the discussion of the processes for the axial term D. One important mechanism has been proposed by Pryce, but here will be given another fourth-order process quadratic both in W_{LS} and $V_{\mathrm{ax}}=c_{20}r^2Y_2^0(\theta\varphi)$:

$$({}^6S-{}^4P-{}^4D-{}^4P-{}^6S)\,.$$
$$W_{LS}V_{\mathrm{ax}}V_{\mathrm{ax}}W_{LS}$$

The contribution of this process is formulated as follows:

$$\begin{aligned} D_1=(1/4)[|\langle{}^6S\,5/2\,J_z=3/2|W_{LS}|{}^4P\,5/2\,J_z=3/2\rangle \\ \langle{}^4P\,5/2\,J_z=3/2\,|V_{\mathrm{ax}}|{}^4D\,L_z=1\;S_z=1/2\rangle|^2 \\ -|\langle{}^6S\,5/2\,J_z=5/2\,|W_{LS}|{}^4P\,5/2\,J_z=5/2\rangle \\ \langle{}^4P\,5/2\,J_z=5/2\,|V_{\mathrm{ax}}|{}^4D\,L_z=1\;S_z=3/2\rangle|^2]/P^2D. \end{aligned}$$

Calculation shows

$$D_1=-(6/35)\,\zeta^2\Delta^2/P^2D, \tag{10}$$

where $\Delta=(1/4)\,(5/\pi)^{1/2}c_{20}\langle r^2\rangle$, $\langle r^2\rangle$ being the average with respect to the radial $3d$-orbital. The process discussed by Pryce gives contribution to D through

$({}^6S-{}^6D(3d^4\,4s)-{}^6S)$:
$W_{SS} \qquad\qquad V_{\mathrm{ax}}$

$$D_2=(24/25)\,\Delta'\,M'/E \tag{11}$$

where $\Delta'=(5/\pi)^{1/2}c_{20}\langle r'^2\rangle$,

$$\langle r'^2\rangle=\int_0^\infty r^4\psi_{3d}(r)\,\psi_{4s}(r)\,dr\,,$$

$$\begin{aligned} M'=\beta^2[3\int_0^\infty(1/r)\,\psi_{3d}(r)\,\psi_{4s}(r)\int_0^r r'^2\psi_{3d}^2(r')\,dr'\,dr \\ -2\int_0^\infty(1/r^3)\,\psi_{3d}^2(r)\int_0^r r'^4\psi_{3d}(r')\,\psi_{4s}(r')\,dr'\,dr] \end{aligned}$$

and E is the energy difference between $3d^4\,4s\,{}^6D$ and $3d^5\;{}^6S$. These two terms should be added to give the dominant part of the total D-value. There may be other processes which contribute to some extent to the D-value, e. g.,

$({}^6S-{}^6P(3d^4\,4d)-{}^6P(3d^4\,4d)-{}^6S)$,
$W_{LS} \qquad\qquad V_{\mathrm{ax}} \qquad\qquad W_{LS}$

$({}^6S-{}^6D(3d^4\,4d)-{}^6S)$.
$W_{SS} \qquad\qquad V_{\mathrm{ax}}$

These processes, however, may play less important role in the contribution to D than stated above, because of greater excitation energies to the intermediate states.

It is known experimentally that the deviation of the spectroscopic splitting factor from the free spin value is very small and the g-factor is almost isotropic. As seen from the above discussions of the ground level splitting, however, there must be an admixture of the excited multiplets and this must result in a deviation of the g-value. The most important processes for this deviation will be third-order ones, quadratic in W_{LS} and linear in the Zeeman term $\sum_i \beta(\boldsymbol{l}_i+2\boldsymbol{s}_i)\mathbf{H}$ via the intermediate multiplet ^4P arising from $3d^5$,

$$\begin{array}{l} (^6S-{}^4P-{}^4P-{}^6S) \text{ and} \\ W_{LS}\,W_H\,W_{LS} \\ (^6S-{}^4P-{}^6S-{}^6S). \\ W_{LS}W_{LS}W_H \end{array}$$

As a perturbing Hamiltonian we introduce:

$$\boldsymbol{H}=\sum_i \xi(i)\,\boldsymbol{l}_i\boldsymbol{s}_i+\sum_i \beta(\boldsymbol{l}_i+2\boldsymbol{s}_i)\mathbf{H}=W_{LS}+W_H, \tag{12}$$

where W_H stands for the Zeeman term. The deviation of the g-value when a magnetic field is applied along the z-axis is given by

$$\Delta g=G(J_z)-G(J_z-1), \tag{13}$$

where

$$\begin{aligned} G(J_z)=[&|\langle {}^6S\,5/2\,J_z|W_{LS}|{}^4P\,5/2\,J_z\rangle|^2 \\ &\langle {}^4P\,5/2\,J_z|\sum_i(\boldsymbol{l}_i+2\boldsymbol{s}_i)|{}^4P\,5/2\,J_z\rangle \\ &-|\langle {}^6S\,5/2\,J_z|W_{LS}|{}^4P\,5/2\,J_z\rangle|^2 \\ &\langle {}^6S\,5/2\,J_z|\sum_i(\boldsymbol{l}_i+2\boldsymbol{s}_i)|{}^6S\,5/2\,J_z\rangle]/P^2. \end{aligned}$$

Calculation yields:

$$\Delta g=-2\zeta^2/P^2. \tag{14}$$

These processes give definitely the minus sign to the deviation of the g-value, being independent upon crystalline potential. These are only third-order processes involving spin-orbit coupling, so far as $3d^5$ is assumed. Third-order processes through excited configurations, e. g., $3d^4 4d\ {}^6P$ (172600 cm^{-1} above ^6S for Mn^{++}), will make less contributions to the g-value deviation than (14) because of higher excitation energies.

§ 4. Discussion

We shall now examine the order of the magnitude of the various processes proposed in section 3 and compare them with experimental data.1),2),3)

The contribution to the splitting parameter a from (8) is calculated to be

$$\begin{aligned} a=&(\mathrm{Dq})^2[2.015+15.94M_0-149.5M_2 \\ &-5.037(M_0-8M_2)^2-0.388(M_0-8M_2)^2(\mathrm{Dq})^2\cdot 10^{-6}] \\ &\cdot 10^{-10}\ \mathrm{cm}^{-1} \quad \text{for Mn}^{++}, \end{aligned} \tag{15}$$

using the values of energy separations between excited quartets and ^6S, i. e., $P=29250$ cm^{-1}, $D=32390$ cm^{-1}, $F=43600$ cm^{-1} and $G=26850$ cm^{-1} 9) and spin-orbit coupling constant $\zeta=400$ cm^{-1} 14), but leaving the crystalline field parameter Dq and the radial parameters of spin-spin interaction within the ion, M_0 and M_2, undetermined. The crystalline field parameter Dq depends upon the charge distributions of the environment ions in crystal around the ion to be considered. The spin-spin interaction parameters, M_0 and M_2, are determined by analyzing the fine structures of the excited levels from spectroscopic data of the free ion, Mn^{++}, while one has no data of Fe^{+++} available. M_0 and M_2 are calculated by the method of the least square, with the analysis of the fine structures of ^4G and ^4F arising from $3d^5$, which are supposed to be explained by the second-order perturbation of the spin-orbit coupling and the first-order perturbation of the spin-spin interaction, while the first-order effect of the spin-orbit coupling vanishes. Using the value $\zeta=400$ cm^{-1}, M_0 and M_2 are calculated to be 0.642 cm^{-1} and 0.016 cm^{-1}, respectively, being considered to be reasonable order of magnitudes. It is, on the other hand, desirable to calculate directly these parameters with the use of radial wave function. Making use of the Slater type wave function, the ratio M_0 to M_2 being 12, M_0 and M_2 are calculated to be 0.0745 cm^{-1} and 0.00621 cm^{-1}, respectively, being much less than those obtained above. It may, however, be conceivable that these parameters depend strongly upon the feature of the radial extension of the wave function and, in this respect it might not be appropriate to use the Slater type function. To evaluate the splitting parameter a, therefore, it may be preferable to use the values of M_0 and M_2 obtained from spectroscopic data. Then (15) becomes

$$a=(\mathrm{Dq})^2(8.526-0.102(\mathrm{Dq})^2\cdot 10^{-6})\cdot 10^{-10}\ \mathrm{cm}^{-1}.$$

The behavior of the splitting parameter a is plotted against the crystalline field parameter

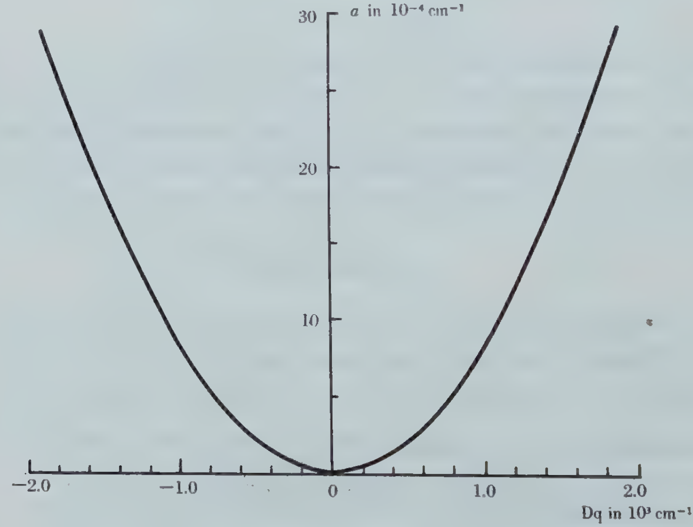

Fig. 2. Calculated splitting parameter a *vs.* Dq for Mn^{++}.

Table I

Ion	Crystal	a in cm^{-1}	D in cm^{-1}	g	Reference
Mn^{++}	$(NH_4)_2Zn(SO_4)_2 6H_2O$	$+0.0095\pm1$	$+0.0243\pm5$	2.000 ± 5	a
	$ZnSiF_6$ $6H_2O$	$+0.0007\pm1$	-0.0179 ± 3	2.000 ± 5	a
	$Mg_3Bi_2(NO_3)_{12}24H_2O$	$+0.0010\pm1$	-0.0211 ± 1	1.997 ± 3	b
	$MgSiF_6$ $6H_2O$	$(+)0.0007$	$(-)0.0274$		c
	MgO	$+0.00186$		2.0014 ± 5	d
	ZnS	-0.00076		2.000	e
Fe^{+++}	$KAl(SeO_4)12H_2O$	-0.0127 ± 2	-0.0103 ± 1	2.003 ± 3	f
	$(NH_3CH_3)Al(SO_4)_2 12H_2O$	$(-)0.010\pm4$	$(-)0.188\pm14$		f
	$RbAl(SO_4)12H_2O$	-0.0134 ± 2	$+0.0022\pm2$	2.003 ± 3	f
	$NH_4Al(SO_4)_2 12H_2O$	$(-)0.0128\pm4$	$+0.016\pm1$		g
	MgO	$+0.0205$		2.0037 ± 7	h

a) B. Bleaney and D. J. E. Ingram, Proc. Roy. Soc. (London) **A205**, (1951), 336.
b) R. S. Trenam, Proc. Phys. Soc. (London) **A66** (1953), 118.
c) T. Arakawa, J. Phys. Soc. Japan **9** (1954), 790.
d) W. Low, Phys. Rev. **105** (1957), 792.
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f) B. Bleaney and R. S. Trenam, Proc. Roy. Soc. (London) **A223** (1954), 1.
g) J. Ubbink, J. A. Poulis and C. J. Gorter, Physica **18** (1951), 361.
h) W. Low, Bull. Am. Phys. Soc. Ser. II, **1** (1956), 283.

Dq in Fig. 2, while experimental data of the values a, D, and g are listed in Table I.

From the analysis of optical absorption spectrum of the complex $MnCl_2 6H_2O$, the crystalline field parameter Dq is obtained to be 1230 cm^{-1} by Tanabe and Sugano.[11] This order of magnitude for Dq is expected to be observed in cases where Mn^{++} ion is surrounded octahedrally by six water molecules. As shown in Fig. 2, the curve of a *vs.* Dq may be considered to represent the correct behavior of the ground state of Mn^{++} in cubic crystalline field and the sign of the splitting parameter a may be definitely positive in the region where Dq is sufficiently small that perturbation procedure is permitted. It may, therefore, be concluded that both the spin-orbit coupling and the spin-spin interaction within the ion give an important contribution to the cubic part of the ground level splitting.

The negative sign of the splitting parameter a, for Mn^{++} in zincblende crystal, reported by Matarrese and Kikuchi[3] does not seem to be conclusive since they noted that the sign was determined from the dependence of the *hfs* pattern upon the magnetic quantum number of the nuclear spin though it is necessary to observe the variation of the relative intensity of lines at lower temperatures for the determination of the absolute sign of the splitting parameter a. The sign of the cubic crystalline potential at the position of cation in zincblende crystal and in MgO crystal are inverted with respect to each other, on the basis of the contribution from the nearest neighbor anions. At first sight, the sign of the splitting parameter a might depend upon the sign of the cubic crystalline potential at the position of the Mn^{++} ion. It is, however, unexpected that such a difference of the sign of the crystalline potential appears in the splitting parameter, since for any configuration just half-filled with electrons there exists no effect of odd powers with regard to the various physical quantities of electron, e. g., charge, magnetic moment, etc. It is, therefore, highly desirable to determine the absolute sign of the splitting parameter a in zincblende crystal at lower temperatures.

For the case of Fe^{+++} ion in the cubic crystalline potential, it might be expected that those processes formulated in (8) would be responsible for the splitting of the ground level. It is not possible to calculate the splitting parameter a on the basis similar to the case of Mn^{++} since there is no detailed spectroscopic data of free Fe^{+++} ion, available for the evaluation of the parameters M_0 and M_2. If there were no essential difference between Mn^{++} and Fe^{+++} ions, as might be expected, regarding the magnitude of the spin-orbit coupling constant ζ, the spin-spin interaction parameters M_0 and M_2 and the crystalline field parameter Dq, the same order of magnitude of the splitting parameter a for both ions might be observed in similar crystals containing Mn^{++} or Fe^{+++} ions, e. g., in MgO. On the other hand, it is reported that the splitting parameter a for Fe^{+++} in several crystals are about ten times larger than those for Mn^{++}, as seen in Table I and the negative signs of the splitting parameter a, except in the case of MgO, are reported. To make the situation clearer, it may be necessary to have more complete spectroscopic data for free Fe^{+++} ion.

D_1 and D_2 are estimated by inserting the values of ζ, and of the energy separation in the denominators of (10) and (11) ($E=62850\ \mathrm{cm}^{-1}$ for Mn^{++} being used), leaving Δ, Δ' and M' as parameters. Δ' differs from Δ only in the averaging of r^2. It is shown that $\langle r'^2\rangle=2.375\langle r^2\rangle$, using the Slater type wave functions, and $\Delta'=2.375\Delta$. There might be no appropriate means to obtain the value of M' from experimental data, however, the calculation of M' is not reliable since it strongly depends upon the overlap of the $3d$ and $4s$ wave functions and one has no exact wave functions. Using the approximate relation between Δ' and Δ, one tentatively obtains:

$$D_1+D_2=-9.90\Delta^2\cdot 10^{-10}+3.62M'\Delta\cdot 10^{-5}\ \mathrm{cm}^{-1}\ \text{for}\ Mn^{++}. \tag{16}$$

Δ is an axial field parameter and is, for example, equal to 8/7 of the overall separation of the orbital levels for one $3d$-electron placed in an axial crystalline field. The first term of (16) contributes definitely negative value to D, regardless of the sign of the crystalline potential. Assuming several reasonable values of M' the behaviors of the D-values are plotted in Fig. 3 against the axial field parameter Δ. On account of the ambiguity of the value of M', it is not possible to give a definite result on D, but the order of magnitude may be correct.

The g-value calculated by the use of (14) are:

$$g=2.0019,\ \text{i. e.,}\ \Delta g=-0.0004\ \text{for}\ Mn^{++},$$

$$g=2.0020,\ \text{i. e.,}\ \Delta g=-0.0003\ \text{for}\ Fe^{+++},$$

where the one-electron spin-orbit coupling constants

$$\zeta=400\ \mathrm{cm}^{-1}\ \text{for}\ Mn^{++}\ \text{and}\ \zeta=440\ \mathrm{cm}^{-1}\ \text{for}\ Fe^{+++14)}$$

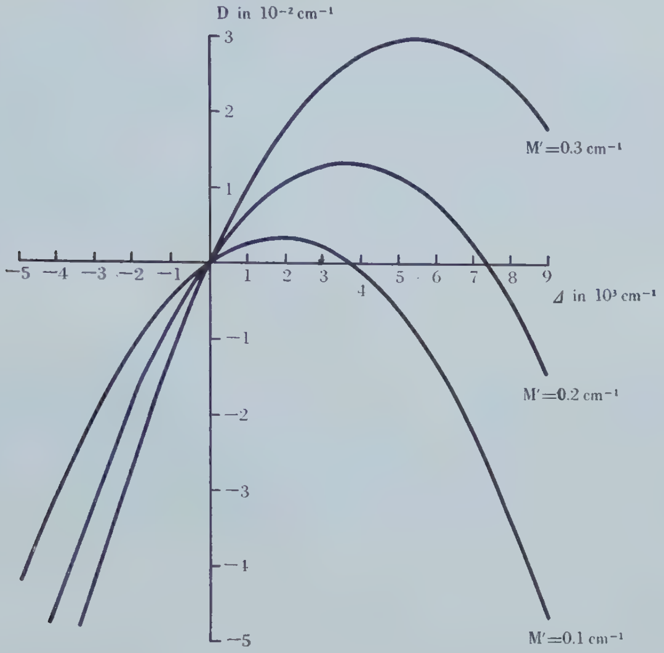

Fig. 3. Calculated D *vs.* Δ curve of Mn^{++}, for several M' values assumed.

are used, and energy separation $P=3.51\cdot10^4\,\mathrm{cm}^{-1}$ is used for Fe^{+++}. These values are in agreement with the observations, though in some cases the observed deviations are slightly greater. As noted above, the processes leading to (14) definitely give a negative deviation from the free spin value (2.0023). On the other hand, there seems to be no observation of a definitely positive deviation which is outside the experimental errors except in one case of Fe^{+++} in MgO crystal (2.0037 ± 0.0007). This was reported by Low[2] who considered this positive deviation to be attributable to the crystalline potential and not to the spin-orbit coupling. On the other hand, Low observed the g-value of Mn^{++} in MgO crystal to be 2.0014 ± 0.0005. However, the cubic crystalline potential may not essentially differ for Mn^{++} and Fe^{+++} in MgO, and it seems difficult to explain these contradictory results. There may not be a third-order process, quadratic in cubic crystalline potential and linear in Zeeman energy. The third-order process, proposed in section 3, would also give reasonable g-values for the rare-earth ions, such as Eu^{++} and Gd^{+++}, which are in ^8S state, since in these rare-earth ions the spin-orbit coupling is stronger and the crystalline potential weaker than in the iron group ions.

In section 2, the overlap of charge clouds and the formation of a partial covalency among the ions were neglected. These effects must be taken in consideration when further detailed treatment are made.

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On The Exciton Problem in Insulating Crystals

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The general problem of configuration interactions for the excited states in ideal insulators is investigated by solving a difference equation which is derived in the same way as in the scattering problem. It is shown that the general picture of exciton falls into three categories: deep case, shallow case, and hydrogenic case, according to different conditions under which the difference equation is to be solved approximately. The structures of the bands of both the singlet and triplet excitons are investigated in these three cases, with particular emphasis on the deep case. The solution derived is intermediate between the "atomic" and "continuum" models hitherto proposed.

Introduction

As is well known, the optical absorption without photocurrent observed in many insulating crystals has been ascribed to excitation of a certain discrete state, the exciton state. The first theoretical work on this subject was done by Frenkel and Peierls[1)] and later works by Slater and Shockley[2)], Wannier[3)], etc. The theoretical study of the exciton is of great importance because, firstly, the exciton problem is regarded as a typical problem of the correlation effect in solids, which cannot be handled within the scheme of one-electron approximation but can be handled only with configuration interactions of the many electron system, and second, the exciton has so far been understood in terms of rough models, i.e., the atomic model[1)], the classical model[4)] and the continuum model[3)], and even the inter-relation between these has not yet been well established. Wannier made an attempt to make this interrelation clear; he, however, confined himself to the continuum model (or the so-called effective mass theory) by replacing the difference equation by a differential equation and got a hydrogenic solution for the exciton, which of course has limited applications to actual crystals. In many cases, it will not be justified to replace the difference equation by a differential equation; by making such an approximation one will often arrive at results that are not only quantitatively erroneous but also qualitatively inacceptable, especially for the lowest exciton state. Slater and Koster have extended Wannier's theory to the case of perturbed periodic lattice[5)] using a difference equation. However, they did not solve the difference equation for the exciton problem; after deriving the difference equation, Slater[6)] merely gave qualitative discussions of the exciton. It is our purpose to make a basic calculation for this problem and to give a semi-quantitative discussion of it.

Recently, an interesting work on exciton band was published by Heller and Marcus[7)].

They considered the exciton as an atomic excitation wave proposed by Frenkel[1] in his early work. According to their result, the structure of the exciton band is determined merely by the dipole-dipole terms of the electron-hole interaction. In Wannier's theory, it was determined solely by the structure of the valence and conduction bands. In actuality, both of them should be effective for the exciton band. In the present paper we shall obtain terms of these two types and show that there are cases where either is dominant or both are effective in determining the structure of the exciton band. We shall consider the spin state but ignore the spin-orbit coupling, so that we shall deal with singlet and triplet excitons separately.

§ 1. Wave functions and energy matrix

In this study we confine ourselves to the problem in simple insulating crystals with a single filled valence band and a single empty conduction band in the ground state. We suppose N atoms and $2N$ electrons. Let $u_{k}(\boldsymbol{r})$ be a Bloch function in the valence band and, $u_{k}^{e}(\boldsymbol{r})$ that in the conduction band. We designate the ground state wave function by Ψ_0. Then we construct the zeroth order excited wave functions in the form :

$$ {}^{M}\Psi_{B}(\boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k})=\frac{1}{\sqrt{(2N)!}}\sum_{\mathrm{P}}(-)^{\mathrm{P}}\,\mathrm{P}\Phi(\boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k})\mathrm{P}\chi\,, \tag{1·1} $$

where

$$ \begin{aligned} \Phi(\boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k}) &= u_{k_1}(\boldsymbol{r}_1)\quad\cdots u_{k}^{e}(\boldsymbol{r}_i)\qquad\cdots u_{k_N}(\boldsymbol{r}_N) \\ &\quad\cdot u_{k_1}(\boldsymbol{r}_{N+1})\cdots u_{k-K}(\boldsymbol{r}_{N+i})\cdots u_{k_N}(\boldsymbol{r}_{2N}) \end{aligned} \tag{1·2} $$

and P means the permutation of order $(2N)!$. The wave function ${}^{M}\Psi_{B}(\boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k})$ is defined by vectors $\boldsymbol{k}-\boldsymbol{K}$, $\boldsymbol{k}$ representing respectively the Bloch function of the valence band from which an electron is missing and the Bloch function of the conduction band to which an electron has been raised. M represents the spin multiplicity, i.e., $M=1$ for the singlet and $M=3$ for the triplet. Thus, for each spin state, we have N^2 wave functions of the zeroth order, from which we get, by a unitary transformation, N^2 new wave functions :

$$ {}^{M}\Psi_{E}(\boldsymbol{K},\ \boldsymbol{R})=\frac{1}{\sqrt{N}}\sum_{k}\exp[-i\boldsymbol{k}\boldsymbol{R}]\,{}^{M}\Psi_{B}(\boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k})\,. \tag{1·3} $$

Each of these represents a so-called excitation wave which corresponds to an electron-hole complex separated by $\boldsymbol{R}$ in space and moving with a wave vector $\boldsymbol{K}$. This statement can easily be verified if one express (1·3) in terms of Wannier functions, which are related to Bloch functions by

$$ \begin{aligned} a(\boldsymbol{r}-\boldsymbol{R}) &= \frac{1}{\sqrt{N}}\sum_{k}\exp[-i\boldsymbol{k}\boldsymbol{R}]\,u_{k}(\boldsymbol{r})\,, \\ a^{e}(\boldsymbol{r}-\boldsymbol{R}) &= \frac{1}{\sqrt{N}}\sum_{k}\exp[-i\boldsymbol{k}\boldsymbol{R}]\,u_{k}^{e}(\boldsymbol{r})\,. \end{aligned} \tag{1·4} $$

Our program is to solve the configuration interaction problem expanding the perturbed wave function, ${}^M\Psi(\boldsymbol{K})$, in basic set of either ${}^M\Psi_B$ or ${}^M\Psi_E$ as follows:

$$\begin{aligned} {}^M\Psi(\boldsymbol{K}) &= \sum_{\boldsymbol{k}} {}^M f_{\boldsymbol{K}}(\boldsymbol{k})\, {}^M\Psi_B(\boldsymbol{k}-\boldsymbol{K}, \boldsymbol{k}) \\ &= \sum_{\boldsymbol{R}} {}^M U_{\boldsymbol{K}}(\boldsymbol{R})\, {}^M\Psi_E(\boldsymbol{K}, \boldsymbol{R}), \end{aligned} \tag{1·5}$$

where, from (1·3) and (1·5),

$$\begin{aligned} {}^M U_{\boldsymbol{K}}(\boldsymbol{R}) &= \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} \exp[i\boldsymbol{k}\boldsymbol{R}]\, {}^M f_{\boldsymbol{K}}(\boldsymbol{k}), \\ {}^M f_{\boldsymbol{K}}(\boldsymbol{k}) &= \frac{1}{\sqrt{N}} \sum_{\boldsymbol{R}} \exp[-i\boldsymbol{k}\boldsymbol{R}]\, {}^M U_{\boldsymbol{K}}(\boldsymbol{R}). \end{aligned} \tag{1·6}$$

We shall seek the enery eigenvalue ${}^M E(\boldsymbol{K})$ and the coefficients ${}^M f_{\boldsymbol{K}}(\boldsymbol{k})$ or ${}^M U_{\boldsymbol{K}}(\boldsymbol{R})$, but before doing this, we shall set up the energy matrix of the $2N$ electron system. The calculation is not essentially different from that which was performed by Slater[8] in his theory of ferromagnetism. To bring out the geneneral picture of the electron-hole model, we shall write it down explicitly, using the method of vector model[9].

Let the energy operater of the $2N$ electron system be

$$\boldsymbol{H} = \sum_i \boldsymbol{H}_b(\boldsymbol{r}_i) + \sum_{(i,j)}{}' \frac{e^2}{r_{ij}}, \tag{1·7}$$

where $\boldsymbol{H}_b(\boldsymbol{r}_i)$ is the Bloch Hamiltonian for the i-th electron, of which the energy and wave function are $W_{\boldsymbol{k}}$ and $u_{\boldsymbol{k}}(\boldsymbol{r}_i)$ for the valence band, $W^e_{\boldsymbol{k}}$ and $u^e_{\boldsymbol{k}}(\boldsymbol{r}_i)$ for the conduction band. Then we have the diagonal term of the energy matrix in the form:

$$\begin{aligned} ({}^M\Psi_B(\boldsymbol{k}-\boldsymbol{K}, \boldsymbol{k})|\boldsymbol{H}|{}^M\Psi_B(\boldsymbol{k}-\boldsymbol{K}, \boldsymbol{k})) &\equiv E_{\text{diag}} \\ = W_{\boldsymbol{k}-\boldsymbol{K}} + W^e_{\boldsymbol{k}} + J^e_{\boldsymbol{k}-\boldsymbol{K},\boldsymbol{k}} &- \tfrac{1}{2} K^e_{\boldsymbol{k}-\boldsymbol{K},\boldsymbol{k}}[1+4(\boldsymbol{s}\cdot\boldsymbol{s})] \\ + \sum_{\substack{\boldsymbol{k}' \\ (\boldsymbol{k}'\neq\boldsymbol{k}-\boldsymbol{K})}}{}' (2W_{\boldsymbol{k}'} + J_{\boldsymbol{k}',\boldsymbol{k}'}) &+ \sum_{\substack{\boldsymbol{k}'\,\boldsymbol{k}'' \\ (\boldsymbol{k}'<\boldsymbol{k}'';\,\boldsymbol{k}',\boldsymbol{k}''\neq\boldsymbol{k}-\boldsymbol{K})}}{}' \sum{}' (4J_{\boldsymbol{k}',\boldsymbol{k}''} - 2K_{\boldsymbol{k}',\boldsymbol{k}''}) \\ + \sum_{\substack{\boldsymbol{k}' \\ (\boldsymbol{k}'\neq\boldsymbol{k}-\boldsymbol{K})}}{}' (2J_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{K}} + 2J^e_{\boldsymbol{k}',\boldsymbol{k}}) &- \sum_{\substack{\boldsymbol{k}' \\ (\boldsymbol{k}'\neq\boldsymbol{k}-\boldsymbol{K})}}{}' (K_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{K}} + K^e_{\boldsymbol{k}',\boldsymbol{k}}), \end{aligned} \tag{1·8}$$

$$\begin{aligned} (\Psi_0|\boldsymbol{H}|\Psi_0) &\equiv E_0 \\ &= \sum_{\boldsymbol{k}'} (2W_{\boldsymbol{k}'} + J_{\boldsymbol{k}',\boldsymbol{k}'}) + \sum_{\substack{\boldsymbol{k}'\,\boldsymbol{k}'' \\ (\boldsymbol{k}'<\boldsymbol{k}'')}}{}' \sum{}' (4J_{\boldsymbol{k}',\boldsymbol{k}''} - 2K_{\boldsymbol{k}',\boldsymbol{k}''}), \end{aligned} \tag{1·9}$$

where

$$\begin{aligned} J_{\boldsymbol{k}',\boldsymbol{k}} &= \iint u^*_{\boldsymbol{k}'}(\boldsymbol{r}_1)\, u^*_{\boldsymbol{k}}(\boldsymbol{r}_2)\, \frac{e^2}{r_{12}}\, u_{\boldsymbol{k}'}(\boldsymbol{r}_1)\, u_{\boldsymbol{k}}(\boldsymbol{r}_2)\, dv_1 dv_2, \\ J^e_{\boldsymbol{k}',\boldsymbol{k}} &= \iint u^*_{\boldsymbol{k}'}(\boldsymbol{r}_1)\, u^{e*}_{\boldsymbol{k}}(\boldsymbol{r}_2)\, \frac{e^2}{r_{12}}\, u_{\boldsymbol{k}'}(\boldsymbol{r}_1)\, u^e_{\boldsymbol{k}}(\boldsymbol{r}_2)\, dv_1 dv_2, \\ K_{\boldsymbol{k}',\boldsymbol{k}} &= \iint u^*_{\boldsymbol{k}'}(\boldsymbol{r}_1)\, u^*_{\boldsymbol{k}}(\boldsymbol{r}_2)\, \frac{e^2}{r_{12}}\, u_{\boldsymbol{k}'}(\boldsymbol{r}_2)\, u_{\boldsymbol{k}}(\boldsymbol{r}_1)\, dv_1 dv_2, \end{aligned} \tag{1·10}$$

$$K^e_{k',k}=\iint u^*_{k'}(\boldsymbol{r}_1)\,u^{e*}_{k}(\boldsymbol{r}_2)\,\frac{e^2}{r_{12}}\,u_{k'}(\boldsymbol{r}_2)\,u^e_k(\boldsymbol{r}_1)\,dv_1dv_2\,.$$

From $(1\cdot 8)$ and $(1\cdot 9)$, we find

$$\begin{aligned}E_{\mathrm{diag}}=E_0&+(W^e_k-W_{k-K})\\&-J^e_{k-K,k}+\tfrac{1}{2}K^e_{k-K,k}[1-4(\boldsymbol{s}\cdot\boldsymbol{s})]\\&+\sum_{k'}(2J^e_{k',k}-K^e_{k',k})\\&-\sum_{k'}(2J_{k',k-K}-K_{k',k-K})\,.\end{aligned}\tag{1·11}$$

It is worth while looking a little into the expression $(1\cdot 11)$, which reveals the picture of the excited electron and hole. $(W^e_k-W_{k-K})$ in the first line of $(1\cdot 11)$ measures the energy of the excited electron as given simply by the band theory, i.e., that which is necessary to take out an electron from state $\boldsymbol{k}-\boldsymbol{K}$ in the valence band and bring it into state $\boldsymbol{k}$ in the conduction band. The terms of the second line can be regarded as the energy of the interaction between the excited electron and the hole, a hole being defined to have opposite charge and opposite spin to those of an electron. One of the last two lines is the energy of interactions of the excited electron with all the electrons in the filled valence band, and the other the corresponding one for the hole. We can eliminate the last additive terms in the following way. We construct combinations,

$$\begin{aligned}&W_{k-K}+\sum_{k'}(2J_{k',k-K}-K_{k',k-K})=W(\boldsymbol{k}-\boldsymbol{K}),\\&W^e_k+\sum_{k'}(2J^e_{k',k}-K^e_{k',k})=W^e(\boldsymbol{k}),\end{aligned}$$

and renormalize the energy bands by taking $W(\boldsymbol{k})$, $W^e(\boldsymbol{k})$ as energies of the valence and conduction bands. Alternatively, we might start with a Bloch Hamiltonian which includes the self-consistent potential of the valence electrons and then set up the energy matrix after subtracting this potential from $\sum'_{(i,j)}(e^2/r_{ij})$. In either case, we arrive at the same electron-hole problem. The reduction from many electron problem to electron-hole problem will be possible in a more general case, in which we consider the perturbations of higher orders as virtual processes; the renormalized bands in this case are the electronic polaron bands[10]. The reducing procedure, however, does not merely affect the band form but it also changes the interaction in the second line of $(1\cdot 11)$, i.e., it changes the coulomb potential between electron and hole from e^2/r to something like $e^2/\kappa r$ through these virtual processes.

Next we set up the non-diagonal elements. These contain the only interaction terms similar to the second line of $(1\cdot 11)$. We can summarize both the diagonal and non-diagonal elements in one formula,

$$\begin{aligned}&(M,\ \boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k}|\boldsymbol{H}|\ M,\ \boldsymbol{k}'-\boldsymbol{K}',\ \boldsymbol{k}')\\&=\delta_{\boldsymbol{K},\boldsymbol{K}'}\Big[\delta_{\boldsymbol{k},\boldsymbol{k}'}\cdot(E_0+W^e(\boldsymbol{k})-W(\boldsymbol{k}-\boldsymbol{K}))\end{aligned}$$

$$-\iint {}^{*}_{k'-\boldsymbol{K}}(\boldsymbol{r}_1)\,u^{e*}_{k}(\boldsymbol{r}_2)\,\frac{e^2}{r_{12}}\,u_{k-\boldsymbol{K}}(\boldsymbol{r}_1)\,u^{e}_{k'}(\boldsymbol{r}_2)\,dv_1dv_2$$

$$+\delta_M\cdot 2\iint u^{*}_{k'-\boldsymbol{K}}(\boldsymbol{r}_1)\,u^{e*}_{k}(\boldsymbol{r}_2)\,\frac{e^2}{r_{12}}\,u_{k-\boldsymbol{K}}(\boldsymbol{r}_2)\,u^{e}_{k'}(\boldsymbol{r}_1)\,dv_1dv_2\Big], \qquad (1\cdot 12)$$

where $\delta_M=1$ for $M=1$ and $\delta_M=0$ for $M=3$. From this expression we can also derive the matrix elements in the ${}^{M}\Psi_E$ system. Thus, making use of $(1\cdot 3)$ and $(1\cdot 4)$, we obtain

$$(M,\ \boldsymbol{K},\ \boldsymbol{R}|\boldsymbol{H}|M,\ \boldsymbol{K}',\ \boldsymbol{R}')$$

$$=\delta_{\boldsymbol{K},\boldsymbol{K}'}\Big[\delta_{\boldsymbol{R},\boldsymbol{R}'}\cdot E_0+\frac{1}{N}\sum_{k}\exp[i\boldsymbol{k}(\boldsymbol{R}-\boldsymbol{R}')](W^e(\boldsymbol{k})-W(\boldsymbol{k}-\boldsymbol{K}))$$

$$+\sum_{\rho}\exp[-i\boldsymbol{K}\boldsymbol{\rho}]$$

$$\times\Big\{-\underset{\text{(Coulomb)}}{\iint a^*(\boldsymbol{r}_1)\,a^{e*}(\boldsymbol{r}_2-\boldsymbol{\rho}-\boldsymbol{R})\,\frac{e^2}{r_{12}}\,a(\boldsymbol{r}_1-\boldsymbol{\rho})\,a^e(\boldsymbol{r}_2-\boldsymbol{R}')\,dv_1dv_2}$$

$$+\delta_M\cdot 2\underset{\text{(Exchange)}}{\iint a^*(\boldsymbol{r}_1)\,a^{e*}(\boldsymbol{r}_2-\boldsymbol{\rho}-\boldsymbol{R})\,\frac{e^2}{r_{12}}\,a(\boldsymbol{r}_2-\boldsymbol{\rho})\,a^e(\boldsymbol{r}_1-\boldsymbol{R}')\,dv_1dv_2}\Big\}\Big]. \qquad (1\cdot 13)$$

§ 2. Derivation of difference equation

In the preceding section we have found the matrix elements of the energy in each of the two systems. Before going over to the perturbation problem, we shall make an approximation for the coulomb and exchange integrals given by $(1\cdot 13)$. Let us assume the following: (1). both integrals are negligible unless $\boldsymbol{R}=\boldsymbol{R}'$, compared with the remaining terms in $(1\cdot 13)$; (2). the exchange term is negligible unless $\boldsymbol{R}=\boldsymbol{R}'=0$, whereas the coulomb term is considerable even for $\boldsymbol{R}\fallingdotseq 0$, $\boldsymbol{R}'\fallingdotseq 0$; (3). the coulomb term is negligible unless $\boldsymbol{\rho}=0$, whereas the exchange term is considerable even for $\boldsymbol{\rho}\fallingdotseq 0$. With these simplifications, we can write down the total Hamiltonian in the form:

$$\boldsymbol{H}=\boldsymbol{H}_e+\boldsymbol{H}_h+\boldsymbol{H}_{\text{int}}+E_0, \qquad (2\cdot 1)$$

$$(M,\ \boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k}|\boldsymbol{H}_e+\boldsymbol{H}_h|M,\ \boldsymbol{k}'-\boldsymbol{K}',\ \boldsymbol{k}')$$

$$=\delta_{\boldsymbol{K},\boldsymbol{K}'}\cdot\delta_{\boldsymbol{k},\boldsymbol{k}'}\cdot(W^e(\boldsymbol{k})-W(\boldsymbol{k}-\boldsymbol{K})), \qquad (2\cdot 2)$$

$$(M,\ \boldsymbol{K},\ \boldsymbol{R}|\boldsymbol{H}_e+\boldsymbol{H}_h|M,\ \boldsymbol{K}',\ \boldsymbol{R}')$$

$$=\delta_{\boldsymbol{K},\boldsymbol{K}'}\cdot\frac{1}{N}\sum_{k}\exp[i\boldsymbol{k}(\boldsymbol{R}-\boldsymbol{R}')](W^e(\boldsymbol{k})-W(\boldsymbol{k}-\boldsymbol{K})), \qquad (2\cdot 3)$$

$$(M,\ \boldsymbol{k}-\boldsymbol{K},\ \boldsymbol{k}|\boldsymbol{H}_{\text{int}}|M,\ \boldsymbol{k}'-\boldsymbol{K}',\ \boldsymbol{k}')$$

$$=\delta_{\boldsymbol{K},\boldsymbol{K}'}\Big[-\frac{1}{N}\sum_{\boldsymbol{Q}}\exp[i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{Q}]\iint|a(\boldsymbol{r}_1)|^2\frac{e^2}{r_{12}}|a^e(\boldsymbol{r}_2-\boldsymbol{Q})|^2dv_1dv_2$$

$$+\delta_M\cdot\frac{1}{N}\sum_{\rho}\exp[-i\boldsymbol{K}\boldsymbol{\rho}]$$

$$\times 2\iint a^*(\boldsymbol{r}_1)\,a^e(\boldsymbol{r}_1)\frac{e^2}{r_{12}}a^{e*}(\boldsymbol{r}_2-\boldsymbol{\rho})\,a(\boldsymbol{r}_2-\boldsymbol{\rho})\,dv_1dv_2\Big],\tag{2·4}$$

$$(M,\ \boldsymbol{K},\ \boldsymbol{R}|\boldsymbol{H}_{\text{int}}|M,\ \boldsymbol{K}',\ \boldsymbol{R}')$$

$$=\delta_{\boldsymbol{K},\boldsymbol{K}'}\Big[-\delta_{\boldsymbol{R},\boldsymbol{R}'}\cdot\iint|a(\boldsymbol{r}_1)|^2\frac{e^2}{r_{12}}|a^e(\boldsymbol{r}_2-\boldsymbol{R})|^2dv_1dv_2$$

$$+\delta_M\cdot\delta_{\boldsymbol{R},\boldsymbol{O}}\cdot\delta_{\boldsymbol{R}',\boldsymbol{O}}\cdot\sum_{\rho}\exp[-i\boldsymbol{K}\boldsymbol{\rho}]$$

$$\times 2\iint a^*(\boldsymbol{r}_1)\,a^e(\boldsymbol{r}_1)\frac{e^2}{r_{12}}a^{e*}(\boldsymbol{r}_2-\rho)\,a(\boldsymbol{r}_2-\rho)\,dv_1dv_2\Big].\tag{2·5}$$

In these expressions we find that $\boldsymbol{H}_e+\boldsymbol{H}_h$ is diagonal with respect to ${}^M\Psi_R$ and $\boldsymbol{H}_{\text{int}}$ is diagonal with respect to ${}^M\Psi_E$. Further we shall refer all the energies to the energy of the ground state E_0, and hence put $E_0=0$.

Now, to go over to the perturbation problem, we rewrite our eigenvalue problem,

$$\boldsymbol{H}\cdot{}^M\Psi(\boldsymbol{K})=E\cdot{}^M\Psi(\boldsymbol{K}),\tag{2·6}$$

in the form,

$${}^M\Psi(\boldsymbol{K})=(E-(\boldsymbol{H}_e+\boldsymbol{H}_h))^{-1}\boldsymbol{H}_{\text{int}}\cdot{}^M\Psi(\boldsymbol{K}),\tag{2·7}$$

and expand ${}^M\Psi(\boldsymbol{K})$ in a series of ${}^M\Psi_E(\boldsymbol{K},\ \boldsymbol{R})$ as given by (1·5). Then we have at once

$${}^MU_{\boldsymbol{K}}(\boldsymbol{R}_i)=\sum_{\boldsymbol{R}_j}(M,\ \boldsymbol{K},\ \boldsymbol{R}_i|(E-\boldsymbol{H}_e-\boldsymbol{H}_h)^{-1}\boldsymbol{H}_{\text{int}}|M,\ \boldsymbol{K},\ \boldsymbol{R}_j)\cdot{}^MU_{\boldsymbol{K}}(\boldsymbol{R}_j)$$

$$=\sum_{\boldsymbol{R}_j}\sum_{\boldsymbol{R}_l}(M,\ \boldsymbol{K},\ \boldsymbol{R}_i|(E-\boldsymbol{H}_e-\boldsymbol{H}_h)^{-1}|M,\ \boldsymbol{K},\ \boldsymbol{R}_l)$$

$$\times(M,\ \boldsymbol{K},\ \boldsymbol{R}_l|\boldsymbol{H}_{\text{int}}|M,\ \boldsymbol{K},\ \boldsymbol{R}_j)\cdot{}^MU_{\boldsymbol{K}}(\boldsymbol{R}_j)$$

$$=\sum_{\boldsymbol{R}_j}(M,\ \boldsymbol{K},\ \boldsymbol{R}_i|(E-\boldsymbol{H}_e-\boldsymbol{H}_h)^{-1}|M,\ \boldsymbol{K},\ \boldsymbol{R}_j)$$

$$\times(M,\ \boldsymbol{K},\ \boldsymbol{R}_j|\boldsymbol{H}_{\text{int}}|M,\ \boldsymbol{K},\ \boldsymbol{R}_j)\cdot{}^MU_{\boldsymbol{K}}(\boldsymbol{R}_j).$$

Making use of (1·3) and (2·3), this equation can be rewritten in the form

$${}^MU_{\boldsymbol{K}}(\boldsymbol{R}_i)=\sum_{\boldsymbol{R}_j}\frac{1}{N}\sum_{\boldsymbol{k}}\frac{\exp[i\boldsymbol{k}(\boldsymbol{R}_i-\boldsymbol{R}_j)]}{E-(W^e(\boldsymbol{k})-W(\boldsymbol{k}-\boldsymbol{K}))}{}^MV_{\boldsymbol{K}}(\boldsymbol{R}_j,\ \boldsymbol{R}_j)\,{}^MU_{\boldsymbol{K}}(\boldsymbol{R}_j),\tag{2·8}$$

where

$${}^MV_{\boldsymbol{K}}(\boldsymbol{R}_j,\ \boldsymbol{R}_j)=(M,\ \boldsymbol{K},\ \boldsymbol{R}_j|\boldsymbol{H}_{\text{int}}|M,\ \boldsymbol{K},\ \boldsymbol{R}_j).\tag{2·9}$$

This is the difference equation describing the excited states of insulators and is very similar to that which was derived by Slater and Koster[5),11)] to determine impurity levels.

§ 3. Solution of difference equation

Let us assume further the following simplification

$$W^e(\boldsymbol{k}) = w_e + \alpha \boldsymbol{k}^2,$$

$$W(\boldsymbol{k}) = w_g + \beta \boldsymbol{k}^2\,;$$

then we have at once

$$\begin{aligned} &W^e(\boldsymbol{k}) - W(\boldsymbol{k}-\boldsymbol{K}) \\ &= (\alpha-\beta)\left(\boldsymbol{k} + \frac{\beta}{\alpha-\beta}\boldsymbol{K}\right)^2 - \frac{\alpha\beta}{\alpha-\beta}\boldsymbol{K}^2 + w_e - w_g\,. \end{aligned}$$

Using this relation and making the substitution

$$^M C_{\boldsymbol{K}}(\boldsymbol{R}_j) = \exp\left[i\frac{\beta}{\alpha-\beta}\boldsymbol{K}\boldsymbol{R}_j\right] {}^M U_{\boldsymbol{K}}(\boldsymbol{R}_j)\,, \qquad (3\cdot1)$$

eq. (2·8) reduces to the form

$$^M C_{\boldsymbol{K}}(\boldsymbol{R}_i) = \sum_{\boldsymbol{R}_j} \frac{1}{N} \sum_{k} \frac{\exp[i\boldsymbol{k}(\boldsymbol{R}_i - \boldsymbol{R}_j)]}{\varepsilon - (\alpha-\beta)\boldsymbol{k}^2} {}^M V_{\boldsymbol{K}}(\boldsymbol{R}_j, \boldsymbol{R}_j) {}^M C_{\boldsymbol{K}}(\boldsymbol{R}_j)\,, \qquad (3\cdot2)$$

where

$$\varepsilon = E - (w_e - w_g) + \frac{\alpha\beta}{\alpha-\beta}\boldsymbol{K}^2. \qquad (3\cdot3)$$

Before discussing the nature of the solution of eq. (3·2), we shall calculate the sum in eq. (3·2). We can convert it into an integral, obtaining

$$I(\boldsymbol{R}) = \sum_{k} \frac{\exp[i\boldsymbol{k}\boldsymbol{R}]}{\varepsilon - \gamma \boldsymbol{k}^2}$$

$$\begin{cases} \simeq \dfrac{V}{(2\pi)^3} \dfrac{2\pi^2}{\gamma} \dfrac{\exp[-(-\varepsilon/\gamma)^{1/2} \cdot |\boldsymbol{R}|]}{|\boldsymbol{R}|}\,, & \text{for } \varepsilon < 0, \ \gamma > 0\,; \qquad (3\cdot4) \\ \propto \dfrac{\exp[\pm i(\varepsilon/\gamma)^{1/2} \cdot |\boldsymbol{R}|]}{|\boldsymbol{R}|}\,, & \text{for } \varepsilon > 0, \ \gamma > 0\,; \qquad (3\cdot5) \end{cases}$$

$$I(0) = \sum_{k} \frac{1}{\varepsilon - \gamma \boldsymbol{k}^2}$$

$$\begin{cases} \simeq \dfrac{V}{(2\pi)^3} \dfrac{4\pi}{\gamma} \left(\dfrac{\pi}{2}x - \xi - x\tan^{-1}\dfrac{x}{\xi}\right), & \text{for } \varepsilon < 0, \ \gamma > 0, \ |\gamma\xi^2| > |\varepsilon|\,; \qquad (3\cdot6) \\ \simeq \dfrac{V}{(2\pi)^3} \dfrac{4\pi}{\gamma} \left(x\tan^{-1}\dfrac{\xi}{x} - \xi\right), & \text{for } \varepsilon < 0, \ \gamma > 0, \ |\gamma\xi^2| < |\varepsilon|\,; \qquad (3\cdot7) \end{cases}$$

where $x=(-\mathcal{E}/\gamma)^{1/2}$, $\gamma=\alpha-\beta$, V is the volume of the crystal and ξ the radius of the sphere having the same volume as the Brillouion zone, i.e., $\xi=(6\pi^2)^{1/3}(N/V)^{1/3}$ for the f.c.c. lattice.

Now we can see from (3·4) and (3·5) that eq. (3·2) reduces to either of two essentially different forms depending on whether we are dealing with the case of $\mathcal{E}<0$ or the case of $\mathcal{E}>0$. In the first case, the sum falls off as $\exp[-(-\mathcal{E}/\gamma)^{1/2}|\boldsymbol{R}_i-\boldsymbol{R}_j|]/|\boldsymbol{R}_i-\boldsymbol{R}_j|$ with increasing $|\boldsymbol{R}_i-\boldsymbol{R}_j|$ and $|\mathcal{E}|$; so does also $I(\boldsymbol{R}_i-\boldsymbol{R}_j)/I(0)$. We show the plot of $I(R_0)/I(0)$ in Fig. 1 as a function of energy for R_0 equal to the primitive translation. Therefore, to the first approximation, eq. (3·2) reduces to

$$^{M}C_{\boldsymbol{K}}(\boldsymbol{R}_i)=\frac{1}{N}\sum_{\boldsymbol{k}}\frac{1}{\mathcal{E}-\gamma\boldsymbol{k}^2}\,{}^{M}V_{\boldsymbol{K}}(\boldsymbol{R}_i,\boldsymbol{R}_i)\,{}^{M}C_{\boldsymbol{K}}(\boldsymbol{R}_i), \tag{3·8}$$

and $\mathcal{E}$ is obtained by solving

$$1-\frac{1}{N}\sum_{\boldsymbol{k}}\frac{1}{\mathcal{E}-\gamma\boldsymbol{k}^2}\,{}^{M}V_{\boldsymbol{K}}(\boldsymbol{R}_i,\boldsymbol{R}_i)=0. \tag{3·9}$$

We denote the solution as ${}^{M}\mathcal{E}_i(\boldsymbol{K})$. The corresponding wave function is obtained to be

$$^{M}\Psi_i(\boldsymbol{K})=A\sum_{\boldsymbol{R}_j}\frac{1}{N}\sum_{\boldsymbol{k}}\frac{\exp[i\boldsymbol{k}(\boldsymbol{R}_j-\boldsymbol{R}_i)]}{{}^{M}\mathcal{E}_i(\boldsymbol{K})-\gamma\boldsymbol{k}^2}\exp\left[-i\frac{\beta}{\alpha-\beta}\boldsymbol{K}\boldsymbol{R}_j\right]{}^{M}\Psi_E(\boldsymbol{K},\boldsymbol{R}_j). \tag{3·10}$$

This tells us that the main part of the wave function for the state of energy ${}^{M}\mathcal{E}_i(\boldsymbol{K})$ is the excitation wave ${}^{M}\Psi_E(\boldsymbol{K},\boldsymbol{R}_i)$ and the rest is a superposition of ${}^{M}\Psi_E(\boldsymbol{K},\boldsymbol{R}_j)$ $(j\neq i)$ with coefficients falling off as $\exp[-(-{}^{M}\mathcal{E}_i(\boldsymbol{K})/\gamma)^{1/2}|\boldsymbol{R}_j-\boldsymbol{R}_i|]/|\boldsymbol{R}_j-\boldsymbol{R}_i|$.

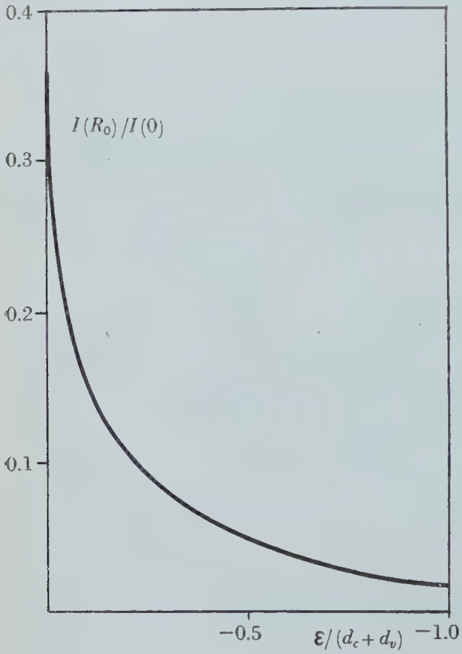

Fig. 1. Plot of $I(R_0)/I(0)$ as a function of energy for R_0 equal to the primitive translation.

In the second case, $\mathcal{E}>0$, it is clear from (3·5) that the sum in eq. (3·2) reduces to a form which is very similar to the Green function in the scattering problem. Thus eq. (3·2) corresponds to the relation between incident and scattering waves. It is well-known that there is no bound state in this case, and the states may be called "ionization continuum". The boundary between the states of exciton and the ionization continuum is given by

$$\mathcal{E}=0, \tag{3·11}$$

i.e.,

$$E\doteqdot w_e-w_g-\frac{\alpha\beta}{\alpha-\beta}\boldsymbol{K}^2. \tag{3·12}$$

Since our main interest is in discrete

states, we shall confine ourselves to them and examine our approximate solution in more detail. We find from (3·6), (3·7), (3·9) and $V/N=6\pi^2/\xi^3$ that

$$\frac{x}{\xi}\tan^{-1}\frac{\xi}{x}-1=\frac{\gamma\xi^2}{3\,{}^{M}V_{\boldsymbol{K}}(\boldsymbol{R}_i,\,\boldsymbol{R}_i)},\quad \text{for}\ \ x/\xi>1\,; \tag{3·13}$$

$$\frac{x}{\xi}\left(\frac{\pi}{2}-\tan^{-1}\frac{x}{\xi}\right)-1=\frac{\gamma\xi^2}{3\,{}^{M}V_{\boldsymbol{K}}(\boldsymbol{R}_i,\,\boldsymbol{R}_i)},\quad \text{for}\ \ x/\xi<1. \tag{3·14}$$

We put $\gamma\xi^2=\alpha\xi^2-\beta\xi^2=d_c+d_v$ where d_c, d_v are the widths of the conduction and valence bands (Fig. 3 (a)), and find $\mathcal{E}/(d_c+d_v)$ from eq. (3·13) and eq. (3·14) as a function of ${}^{M}V_{\boldsymbol{K}}(\boldsymbol{R}_i,\boldsymbol{R}_i)/(d_c+d_v)$ only if we have the condition

$$|{}^{M}V_{\boldsymbol{K}}(\boldsymbol{R}_i,\,\boldsymbol{R}_i)|>\tfrac{1}{3}(d_c+d_v). \tag{3·15}$$

If this is not satisfied, no solution exists. The behaviour of the solution is shown in Fig. 2, where ${}^{M}V_{K}(\boldsymbol{R}_i,\boldsymbol{R}_i)$ is simplified as ${}^{M}V$. For sufficiently large $|{}^{M}V|$, $\mathcal{E}$ is linear in ${}^{M}V$, i.e., we have

$${}^{M}\mathcal{E}(\boldsymbol{K})=\tfrac{1}{2}(d_c+d_v)+{}^{M}V, \tag{3·16}$$

i.e., by virtue of eq. (3·3),

$${}^{M}\mathcal{E}(\boldsymbol{K})=\tfrac{1}{2}(d_c+d_v)+{}^{M}V+w_e-w_g-\frac{\alpha\beta}{\alpha-\beta}\boldsymbol{K}^2. \tag{3·17}$$

For $|{}^{M}V|$ slightly greater than $\frac{1}{3}(d_c+d_v)$, $\mathcal{E}$ is roughly proportional to $({}^{M}V+\frac{1}{3}(d_c+d_v))^2$ (${}^{M}V$ is negative in all cases), and

$$\begin{aligned}{}^{M}\mathcal{E}(\boldsymbol{K})\simeq&-\left(\frac{6}{\pi}\right)^2(d_c+d_v)^{-1}({}^{M}V+\tfrac{1}{3}(d_c+d_v))^2\\&+w_e-w_g-\frac{\alpha\beta}{\alpha-\beta}\boldsymbol{K}^2.^{*}\end{aligned} \tag{3·18}$$

We shall later (in § 4) come to discussing these solutions and show that our results give a useful information for the actual problems, as far as we are dealing with the lowest exciton band, either singlet or triplet.

The energy of the lowest exciton band is obtained by substituting ${}^{M}V_0(\equiv{}^{M}V_{\boldsymbol{K}}(0,\,0))$ for ${}^{M}V$ where ${}^{M}V_0$ is assumed to have the largest absolute value of all the ${}^{M}V$. From (2·5), we have

$$\begin{aligned}{}^{1}V_0=&-\iint a^{*}(\boldsymbol{r}_1)\,a(\boldsymbol{r}_1)\frac{e^2}{r_{12}}a^{e*}(\boldsymbol{r}_2)\,a^{e}(\boldsymbol{r}_2)\,dv_1dv_2\\&+2\iint a^{*}(\boldsymbol{r}_1)\,a^{e}(\boldsymbol{r}_1)\,\frac{e^2}{r_{12}}a^{e*}(\boldsymbol{r}_2)\,a(\boldsymbol{r}_2)\,dv_1dv_2\end{aligned}$$

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* This approximation is not so useful as (3·17), because $I(\boldsymbol{R})/I(0)$ falls off very slowly with $|\boldsymbol{R}|$ for such an energy as expressed by (3·18).

$$+2\sum_{(\rho\neq 0)}{}'_{\rho} \exp[-i\boldsymbol{K}\boldsymbol{\rho}]\iint a^*(\boldsymbol{r}_1)\, a^e(\boldsymbol{r}_1)\, \frac{e^2}{r_{12}} a^{e*}(\boldsymbol{r}_2-\boldsymbol{\rho})\, a(\boldsymbol{r}_2-\boldsymbol{\rho})\, dv_1 dv_2\,, \tag{3·20}$$

$${}^3V_0=-\iint a^*(\boldsymbol{r}_1)\, a(\boldsymbol{r}_1)\frac{e^2}{r_{12}} a^{e*}(\boldsymbol{r}_2)\, a^e(\boldsymbol{r}_2)\, dv_1 dv_2\,. \tag{3·21}$$

When we are dealing with the lowest exciton with deep energy expressed by (3·16) or (3·17) (we call such an exciton "deep exciton"), in the singlet case the band structure is determined by the sum of two parts: one is

$$-\frac{\alpha\beta}{\alpha-\beta}\boldsymbol{K}^2=\frac{\hbar^2}{2(m_e+m_h)}\boldsymbol{K}^2, \quad (\alpha=\hbar^2/2m_e,\ \beta=-\hbar^2/2m_h) \tag{3·22}$$

which is equivalent to the result obtained by Wannier's continuum theory,[5] and the other is

$$2\sum_{(\rho\neq 0)}{}'_{\rho} \exp[-i\boldsymbol{K}\boldsymbol{\rho}]\iint Q^*(\boldsymbol{r}_1)\frac{e^2}{r_{12}} Q(\boldsymbol{r}_2-\boldsymbol{\rho})\, dv_1 dv_2\,, \tag{3·23}$$

$$(Q(\boldsymbol{r})=a^{e*}(\boldsymbol{r})\, a(\boldsymbol{r}))$$

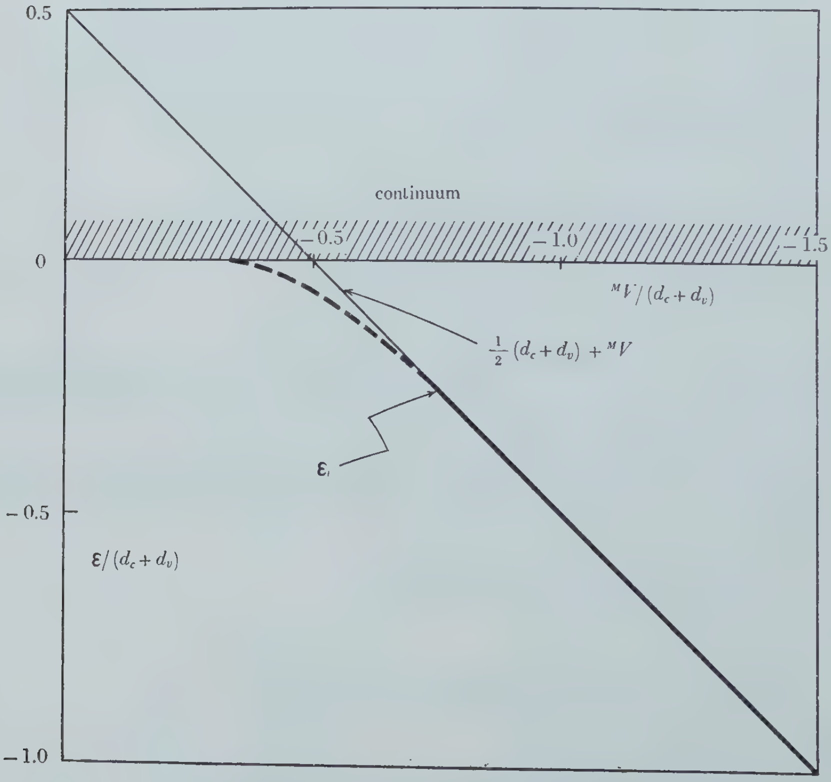
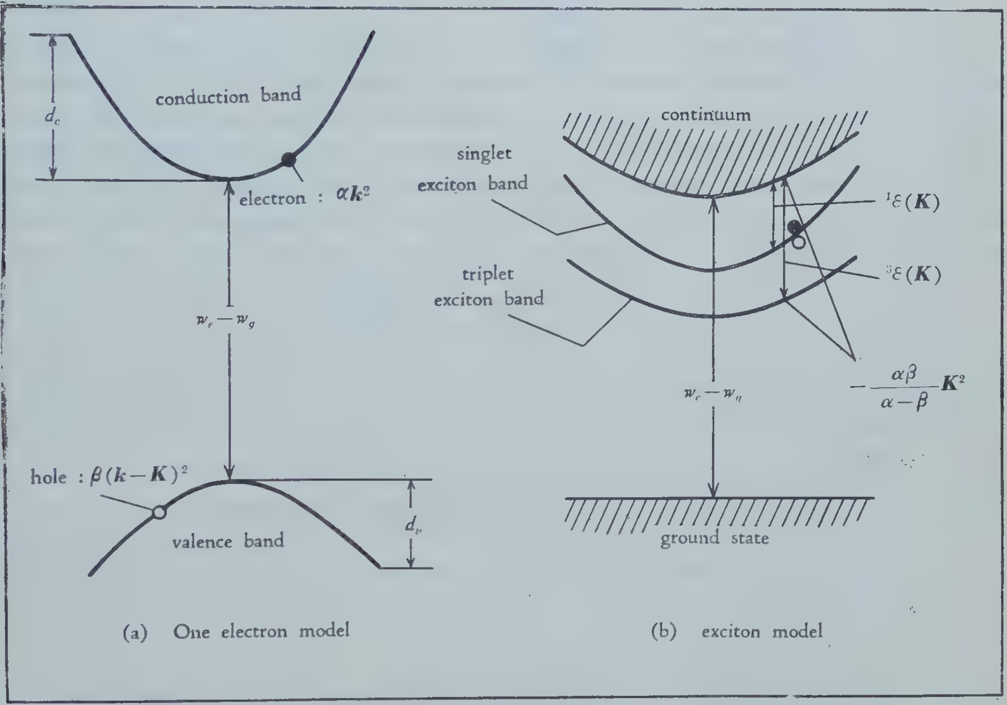


Fig. 2. Energy ε as a function of ${}^MV$.

which is almost equivalent to the dipole-dipole terms presented by Heller and Marcus[7]. In the triplet case, on the other hand, the structure is determined only by (3·22)*. The separation between the singlet and triplet exciton bands is given by (3·23) plus twice the exchange integral between the two Wannier functions on the same unit cell. This separation can in many cases (for a general **K**) be taken as positive, so that the triplet exciton band lies below the singlet band. In some other cases (and for some **K**), however, we anticipate that it is not so. (see § 4)

When the energy of the lowest exciton is shallow (we call this "shallow exciton"), the modification of the straight line as indicated in Fig. 2 affects qualitatively the situation mentioned above. The first part, (3·22), remains unchanged, but the effect of the second part and the singlet-triplet separation diminish by this modification.



(a) One electron model (b) exciton model

Fig. 3. Schematic representation of the electronic excitation in ideal insulators.

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* It is worth while looking a little into the meaning of (3·23), from which the difference in the nature of the singlet and triplet excitons comes. Consider the special case of $d_v = 0$ ($m_h = \infty$), which actually corresponds to the case of X-ray exciton[16] (the exciton due to X-ray absorption). Then (3·22) vanishes, so that the triplet exciton behaves as a localized exciton.[4], [17] Although the hole has an infinite effective mass, the singlet exciton can still propagate through the lattice, as it has a band expressed by (3·23). The physical picture of this propagation is this: an excitation created in one unit cell is annihilated accompanied by the simultanious creation of an excitation in another unit cell. Such a process is forbidden in the triplet case but is allowed in the singlet case.

§ 4. Discussion and remarks

We have found the solution of our difference equation to the first approximation, under the condition of inequality (3·15). We shall discuss this solution. At first ${}^MV$ will be assumed as

$$ {}^MV_0 \neq 0, \quad {}^MV_1 = {}^MV_2 = \cdots = 0. \tag{4·1}$$

Then our eigen-value problem (3·2) will exactly be replaced by

$$\frac{{}^MV_0}{N} \sum_{k} \frac{1}{\mathcal{E} - \gamma \boldsymbol{k}^2} = 1 . \tag{4·2}$$

We have only one solution of this equation, $\mathcal{E}_0(\boldsymbol{K})$, only if the condition

$$|{}^MV_0| > \tfrac{1}{3}(d_c + d_v) \tag{4·3}$$

is satisfied. This means that the discrete level, or the exciton level, does not begin to emerge from the continuum unless $|{}^MV_0|$ reaches a critical value, $\frac{1}{3}(d_c+d_v)$* ; in other words, if either of the effective masses of the electron and hole is small, or both are small (and of the same sign), there is no exciton state so far as the interaction energy between the electron and hole is not sufficiently large. This statement is typical of the correlation effects in three dimensional space. We have one singlet and one triplet exciton state under the condition of (4·3), and the corresponding wave functions are of the form

$${}^MU_{\boldsymbol{K}}(\boldsymbol{R}_j) = A_0 \cdot \exp\left[-i \frac{\beta}{\alpha - \beta} \boldsymbol{K}\boldsymbol{R}_j \right] \frac{\exp[-(-{}^M\mathcal{E}_0(\boldsymbol{K})/(\alpha-\beta))^{1/2} \boldsymbol{R}_j]}{|\boldsymbol{R}_j|} . \tag{4·4}$$

The assumption of (4·1) may not correspond to actual situations for most solids, except perhaps in metals where it is likely that

$$|{}^MV_0| \gg |{}^MV_1|, \; |{}^MV_2|, \cdots.$$

Next, we concern ourselves with the following cases :

$$|{}^MV_0| > \tfrac{1}{3}(d_c+d_v) > |{}^MV_1|, \; |{}^MV_2|, \cdots, \tag{4·5}$$

$$\tfrac{1}{3}(d_c+d_v) > |{}^MV_0|, \; |{}^MV_1|, \cdots, \tag{4·6}$$

$$|{}^MV_0|, \cdots |{}^MV_n| > \tfrac{1}{3}(d_c+d_v) > |{}^MV_{n+1}|, \cdots . \tag{4·7}$$

These can occur more likely for the exciton in insulators. In the case of (4·5), we can start by taking only ${}^MV_0$, then replace our difference equation approximately by eq. (3·8) putting $\boldsymbol{R}_i = 0$; we again find eq. (4·2). Thus, we have, as in the previous case, the exciton state expressed by (4·4), which certainly is the lowest exciton state. It must be noted, however, that we arrive at eq. (4·2) by considering the behaviour of $I(\boldsymbol{R}_j)/I(0) \cdot {}^MV_j/{}^MV_0$, i.e., that the falling off of $\exp[-(-\mathcal{E}/\gamma)^{1/2}|\boldsymbol{R}_j|]/|\boldsymbol{R}_j|$ is intensified by the falling off of ${}^MV_j/{}^MV_0$ so far as we are dealing with the lowest exciton state. On the other hand, if we start with any one of ${}^MV_1$, ${}^MV_2, \cdots$, we have no solution of eq. (3·8).

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* If $\beta > 0$, the critical value must be replaced by $\frac{1}{3}(d_c - d_v)$.

We cannot, however, conclude, contrary to the previous case of (4·1), the non-existence of exciton states other than the lowest one. In this case, we cannot replace our difference equation (3·2) by eq. (3·8), but we must solve the difference equation (3·2) considering all of ${}^{M}V_0$, ${}^{M}V_1$, ${}^{M}V_2, \cdots$. Then we may expect that some or many members of the higher exciton states, of which the wave functions are much more spread out than (4·4), will arise above the lowest one. These states may roughly be described by the continuum theory.

In the case of (4·6), no solution like (4·4) exists, and the whole spectrum of the exciton is more like that given by the continuum theory (we call this "hydrogenic exciton"). In this case, we may expect that the separation between the singlet and triplet exciton bands is almost zero and both bands are determined only by (3·22). In the case of (4·7), the situation will be somewhat like that of (4·5), although actually we must take up all the ${}^{M}V_0, \cdots {}^{M}V_n$ and solve the secular equation which is at present beyond our scope. We may expect that if $|{}^{M}V_0|$ is sufficiently large, our simple approximation (4·2) is again useful for the lowest state. This is because our approximation gives the state of s-like symmetry (for ${}^{M}U_{\boldsymbol{K}}$), and in many cases such a state may actually be the lowest state.

Experimental values of the lowest exciton states (singlet, $\boldsymbol{K}=0$) for alkali halide crystals, given in terms of the optical dissociation energy are -1.9 e.v. (NaCl), -1.8 e.v. (KCl), -1.5 e.v. (RbCl), etc.[12]* The situation in these cases will certainly correspond to that of our deep exciton expressed by (3·16) and the condition $|{}^{M}V_0| > \frac{1}{3}(d_c+d_v)$.** It has generally been believed that the atomic model which consciously disregards excitations from one cell to another can give reasonable explanation of the excitons in alkali halides. To understand why it is so, we have only to notice that our solution ((3·16) and (4·4)) for sufficiently deep exciton much resembles the atomic model. For cuprous oxide, the experimental value is -0.14 e.v. (in yellow series) and -0.17 e.v. (in green series)[13], and the situation will be that of our shallow exciton $(|{}^{M}V_0| \gtrsim \frac{1}{3}(d_c+d_v))$, or more like that of the hydrogenic exciton $(|{}^{M}V_0| < \frac{1}{3}(d_c+d_v))$. It is interesting that Gross[13] observed almost complete hydrogenic energy spectra in Cu_2O.***

Finally, we shall make remarks regarding the nature of the exciton bands in our deep case. We consider the dynamical nature of excitons : the creation of excitons by photons, the scattering with lattice vibrations, and mechanism for the nonradiative decay. We have already mentioned that the structure of the singlet exciton band is expressed by (3·22) plus (3·23), both terms being of the same order of magnitude, while the triplet exciton band is expressed only by (3·22) ; the energy separation of two bands is twice the exchange terms in (3·20) plus (3·23). The term (3·23) is zero at $\boldsymbol{K}=0$,[*] but it increases or decreases with $|\boldsymbol{K}|$ depending on the direction of $\boldsymbol{K}$.[**] Therefore, we can

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* These values are somewhat uncertain, as it is not precisely known at which point in the absorption spectrum excitons cease and the states of continuum start.

** A rough estimate of the magnitude of ${}^{M}V$ for NaCl will be ${}^{M}V_0 \approx -(e^2/r_0) = -5.1$ e.v., ${}^{M}V_1 \approx (-e^2/\kappa_0\sqrt{2}\,r_0) = -1.6$ e.v. where r_0 is the Na—Cl distance and κ_0 is the optical dielectric constant. It is assumed that, for such a small distance r_0, ${}^{M}V_0$ is close to $-e^2/r_0$ rather than to $-e^2/\kappa_0 r_0$ as is commonly supposed. We substitute $-\varepsilon(0) \approx 1.9$ e.v. and ${}^{M}V_0 \approx -5.1$ e.v. in our relation $\varepsilon = \frac{1}{2}(d_c+d_v) + {}^{M}V_0$ and find $d_c+d_v \approx 6.4$ e.v. Thus, $|{}^{M}V_0| \approx 5.1 > \frac{1}{3}(d_c+d_v) \approx 2.1 > |{}^{M}V_1| \approx 1.6$, which conforms to our case (4·5).

*** We have made these two assumptions in our argument: the simple energy expression for a valence and a conduction band, and the diagonal form of $\boldsymbol{H}_{\text{int}}$ with respect to ${}^{M}\Psi_{E}$. The former does not seem to be a strong restriction for our discussion, i.e., replacing $W(\boldsymbol{k})$ and $W^{e}(\boldsymbol{k})$ by a more complicated form will perhaps modify the condition (4·3) a little quantitatively, though this may affect the structure of exciton band. However, it might be possible that if the latter is not satisfied our discussion stated in § 4 is influenced qualitatively. The rigorous justification for this assumption in actual problems is at present beyond our scope.

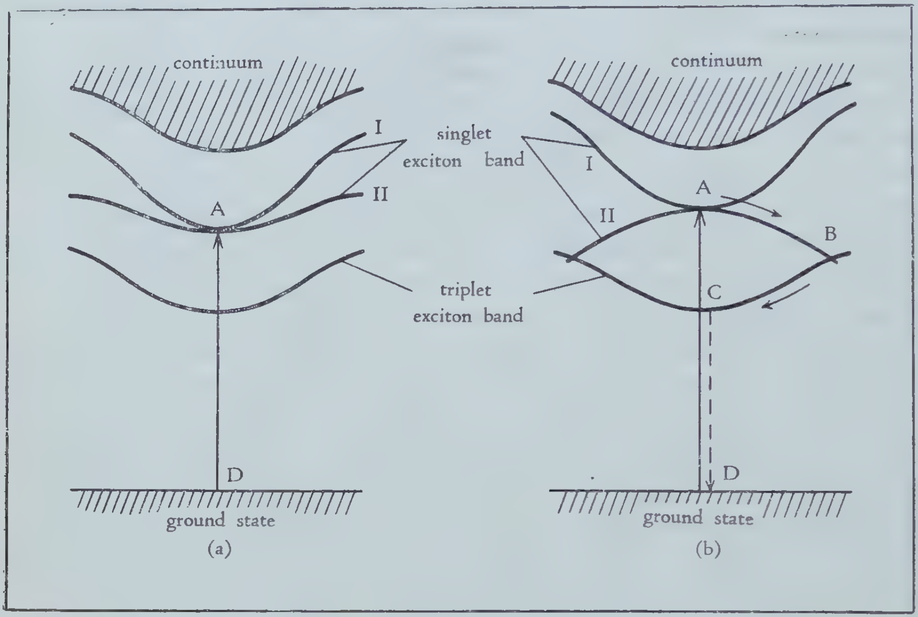

Fig. 4. Schematic representation of the exciton bands for the deep exciton.

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[*] According to Heller and Marcus, the term (3·23) can be expanded in inverse power of lattice vectors, leading to

$$-2|\boldsymbol{\mu}|^2\sum_{\rho}{}' \frac{3\cos^2(\boldsymbol{\mu},\boldsymbol{\rho})-1}{|\boldsymbol{\rho}|^3}e^{-i(\boldsymbol{K}\boldsymbol{\rho})}, \qquad (3\cdot 23')$$

where

$$\boldsymbol{\mu}=e\int \boldsymbol{r}Q(\boldsymbol{r})\,dv.$$

For a cubic lattice, we can replace the term $\cos^2(\boldsymbol{\mu},\boldsymbol{\rho})$ by $\overline{\cos^2(\boldsymbol{\mu},\boldsymbol{\rho})}=\frac{1}{3}$ if $\boldsymbol{K}=0$, so that (3·23′) vanishes at $\boldsymbol{K}=0$ regardless of the direction of dipole $\boldsymbol{\mu}$.

[**] Heller and Marcus have shown that the energy of perpendicularly polarized exciton ($\boldsymbol{\mu}\perp\boldsymbol{K}$) in simple cubic lattice increases with $|\boldsymbol{K}|$ for prominent directions of propagation (e.g., (100), (110) and (111)). The effective mass of these excitons is therefore positive. They have further shown that for excitons polarized parallel to the propagation vector ($\boldsymbol{\mu}\|\boldsymbol{K}$) the converse is true, i.e., they have a negative mass. However, in the present theory $\boldsymbol{\mu}$ is determined by the Wannier functions of single valence and single conduction bands and hence has a fixed direction; therefore we have from (3.23′) an energy that increases with $|\boldsymbol{K}|$ for directions perpendicular to $\boldsymbol{\mu}$ and decreases with $|\boldsymbol{K}|$ for the direction parallel to $\boldsymbol{\mu}$. If we extend our theory to degenerate valence bands, we should have dipole moments of three types and hence find a family of ascending energy curves and a family of descending energy curves for all prominent directions of $\boldsymbol{K}$.

distinguish between two cases (a) and (b) of Fig. 4 taking the exchange term in (3·20) to be positive; for curve I, (3·23) is zero at point A ($\boldsymbol{K}=0$) and is assumed to increase with $|\boldsymbol{K}|$ corresponding to a certain direction of propagation, and for curve II to decrease with $|\boldsymbol{K}|$ corresponding to another direction; in the case (b), (3·23) is assumed to vary more rapidly than in the case (a), so that it may happen in the case (b) that at point B curve II crosses with, or comes close to, the triplet band, which increases with $|\boldsymbol{K}|$ for any direction, as far as $m_e+m_h>0$.

An exciton which is initially created by a photon is singlet with $\boldsymbol{K}=0$, triplet states being forbidden if spin-orbit interactions are ignored. However, excitons accompanied by a simultaneous emission or absorption of lattice phonons are also created at points $\boldsymbol{K}\fallingdotseq 0$ in the singlet band. They may presumably pick up momenta through interactions with lattice vibrations. Therefore, in case (b), they will move in the singlet band from A to B, and at or near point B they will convert to triplet excitons through spin-orbit interactions or some other effects. The triplet excitons thus created also will move down the triplet band and reach a point C. Some of them will decay through spin-orbit interactions from the neighbourhood of C to the ground state D, but others will be wandering in the triplet band for some time and decay with multiple phonon emissions (or through other effects), i.e., make nonradiative decays. If the spin-orbit interactions are weak, most of them will make nonradiative decays, while if the spin-orbit interactions are strong, luminescence corresponding to C → D will occur and the wandering time will be shorter.

Various mechanisms14) by which excitons can decay without luminescence have been conceived. However, all but one proposed by Overhauser15) postulate the existence of lattice imperfections, while the above explanation does require no lattice imperfections. The exciton created at point A will have a lifetime of about 10^{-8} sec. before emitting a photon. Therefore, for the usual mechanisms, the exciton must encounter a lattice imperfection in a time shorter than 10^{-8} sec. to decay without luminescence. In our mechanism the exciton must move out from a certain neighbourhood of A in a time shorter than 10^{-8} sec, but the decay time can be much longer than 10^{-8} sec. We have yet no experiment as to the triplet exciton. It might be interesting to observe the triplet exciton by the method of para-magnetic resonance.

In conclusion the author would like to express his sincere thanks to Professor T. Nagamiya, Professor K. Husimi and Mr. Y. Kitano for their valuable discussions.

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Ionization Loss near the Origin of an Electron Pair of Very High Energy

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Ionization loss of negaton-positon pair is calculated as a function of their distance. This result will be useful to measure the energy of a γ-ray above 10^{10} eV from the reduction of ionization near the origin of the created pair. The reduction is due to the interference between the electro-magnetic field of negaton and that of positon. We adopted a similar method to that of Fermi who had treated the energy loss of single electron taking into account the polarizability of the medium. Our result is not so different from Čudakov's[4] so far as the distance between negaton and positon is small.

§ 1. Introduction

Recent experiments are directed towards the exploitation of higher and higher energy regions. This fact stimulated us to investigate the new techniques to measure the energy in these regions. Let us consider, for example, the method of measuring the energy of γ-ray. One of them consists in observing the magnitude of cascade shower with a specially designed emulsion chamber.[1] For the development of the cascade shower, reliable theories have already been established. Another method, which is expected to take a complementary role to the above mentioned one, is to measure the angle of divergence of the negaton-positon pair produced by the γ-ray in question. The probability distribution of the angle of divergence was studied by A. Borsellino.[2] Rougly Speaking, the most probable angle is $\sim 4\, mc^2/E$, m and E being the rest mass of electron and the energy of γ-ray, respectively.

Although we cannot discriminate between two tracks whose angle is smaller than, say, 10^{-4} radian (which corresponds to 50 Gev γ-ray), we can, with the aid of some appropriate theories, infer the magnitude of this angle from the reduction of ionization observed as the function of path-length from the origin of the pair (reduction from the case of two independent particles), since this reduction depends on the distance between negaton and positon.

D. H. Perkins[3] made the ionization measurement on a number of electron-pair tracks.

He has shown that with pairs of the energy $\sim 10^{11}$ eV the reduction is measurable for the path-length $\lesssim 100\,\mu$. A. E. Čudakov[4)] obtained a formula for the reduction of ionization, based on a simple and classical argument. Recently, W. Wolter and M. Mięsowicz[5)] have reported that the energy measurement based on the A. E. Čudakov's formula does not contradict with the results from the energy development of cascade. As a measure of ionization, they took the coefficient of the exponential distribution of gap lengths. Their result (cf. Fig. 6) seems to show that the statistical error does not prevent us at least from knowing the order of magnitude of the energy.

Now, after the suggestion of M. Koshiba, we tried to examine the reduction effect in more detail than in the work of A. E. Čudakov. Our project is to treat this problem on a level with the theory of ionization of a single particle, i. e. to treat it not classically, and to take into account the polarizability of the medium. Our result coincides with that of Čudakov, if the distance between negaton and positon is small, i. e. in the region where Čudakov's formula has its validity.

§ 2. Formulation of the problem

(i) Many theoretical works have been given by various authors aiming at the description of the averaged energy loss of a single ionizing particle. The work of Sternheimer,[6)] based on the original idea of Fermi,[7)] is in satisfactory agreement with the experimental results. But recently, P. Budini and L. Taffara[8)] pointed out that difficulties arise when this theory of energy loss is invoked to explain the phenomena such as the formation of tracks in nuclear emulsions, since the above theory does not tell, for example, what part of the total loss contributes to the ionization.

Though we are not blind to this fact, here we will not enter into the discussion of such a detailed behavior of ionization processes. We follow the idea of Fermi and assume that the ionization of negaton-positon pair is proportional to (or may be adjusted to) the average energy loss by the proper choice of a parameter: the average ionization energy.

(ii) Consider a negaton-positon pair created by a high energy γ-ray, the paths of two particles* forming a small angle θ. When we limit our attention to a small portion of the paths, we may safely regard them as the segments of two parallel lines separated by a distance $d=x\theta$, x being the path-length from the origin of the pair to this part. We consider only the region of paths, in which the multiple scattering of electrons does not essentially effect on the separation of electrons. (See § 5 (ii)). For the calculation of the grain density due to the ionization, this model will be sufficient.

(iii) Now, we consider the case in which the distance d is somewhat larger than the atomic size $b_1 \sim 10^{-8}$ cm.

Following Fermi, we devide the whole medium into two regions, I and II.

The region I consists of the inner part of two cylinders (radius b_1) whose axes coincide with the paths of electrons (Fig. 1).

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* Taking the Lorentz contraction of the time into account, it is easily shown that the spread of an electron-wave packet in flight (with the velocity nearly equal to c) is so small that a classical idea such as "path of particle" holds with sufficient accuracy.

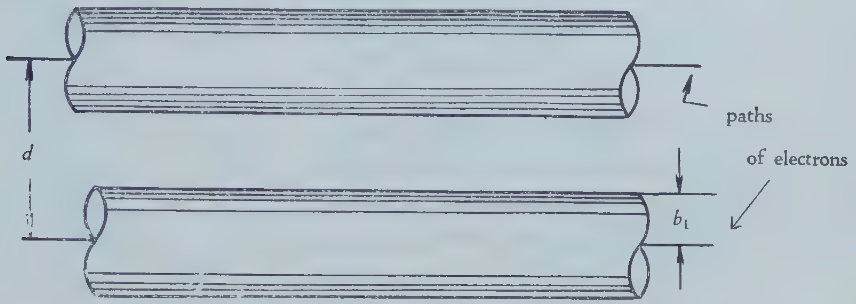

Fig. 1

Since in this region the atomic structure of the medium must be taken into account, Fermi used the theory of Bethe-Bloch, neglecting the effect of polarizability. Instead, we will use the method of Weizsäcker-Williams,[10) which is easily shown to reproduce the result of Bethe-Bloch if applied to the problem of a single electron.*

The region II is the outside part of the cylinders, which may be regarded as a continuum of the medium represented by a dispersion formula. For the problem of electron pair, it is anticipated that the polarization effect is not so important because the electric field is, roughly speaking, dipolar and of short range. Thus, it will be sufficient to use the dispersion formula in the approximation of a single oscillator.

§ 3. The region I

(i) Let us calculate the energy loss of the negaton-positon pair per unit path-length $-\Delta_{\mathrm{I}} E/\Delta x$ which is caused by their interaction with the atoms in the region I. Another part $-\Delta_{\mathrm{II}} E/\Delta x$ caused by the interaction with the region II will be calculated in the next section.

We neglect here the polarization effect, especially the reaction of the region II. Since the energy loss is small compared with the kinetic energy, we may, for a time, assume the velocity of electrons to be constant.** Adopting the Weizsäcker-Williams method, we should at first find the intensity distribution of the equivalent photons surrounding the electrons. (See Fig. 2). Fourier components of the field can immediately be written down (in the limit $v \to c$; v being the common velocity of two electrons),

$$\begin{aligned} \boldsymbol{E}_\nu &= (2e/\pi c^2\gamma)\nu\{K_1(\nu r_1/c\gamma)\,\boldsymbol{i}_1 - K_1(\nu r_2/c\gamma)\,\boldsymbol{i}_2\} \\ \boldsymbol{H}_\nu &= (2e/\pi c^2\gamma)\nu\{K_1(\nu r_1/c\gamma)\,\boldsymbol{j}_1 - K_1(\nu r_2/c\gamma)\,\boldsymbol{j}_2\}, \end{aligned} \tag{3·1}$$

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* For the precise reproduction, we have to take $b_0 = \sqrt{4/3.17}\sqrt{mc^2/E_{\max}}(\hbar/mc)$, where $E_{\max}$ is the maximum transferable energy. See § 3, for the meaning of b_0.

** In addition, we may assume the velocity of the two electrons to be equal since $|v_+ - v_-| \sim c(mc^2/E)^2$.

where e is the charge of positon, ν the frequency, and $\gamma=1/\sqrt{1-(v/c)^2}$. We have neglected the component of the electric field vertical to the paper in Fig. 2, since this has no contribution to the average flow of photons.

The electro-magnetic field in an extreme Lorentz-contraction may be compared to that of the plane waves of light* propagating with electrons, energy flow of which, if integrated over a long time interval, is given by

$$\int_{-\infty}^{\infty} S dt=(c/4)\int_0^{\infty}(\boldsymbol{E}_\nu\times\boldsymbol{H}_\nu)_x d\nu$$

$$=(e^2/\pi^2c^3\gamma^2)\int_0^{\infty}d\nu\,\nu^2\{K_1^2(\nu r_1/c\gamma)+K_1^2(\nu r_2/c\gamma)$$

$$-2K_1(\nu r_1/c\gamma)K_1(\nu r_2/c\gamma)\cos\varphi\} \qquad (3\cdot 2)$$

$$\equiv\int_0^{\infty}I(\nu)d\nu$$

Thus the effect of the electromagnetic field of the rapid electron pair, accumulated over a long time interval, is equivalent to the sweep of light waves with the intensity distribution $I(\nu)$ given by $(3\cdot 2)$.

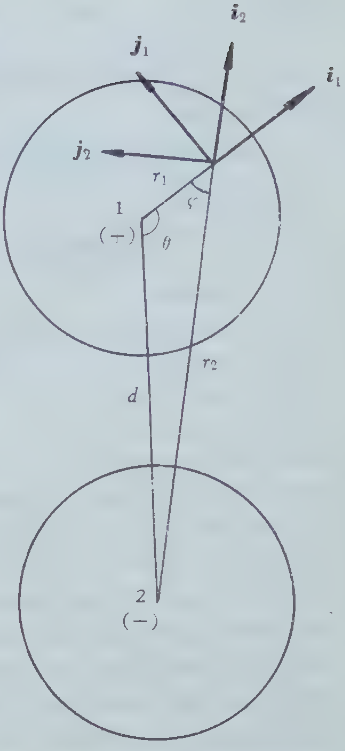

Fig. 2. The section of the cylinders of Fig. 1, electrons moving towards the reader.

The light waves will excite or ionize the atoms. The amount of energy $-\Delta_1 E$, transferred in this way to the atoms in the interval Δx (and also in the region I) of the medium, is

$$-\Delta_1 E/\Delta x=2\int_0^{2\pi}d\theta\int_{b_0}^{b_1}r_1dr_1N\sum_n(4\pi^2/3)(e^2/\hbar c)\nu_n|\boldsymbol{X}_{on}|^2I(\nu_n),$$

where the factor " 2 " in front corresponds to two cylinders. Or in terms of the oscillator strength f_n;

$$-\Delta_1 E/\Delta x=2\int_0^{2\pi}d\theta\int_{b_0}^{b_1}r_1dr_1 2\pi^2N(e^2/mc)\sum_n f_nI(\nu_n), \qquad (3\cdot 3)$$

where n symbolizes the excited state, N is the number density of atoms in the medium, and $b_0\sim\hbar/mc$ is the cut-off length characteristic of the Weizsäcker-Williams method.

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* The component of electric field vertical to the paper in Fig. 2, i. e. the longitudinal component is smaller than the transversal one by a factor $\sim 1/\gamma$. In addition, it is easily shown that $\boldsymbol{E}_\nu$ and $\boldsymbol{H}_\nu$ in (3·1), i. e. the transverse fields are perpendicular to each other, and have almost the same magnitude.

(ii) Substituting (3·2) in (3·3), we obtain

$$-\Delta_1 E/\Delta x = 2(2Ne^4/mc^2)(c\gamma)^{-2}\sum_n f_n \nu_n^2 \int_0^{2\pi} d\theta \int_{b_0}^{b_1} dr_1$$

$$\times r_1\{K_1^2(\nu_n r_1/c\gamma) + K_1^2(\nu_n r_2/c\gamma) - 2K_1(\nu_n r_1/c\gamma)K_1(\nu_n r_2/c\gamma)\cos\varphi\}. \tag{3·4}$$

Taking into account that f_n becomes very small if ionization to a state of high energy is concerned, we find for the terms important in the above summation*

$$\nu_n r_1/c\gamma \sim \nu_n r_2/c\gamma \sim \nu_n b_1/c\gamma < \nu_K b_1/c\gamma \sim 10^{-3}$$

(for 10 Bev electron energy)

where $\hbar\nu_K$ is the ionization energy of the K-shell electrons.

Then, using the approximate form $K_1(z) \sim 1/z$, we obtain for the first term in (3·4)

$$\int_0^{2\pi} d\theta \int_{b_0}^{b_1} dr_1 r_1 (c\gamma/\nu_n)^2 r_1^{-2} = 2\pi(c\gamma/\nu_n)^2 \log(b_1/b_0),$$

and for the second

$$\int_0^{2\pi} d\theta \int_{b_0}^{b_1} dr_1 dr_1 (c\gamma/\nu_n)^2 (d^2 + r_1^2 - 2dr_1\cos\theta)^{-1}$$

$$= \int_{b_0}^{b_1} (c\gamma/\nu_n)^2 2\pi r_1 dr_1 (d^2 - r_1^2)^{-1}$$

$$\simeq -\pi(c\gamma/\nu_n)^2 \log(1 - b_1^2/d^2), \quad (b_0 < d).$$

The third term is found to give a smaller contribution than the others by a factor $(\nu_n b_1/c\gamma)^2$, and so is neglected. Thus, it can be stated that the reduction effect is not important for the region I.

Since the ν_n^2 has cancelled out in the $\sum_n\cdots$, we can use the Thomas-Reiche-Kuhn sum-rule :[11)]

$$\sum_n f_n = Z$$

with atomic number Z of the atoms in question.

Thus denoting the number density of electrons NZ by n, we arrive at the following expression :

$$-\Delta_1 E/\Delta x = 2(2\pi ne^4/mc^2)\{2\log(b_1/b_0) - \log(1 - b_1^2/d^2)\} \tag{3·5}$$

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* When we make $d \to \infty$ in § 5, the terms, containing r_2 as arguments, themselves become unimportant. This situation is already reflected on the 1st term in the expansion.

§ 4. The region II

(i) In the case of polarizable medium, Fermi calculates the energy overflowing into the region II, and identifies it with the energy loss of an electron which is caused by the interaction with atoms in the region II (including Čerenkov radiation). Let us proceed along the same line.

Assume the dispersion law in the medium to be that of the single oscillator model,*

$$\boldsymbol{E}_\nu = 4\pi\Gamma(\nu)\boldsymbol{P}_\nu,\quad \Gamma(\nu) = (m/4\pi ne^2)(\nu_0^2-\nu^2-2ip\nu). \tag{4·1}$$

Then the electro-magnetic field of negaton-positon pair becomes[7] (see Fig. 3)

$$E_x = (e/\pi v^2)\int_{-\infty}^{\infty} i\nu\, d\nu\, (\Gamma/(1+\Gamma)-\beta^2) \times \{K_0(kr_1)-K_0(kr_2)\}\, e^{i\nu(x/v-t)} \tag{4·2}$$

$$H_\theta = -(e/\pi c)\int_{-\infty}^{\infty} i\nu\, d\nu\, k \times \{K_1(kr_1)-K_1(kr_2)\cos\varphi\}\, e^{i\nu(x/v-t)}$$

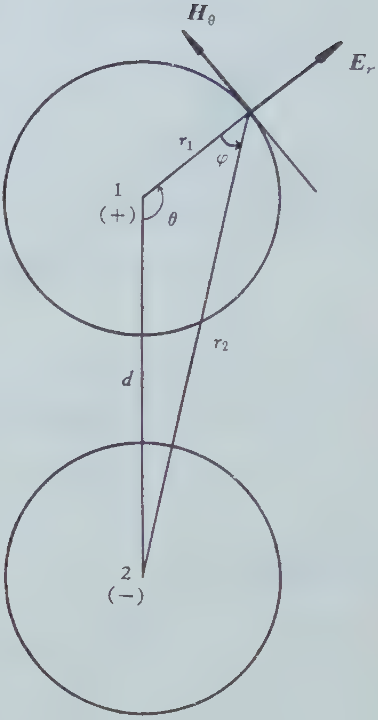

Fig. 3

where x-axis is directed towards the reader, $\beta \equiv v/c$ and

$$k^2 \equiv \nu^2(1-\beta^2)/v^2 - \nu^2/c^2\Gamma(\nu),\quad \mathrm{Re}\, k > 0. \tag{4·3}$$

Forming the Poynting vector outwards and integrating it over the cylindrical surfaces, $r_1 = b_1$ and $r_2 = b_1$, we obtain[7]

$$-\Delta_{II}E/\Delta x = 2(4ne^4/m\pi v^2(\varepsilon-1))\,\mathrm{Re}\int_0^{2\pi} b_1\, d\theta \int_0^{\infty} \{(1-x^2-2i\eta x)/(\varepsilon-x^2-2i\eta x)-\beta^2\}\, ix\, dx \times k^*\{K_1(k^*b_1)-K_1(k^*r_2)\cos\varphi\}\{K_0(kb_1)-K_0(kr_2)\} \tag{4·4}$$

where

$$\varepsilon \equiv 1+(\nu_p/\nu_0)^2,\quad \nu_p^2 \equiv 4\pi ne^2/m:\ \text{(plasma frequency)}^2.$$

$$\eta \equiv p/\nu_0,\quad x \equiv \nu/\nu_0,\quad r_2 \equiv (b_1^2+d^2-2b_1 d\cos\theta)^{1/2}.$$

(ii) Now, η may be regarded as the ratio of the width of the absorption band to the absorption energy. Since η is of the order of $1/100$ for AgBr,[12] for example, we may take the limiting value of (4·4) as $\eta \to 0$. As $\eta \to 0$, the pole of the integrand of (4·4) approaching the real axis, we should deform the path of integration as indicated in Fig. 4, and then we have a contribution from the residue $-i\pi \operatorname{Res}(\sqrt{\varepsilon})$. In the limit $\eta \to 0$, k is given by

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* The many oscillator model is equivalent to the single oscillator model if the approximation $K_0(z) \sim \frac{1}{2}\log(4/3.17z^2)$ is allowed. Then, $\log\nu_0 = \sum_n (f_n/Z)\log\nu_n$, Sternheimer[6] gave $\hbar\nu_0 = 21.0$ ryd. for emulsion.

$$k=-igx\{(x^2-a)/(1-x^2)\}^{1/2}$$

$$(\equiv -il),$$

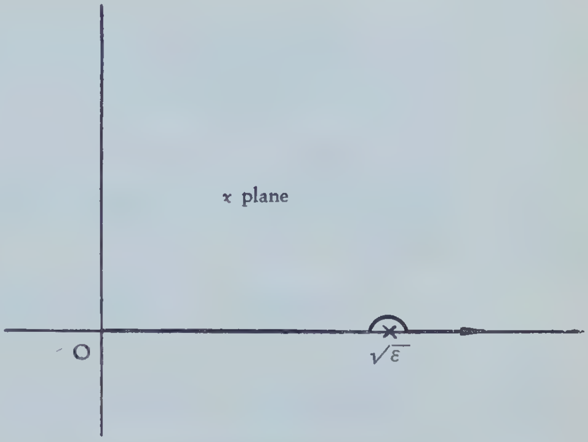

Fig. 4. The path of the integration in (4·4).

where

$$a=(c^2-v^2\varepsilon)/(c^2-v^2),$$

$$g^2=(\nu_0^2/v^2)(1-\beta^2)$$

$$\operatorname{Re} k \to +0,$$

and is imaginary for $0 \leqq x < 1$, real for $1 \leqq x$, since for extremely relativistic electrons $v > c/\sqrt{\varepsilon}$. Thus, noting that the modified Bessel functions for real argument are also real, we obtain for (4·4),

$$-\Delta_{\text{II}}E/\Delta x = A+B \tag{4·6}$$

$$A \equiv 2\cdot(4ne^4/m\pi v^2)(\varepsilon-1)^{-1}\int_0^{2\pi} b_1\,d\theta \int_0^1 dx\{(1-x^2)/(\varepsilon-x^2)-\beta^2\}$$

$$\times \operatorname{Re} ix(il)\{K_1(ilb_1)-K_1(ilr_2)\cos\varphi\}\{K_0(-ilb_1)-K_0(-ilr_2)\},$$

$$B \equiv 2\cdot(2ne^4/mv^2)\int_0^{2\pi} b_1\,d\theta\,\mu\{K_1(\mu b_1)-K_1(\mu r_2)\cos\varphi\}$$

$$\times\{K_0(\mu b_1)-K_0(\mu r_2)\},$$

B being the contribution from the residue at $x=\sqrt{\varepsilon}$, and

$$\mu=\nu_0\sqrt{\varepsilon}/v.$$

(iii) Now, let us calculate the integrals.

By using the expansion of Bessel functions[13] (cf. Fig. 5),

$$K_0(\mu r_2)=I_0(\mu b_1)K_0(\mu d)+2\sum_{m=1}^{\infty} I_m(\mu b_1)K_m(\mu d)\cos m\theta,$$

$$K_1(\mu r_2)\cos\varphi=-K_1(\mu r_2)\cos(\theta+\psi)=-\sum_{m=-\infty}^{\infty} I_m(\mu b_1)K_{m+1}(\mu d)\cos(m+1)\theta. \tag{4·7}$$

we obtain for B,

$$B=2(2ne^4/mv^2)2\pi\mu b_1[\{K_1(\mu b_1)+I_1(\mu b_1)K_0(\mu d)\}$$

$$\times\{K_0(\mu b_1)-I_0(\mu b_1)K_0(\mu d)\}$$

$$-\sum_{m=1}^{\infty}I_m(\mu b_1)I_{m-1}(\mu b_1)\{K_m(\mu d)\}^2]. \qquad (4\cdot8)$$

For A, the integration over θ is performed just in the same way as for B, and there comes in question,

$$\mathrm{Re}\cdot 2\pi\cdot ix(+il)[\{K_1(ilb_1)+I_1(ilb_1)K_0(ild)\}$$

$$\times\{K_0(-ilb_1)-I_0(-ilb_1)K_0(-ild)\}$$

$$-\sum_{m=1}^{\infty}I_{m-1}(ilb_1)I_m(-ilb_1)|K_m(ild)|^2]$$

$$=\mathrm{Re}\cdot 2\pi xl[(\tfrac{1}{2}\pi/lb_1)\{J_0(ld)-1\}$$

$$-\{I_1(ilb_1)I_0(-ilb_1)|K_0(ild)|^2$$

$$+\sum_{m=1}^{\infty}I_{m-1}(ilb_1)I_m(-ilb_1)|K_m(ild)|^2\}], \qquad (4\cdot9)$$

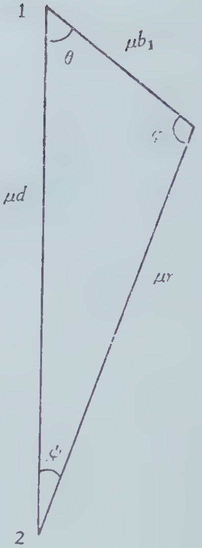

Fig. 5

where we have used the Wronskian relations[13]:

$$\mathrm{Re}\{K_1(iy)K_0(-iy)\}=\tfrac{1}{2}(\pi^2/4)i\{H_0^{(1)}(y)\cdot dH_0^{(2)}(y)/dy-H_0^{(2)}(y)\cdot dH_0^{(1)}(y)/dy\}$$

$$=\tfrac{1}{2}(\pi^2/4)i(-4i/\pi y)=\pi/2y,$$

and

$$\mathrm{Re}\{I_1(iy)K_0(-iy)K_0(iz)-I_0(-iy)K_1(iy)K_0(-iz)\}$$

$$=\tfrac{1}{2}\{-I_1(-iy)K_0(-iy)K_0(iz)-I_0(iy)K_1(iy)K_0(-iz)$$

$$-I_1(iy)K_0(iy)K_0(-iz)-I_0(-iy)K_1(-iy)K_0(iz)\}$$

$$=-(i/2y)\{K_0(iz)-K_0(-iz)\}$$

$$=-(\pi/2y)J_0(z), \quad \because \quad K_0(-iz)=\tfrac{1}{2}\pi iH_0^{(1)}(z)$$

$$=\tfrac{1}{2}\pi i\{J_0(z)+iY_0(z)\}.$$

The second curly bracket in the square bracket in (4·9) is pure imaginary and so gives us no contribution, since $I_m(\pm iy)=e^{-(1/2)m\pi i}J_m(\mp y)$.

Substituting (4·9) in A of (4·6) we obtain

$$A=2(4ne^4/m\pi v^2)(\varepsilon-1)^{-1}2\pi^2\tfrac{1}{2}$$

$$\times\int_0^1 dx\, x\{(1-x^2)/(\varepsilon-x^2)-\beta^2\}\{-1+J_0(ld)\}$$

$$=2(2\pi ne^4/mv^2)[\log\{\varepsilon/(\varepsilon-1)\}$$

$$-\tfrac{1}{2}\{(1-\beta^2)/(\varepsilon-1)\}\{1-\lambda d\cdot K_1(\lambda d)\}$$

$$+2K_0(\lambda d\sqrt{\varepsilon/(\varepsilon-1)})-2K_0(\lambda d)], \qquad (4\cdot10)$$

$$\lambda \equiv g\sqrt{-a} = (\nu_0/c)\sqrt{\varepsilon - 1}, \quad \lambda\sqrt{\varepsilon/(\varepsilon-1)} = \mu, \quad \text{for} \quad v \to c. \tag{4·11}$$

In the calculation of (4·10), we have adopted the approximation for (4·5):

$$l \equiv gx\sqrt{(x^2-a)}/(1-x^2) \doteqdot gx\sqrt{(-a)}/(1-x^2),$$

since for extremely relativistic electrons $v > c/\sqrt{\varepsilon}$, $-a$ is so large that $x(0 \leqq x \leqq 1)$ may safely be neglected. We have changed the variable of integration x to $t = x/\sqrt{1-x^2}$, and then used the formula[11)]:

$$\int_0^\infty \frac{x^{\nu+1} J_\nu(ax)}{(x^2+k^2)^{\mu+1}}\, dx = \frac{a^\mu k^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ak).$$

(iii) The sum of A and B, which is $-\Delta_{11} E/\Delta x$, (4·6), looks somewhat complicated. We mention here some facts that simplify the situation. Firstly, if it is allowed to expand the last term in B^*, (4·8), we obtain the following expression:

$$\begin{aligned}
&\mu b_1 \sum_{m=1}^{\infty} I_m(\mu b_1)\, I_{m-1}(\mu b_1)\, \{K_m(\mu d)\}^2 \\
&\quad \simeq \mu b_1 \sum_{m=1}^{\infty} \{(\tfrac{1}{2}\mu b_1)^m/m!\}\, \{(\tfrac{1}{2}\mu b_1)^{m-1}/(m-1)!\}\, \{\tfrac{1}{2}(m-1)!/(\tfrac{1}{2}\mu d)^m\}^2 \\
&\quad = \tfrac{1}{2}\sum m^{-1}(b_1/d)^{2m} \\
&\quad = -\tfrac{1}{2}\log(1 - b_1^2/d^2),
\end{aligned} \tag{4·12}$$

which has the counter-term to cancel in (3·5).

Secondly, expanding also the functions in B whose argument is μb_1, we obtain

$$\begin{aligned}
&\mu b_1 \{K_1(\mu b_1) + I_1(\mu b_1) K_0(\mu d)\}\, \{K_0(\mu b_1) - I_0(\mu b_1) K_0(\mu d)\} \\
&\quad \doteqdot \tfrac{1}{2}\log(4/3.17\, \mu^2 b_1^2) - K_0(\mu d),
\end{aligned} \tag{4·13}$$

where the terms of order μb_1 are neglected compared with 1.

Thirdly, in A (4·10), we may neglect the term proportional to $(1-\beta^2)/(\varepsilon-1)$, since this factor is smaller thon 10^{-}[illegible] for the electrons of energy above 10 Bev. in AgBr.

§ 5. Results and discussions

(i) Now, we have arrived at the goal. Using the equations (3·5); (4·8), (4·12), (4·13) and (4·10) altogether, we get for the energy loss of the relativistic electron pair in a polarizable medium

$$\begin{aligned}
-\Delta E/\Delta x &= -\Delta_1 E/\Delta x - \Delta_{11} E/\Delta x \\
&= 2(2\pi n e^4/mc^2)[-2\log\tfrac{1}{2}1\cdot78\,\mu b_0 - 2K_0(\lambda d) + \log\{\varepsilon/(\varepsilon-1)\}] \\
&= 2(4\pi n e^4/mc^2)[\tfrac{1}{2}\log(4c^2/3.17\,\nu_p^2 b_0^2) - K_0(\nu_p d/c)],
\end{aligned} \tag{5·1}$$

because, as is already mentioned,

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* $\mu b_1 \sim \{\hbar\nu_0/(e^2/2a_0)\}\{e^2/\hbar c\}\{b_1/2a_0\} \sim 21/137$, a_0: Bohr radius.

$$\lambda=(\nu_0/c)\sqrt{\varepsilon-1}=\nu_p/c, \quad \mu=(\nu_0/c)\sqrt{\varepsilon}. \tag{5·2}$$

The final result has a rather simple form, and in addition, it does not depend on b_1, i. e. on the division of the medium into the atomic and the continuum region. It is a very remarkable fact. The curves of (5·1) are given as a function of $x\,(d=x\Theta)$ in Fig. 6.

Let us examine the two limiting forms of our result (5·1). For the electron pair separated far away, $d\to\infty$,

$$-\Delta E/\Delta x=2\,(2\pi ne^4/mc^2)\cdot\log\left(4c^2/3.17\,\nu_p^{\,2}b_0^{\,2}\right) \tag{5·3}$$

which is twice the energy loss of a single electron, as is expected.

For small d, such that $\nu_p d/c\ll 1$,

$$-\Delta E/\Delta x=2\,(4\pi ne^4/mc^2)\log\left(d/b_0\right) \tag{5·4}$$

which is in agreement with Čudakov's result. Though he did not take the polarizability into account, this agreement for small d is not fortuitous, because the electro-magnetic field of such a dipolar source has a range so short that the polarization of the medium is not effective.

(ii) Estimation of the Energy of a γ-ray.

As already pointed out by Wolter et al.,5) this reduction of ionization is useful to estimate the angle of divergence of an electron pair and then the energy of the parent γ-ray. It is true that directly the reduction of ionization is determined by the separation

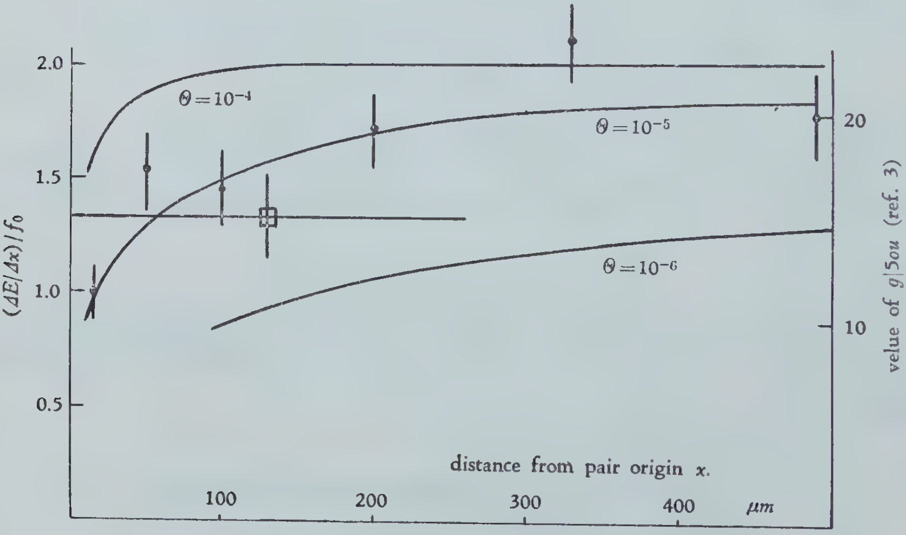

Fig. 6. (5·1) is graphed in the case of emulsion ($\hbar\nu_p=2.82$ Ryd.) and cut off energy $E_{\max}=5$ KeV, so $b_0=11.2\times$ (Compton wave length). The formula (5·1) reduces to $-\Delta E/\Delta x=(8\pi ne^4/mc^2)\times[7.18-K_0(1.94\times10^6\,d)]$, where $d=x\Theta$, and x is the path length in cm. f_0 corresponds to the single relativistic electron. In the case $\Theta\lesssim10^{-5}$, the curve can be approximated by (5·4). The experimental values are of Perkins,3) and of Wolter et. al.5)

d of the pair electrons, and according to Lohrmann,[14] the separation d up to distance $x \sim 100\,\mu$ from the origin of the pair is related with the angle of divergence θ as $d = x\theta$. (whereas at distance $x > 1000\,\mu$, the coulomb multiple scattering disturbs this relation.)

As to the relation between the angle of divergence and the energy of the parent γ-ray, one may refer to Borsellino[2] and Lohrmann.[14]

In fact, Wolter and Mięsowicz estimated the energy of a γ-ray in this way, obtaining the result that does not contradict with the value estimated from the energy development of the cascade. Their result (Fig. 6) seems to show that the statistical error of grain count does not prevent us at least from knowing the order of magnitude of the energy. As a measure of ionization, they took the coefficient of the exponential distribution of gap lengths.

The authors would like to express their sincere thanks to Professor H. Kanazawa, Professor K. Ono, and in particular to Professor M. Nogami for their valuable discussions. Their thanks are also due to Dr. M. Koshiba for suggesting this problem, and to the members of their institute for discussions. They are indebted to Dr. S. Endo who carefully read the manuscript and gave us valuable criticisms.

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Note added in proof. There appeared a paper on the same theme as ours: G. Yekutieli, Nuov. Cim. **5** (1957), 1381. His result is almost the same as ours. Some differences: (i) The treatment of the region I (ii) The adequacy of his expansion of the Bessel functions for region II is not self-evident. His result is essentially the same as Cudakov's, while ours is somewhat improved.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On the Line Width in Ferromagnetic Resonance

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August 2, 1957

In spite of several suggestions, the mechanism of the line width in ferromagnetic resonance is imperfectly understood. So far there have been many theories about it principally from the microscopic standpoints. But in view of the fact that the resonance condition depends critically on the demagnetization factors, and therefore on the geometry of the sample, it is likely that the resonance line width should be determined also by the macroscopic property of the sample. In this Letter we propose a new mechanism which may contribute to the line width and the relaxation phenomena at low temperatures.

In his microscopic theory for deriving the Kittel's fundamental formula,[1] van Vleck[2] has assumed that the demagnetization field is uniform in all portions of the sample. This means that the Zeeman frequency of each spin caused by the external field H which is set up along the z axis shifts equally by a factor corresponding to the uniform demagnetization field. The above presumption seems to be not always true, and the demagnetization field must be different for different portions of the specimen; it is considered to be more intense in the edge than in the interior of the sample. Thus each spin precesses with different Lamor frequency, giving rise to the finite width of the resonance lines. Of course the distribution of the uniform value of the demagnetization field is considered to be most frequent, and it might correspond to the peak of the resonance lines which is determined by the Kittel-van Vleck formula. The line width thus caused is evidently independent of temperature, and hence it seems to contribute to the explanation of the constant line width.

Considering the non-uniformity of the demagnetization field, the resonance frequency ω_j for the j-th part of the sample is

$$\omega_j^2=\gamma^2\{H+(N_{jx}-N_{jz})M\}\times\{H+(N_{jy}-N_{jz})M\}, \qquad (1)$$

where M is the z component of the intensity of the magnetization, γ the gyro-magnetic ratio and N_j the demagnetization factor which depends on j in opposition to van Vleck's assumption. If we write

$$N=N_j+\Delta N_j, \qquad (2)$$

the first term N is the average value of N_j and equivalent to the one used by van Vleck. We have no actual knowledge of the distribution of ΔN_j or ω_j, which might result from the inhomogeneities of the domain, the elastic properties, etc., and not always from the presence of the impurity. So we cannot know the line width quantitatively. Here we shall assume

$$\Delta N_j = C_j N, \tag{3}$$

where C_j is some constant, and the above estimation seems to be reasonable except the case of the saturation of ΔN_j[3]. Then equation (1) may be written in the form

$$\omega_j^2 = \gamma^2 \{H_1^e + (\Delta N_{jx} - \Delta N_{jz}) M\}$$
$$\times \{H_2^e + (\Delta N_{jy} - \Delta N_{jz})\} M\}, \tag{4}$$

where

$$H_1^e = H + (N_x - N_z) M,$$
$$H_2^e = H + (N_y - N_z) M.$$

As have been done in many experiments, we shall hereafter denote H_j in place of ω_j for describing the resonance condition for each part of the sample. In order to obtain the geometry dependency of the line width, we shall consider the case when a magnetic field is applied perpendicular and parallel to the plane of a disk and compare the line width.

(I) $H \perp$ plane of the specimen:

$$N_z = 4\pi, \quad N_x = N_y = 0,$$
$$H_1^e = H_2^e = H - 4\pi M;$$
$$\Delta H_j = H_j - H_0 = -(\Delta N_{j1} + \Delta N_{j2}) M/2$$
$$+ \sqrt{(\Delta N_{j1} - \Delta N_{j2})^2 M^2/4 + \omega^2 \gamma^{-2}} - \omega \gamma^{-1}, \tag{5}$$

where H_0 is the resonance field for the peak and it is given by

$$H_0 = 4\pi M + \omega \gamma^{-1}, \tag{6}$$

and where

$$\Delta N_{j1} = \Delta N_{jx} - \Delta N_{jz},$$
$$\Delta N_{j2} = \Delta N_{jy} - \Delta N_{jz}. \tag{7}$$

(II) H // plane of the specimen:

$$N_y = 4\pi, \quad N_x = N_z = 0,$$
$$H_1^e = H, \quad H_2^e = H + 4\pi M$$
$$\Delta H_j = -(\Delta N_{1j} + \Delta N_{2j}) M/2$$
$$+ \sqrt{(\Delta N_{1j} - \Delta N_{2j} - 4\pi)^2 M^2/4 + \omega^2 \gamma^{-2}}$$
$$- \sqrt{4\pi^2 M^2 + \omega^2 \gamma^{-2}} \tag{8}$$

where

$$H_0 = -2\pi M + \sqrt{4\pi^2 M^2 + \omega^2 \gamma^{-2}}. \tag{9}$$

By considering the equation (3), in the case (I), $\Delta N_z > \Delta N_x$, ΔN_y and in the case (II), $\Delta N_y > \Delta N_x$, ΔN_z. From (5) and (8) we see that the line width is broader in the case (I) than that in the case (II), and this seems to be verified in experiments owing to Bloembergen and Wang.[4] Further as is easily imagined, C_j in the equation (2) is larger in ferrites than in metals, and it seems to be also verified by their experiments. For the absolute magnitude of the line width, we might say that line width is of the same order of magnitude as the shift by the demagnetization field or less than that.

The shape dependence of the line width seems to be not explained by the microscopic point of view such as the spin wave picture.

In conclusion the author wishes to express his cordial thanks to Prof. Tomita for his kind interest and illuminating discussions.

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E2 + M1 Mixed Transitions in Heavy Odd A Nuclei

M. Kawamura

Prog. Theor. Phys. **18** (1957), 87 (*L*)

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Some Considerations on the High-Energy Electron-Proton Scattering and the Nucleon Mass Difference

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The recent high-energy electron-proton scattering experiments at Stanford are analysed in connection with the small mass difference of the nucleon. If nucleons are rigid bodies, then it is shown that a consistent interpretation is given when we ascribe to the electron a finite size of the same order as those of nucleons. Discussions of this result are extended to other related phenomena.

§ 1. Introduction

The recent experiments on the scattering of high-energy electron by hydrogen at Stanford[1)2)] will be analysed in connection with the neutron-proton mass difference. For this purpose, we shall first consider the scattering of high-energy electron by the proton. The experimental scattering cross section does not agree with the theoretical value where the radiative corrections and the existence of the large anomalous magnetic moment (a.m.m.) of the proton are taken into account. For this discrepancy between the experimental and the theoretical differential cross sections, two alternative explanations are presented[1)] :

(I) The proton is not a point, but has a finite extent. The Coulomb law itself holds even at small distances.

(II) At small distances there is a break-down of the Coulomb law, but the particle retains the point character.

Phenomenologically, these alternative interpretations cannot be distinguished from each other by the experiment of the electron-proton scattering alone. We shall take the first possibility and talk in terms of the finite-extent in the following[*)]. The theoretical scattering cross section for the electron and the proton with finite sizes, besides the radiative corrections (Schwinger correction[3)]), can be given by the Rosenbluth' formula[4)] :

$$\frac{d\sigma}{d\Omega}=\left(\frac{e^2}{8\pi E}\right)^2\frac{\cos^2\dfrac{\theta}{2}}{\sin^4\dfrac{\theta}{2}}\frac{1}{1+2\dfrac{E}{M}\sin^2\dfrac{\theta}{2}}$$

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[*] Of these two alternatives, (I) may be regarded as the wider standpoint. This will be seen if one recalls the fact that the second picture can be reduced to the first one when the sizes of all the charged particles interacting with the photon are the same.

$$\times\left\{F^2+\frac{q^2}{4M^2}\left[2(F+\mu_pG)^2\tan^2\frac{\theta}{2}+\mu_p{}^2G^2\right]\right\}H^2, \tag{1·1}$$ [*]

where F, G and H are form factors introduced to take account of the diffuse distributions of the proton charge, the proton a.m.m. and the electron charge, respectively.

As will be seen from (1·1), the scattering experiment cannot give any individual information about each form function F, G or H, but it tells only the behaviors of the products

$$FH \quad \text{and} \quad GH. \tag{1·2}$$

Thus as far as we investigate the electron-proton scattering alone, we cannot distinguish following two possible refinements from each other :

(I–I) The nucleon, having strong interactions with other fields, has diffuse charge and diffuse a.m.m. On the contrary, the charge distribution of the electron, which has no strong interaction with other fields, is a point.

(I–II) Both the nucleon and the electron have their finite sizes.

Fortunately we can further find other phenomena appropriate for the detailed discussions about this problem. The small mass differences in the elementary particles will be helpful for this purpose. For the present we shall confine ourselves to the neutron-proton mass difference.

In order to proceed our investigation we shall take two assumptions :

(A) The neutron-proton mass difference is due solely to the electromagnetic effects[5]. This assumption is not unreasonable if one recalls a successful attack by Gell-Mann and Nishijima[6] to the variety of the strange particle processes under the charge independence hypothesis.

(B) The propagation functions of the nucleon, in the strong interaction with other fields, can statisfactorily be approximated by the conventional propagation function for the point nucleon. The interactions of the nucleon and of the electron with the photon are respectively described by the Hamiltonians[**]

$$\left\{\begin{aligned} H_{\text{proton}}=&-ie\int dxdy\bar{\psi}_p(x)\gamma_\mu\psi_p(x)F_p(\boldsymbol{x}-\boldsymbol{y})\delta(x_0-y_0)A_\mu(y)\\ &-\frac{1}{2}\left(\frac{e\mu_p}{2M}\right)\int dxdy\bar{\psi}_p(x)\sigma_{\mu\nu}\psi_p(x)G_p(\boldsymbol{x}-\boldsymbol{y})\delta(x_0-y_0)F_{\mu\nu}(y),\end{aligned}\right.$$

---

[*] For the notations, see reference 1).

[**] One may want to take covariant (4-dimensional) form functions instead of 3-dimensional ones, which we took here only for simplicity's sake. As will be seen in § 4, this causes no essential difference for the present problem. Even for the acceptance of covariant form functions, the Hamiltonian like (1·3) injures the microscopic causality etc., of the usual quantum field theory. This difficulty however seems to be solved at the more profound root, and before we enter into these problems, it might be necessary to make a rough but intuitively clear picture on the particle structure. We expect that our treatment serve to give a clue to the future theory.

$$\left\{\begin{aligned} H_{\mathrm{neutron}} &= -\frac{1}{2}\left(\frac{e\mu_n}{2M}\right)\int dxdy\bar{\psi}_n(x)\,\sigma_{\mu\nu}\psi_n(x)\,G_n(\boldsymbol{x}-\boldsymbol{y})\,\delta(x_0-y_0)\,F_{\mu\nu}(y)\,^{*)}, \\ H_{\mathrm{electron}} &= -ie\int dxdy\bar{\psi}_e(x)\,\gamma_\mu\psi_e(x)\,H(\boldsymbol{x}-\boldsymbol{y})\,\delta(x_0-y_0)\,A_\mu(y), \\ \sigma_{\mu\nu} &= \frac{1}{2i}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu) \quad \text{and} \quad F_{\mu\nu}(y)=\partial_\mu A_\nu(y)-\partial_\nu A_\mu(y). \end{aligned}\right. \tag{1·3}$$

This assumption (B) means that the nucleon is a rigid body.

In Section 2 we shall calculate the neutron-proton mass difference, and show that, by comparing the results with those from the electron-proton scattering, a possibility (I–I) would be excluded. In Section 3, a plausible interpretation is given from the standpoint (I–II). It might be suggested that the spatial extents of the electron and of the nucleon are of the same order. It will be shown in Section 4 that these spatial extents are compatible with the experimental results of the a.m.m. of the electron and of the Lamb-Retherford shift of the energy levels in hydrogen.

§ 2. Neutron-proton mass difference

Assuming (A) and (B) our calculation of the neutron-proton mass difference will be seen to result in that the standpoint (I–I) might not explain the electron-proton scattering and the neutron-proton mass difference in a consistent way.

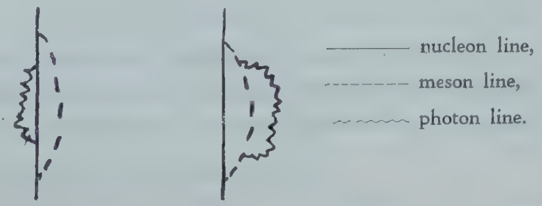
With (A) and (B), the nucleon mass difference is given in e^2-approximation[5]:

$$\begin{aligned} \varDelta M = \frac{e^2}{(2\pi)^4}\int\frac{dk}{k^2}\Bigg\{&\left[\gamma_\mu F_p(\boldsymbol{k})+\frac{i\mu_p}{4M}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)G_p(\boldsymbol{k})\right] \\ &\times\frac{1}{\gamma p-\gamma k-iM}\left[\gamma_\mu F_p(\boldsymbol{k})-\frac{i\mu_p}{4M}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)G_p(\boldsymbol{k})\right] \\ &-\left(\frac{\mu_n}{4M}\right)^2 G_n^{\,2}(\boldsymbol{k})\,(\gamma_\mu\gamma k-\gamma k\gamma_\mu)\frac{1}{\gamma p-\gamma k-iM}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)\Bigg\}\Bigg|_{\gamma p=iM}, \end{aligned} \tag{2·1} ^{**)}$$

---

[*] In this Hamiltonian, the charge distribution of the neutron is assumed to be zero, i.e. $F_n(\boldsymbol{x})\equiv 0$. This does not contradict with the fact that the observed electron-neutron interaction is explained by the so-called Foldy term alone within the experimental error. See for example, H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (1955) Vol. II, §46f.

[**] According to the assumption (B), in eq. (2·1) the following sorts of the Feynman diagram are omitted:



Contributions of these classes of processes should also be estimated.

where M is the mass of the nucleon in interaction with all the other fields except the photon ; and μ_p and μ_n are the a.m.m. of the proton and the neutron in nuclear magneton respectively[7] :

$$\mu_p=1.79274 \qquad \text{and} \qquad \mu_n=-1.91314.$$

F_p, G_p and G_n are the Fourier transforms of the form functions for the diffuse distribution of the proton charge and the diffuse distributions of the a.m.m. of the proton and of the neutron respectively.

Assuming the spherically symmetric form functions and noting that[8]

$$\int dk=\int_0^\infty dKK^2\int d\Omega\int_{-\infty}^\infty dk_0\,,$$

we get, exclusive of the final integration over K, for the neutron-proton mass difference ΔM:

$$\begin{aligned}\Delta M=\frac{e^2}{(2\pi)^2}M\Bigg\{&-\int_0^\infty dRF_p^2(MR)\left[R+\frac{2}{\sqrt{R^2+1}}-\sqrt{R^2+1}\right]\\&+3\mu_p\int_0^\infty dRF_p(MR)G_p(MR)\left[\sqrt{R^2+1}-\frac{1}{\sqrt{R^2+1}}\right]\\&+\frac{5}{4}\mu_p^2\int_0^\infty dR\,G_p^2(MR)\left[\sqrt{R^2+1}-\frac{1}{\sqrt{R^2+1}}\right]\\&-\frac{5}{4}\mu_n^2\int_0^\infty dRG_n^2(MR)\left[\sqrt{R^2+1}-\frac{1}{\sqrt{R^2+1}}\right]\Bigg\},\end{aligned} \tag{2·2}$$

where $R=K/M$. The first three terms in (2·2) concern with the proton, and the last term is the contribution from the neutron by the a.m.m.—a.m.m. interaction through the photon. Inthe special case of

$$F_p=G_p=G_n=1\,,$$

(2·2) tends to the mass difference for the conventional point model of the nucleon.

For the moment we assume that :

(C) The spatial extent of the a.m.m. of the neutron is the same as that of the proton, namely

$$G_p=G_n, \tag{2·3}$$

which will presently be shown harmless to the conclusion of this paper.

As was demonstrated from the analysis of the experimental data of the electron-proton scattering worked out by Chambers and Hofstadter[1],

$$G_pH=F_pH \qquad \text{or} \qquad G_p=F_p\,, \tag{2·4}$$

which is also to be invoked.

Thus we have calculated the neutron-proton mass difference for different types of form functions and determined the nucleon radius $\boldsymbol{b}$. The rms radius*) of the proton $\boldsymbol{a}$, obtained from the electron-proton scattering assuming the point electron, is also included in Table I for comparison. There is seen a clear-cut difference between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Table I. Summary of the models and values of rms radii of the nucleon. Equal radii for F_p, G_p and G_n are assumed. The second column presents the rms radius $\boldsymbol{a}$ of the proton determined by Chambers and Hofstadter from the electron-proton scattering, assuming the point electron. In the 3rd column the rms radius $\boldsymbol{b}$ of the nucleon is given, which is determined from the neutron-proton mass difference.

type of form fnuction	$\boldsymbol{a}$ *e-p* scattering (in 10^{-14}cm)	$\boldsymbol{b}$ *n-p* mass difference (in 10^{-14}cm)
e^{-r^2}	7.2±0.5	3.5
r^2e^{-r}	7.5±0.5	3.6
$r\,e^{-r}$	7.8±0·5	3.8
e^{-r}	8.0±0.5	4.2
$\frac{1}{r}e^{-r}$	—	5.5

Fig. 1 describes $\varDelta M$ as a function of the nucleon rms radius when we take, as an example, an exponential function for the form factors**) i.e.

$$F_p(r)=G_p(r)=G_n(r)=(3\sqrt{3}/\pi\boldsymbol{b}^3)\exp(-2\sqrt{3}\,r/\boldsymbol{b}).$$

A glance over the figure shows that a considerable fluctuation in $\varDelta M$ is insensible for the value of $\boldsymbol{b}$. If we take as the rms radius of the nucleon the value $\boldsymbol{a}=(8.0\pm0.5)\times10^{-14}$cm, then we obtain

$$\varDelta M<0\,,$$

contrary to the observed value

$$\varDelta M=2.53\,m \qquad (m:\ \text{rest energy of the electron}).$$

This means that the point electron model has to be abandoned. The situation is quite independent of the type of distributions, as will be seen from the table.

The above result is obtained under the assumption (C)—$G_p(r)=G_n(r)$, which is now examined. The fourth term in $(2\cdot2)$, that is the only contribution from the interaction of the neutron with the photon, is negative definite for any $G_n(r)$. The examination is completed by running the rms radius for the a.m.m. of the neutron from zero to infinity. Thus assuming that $F(r)=G_p(r)\propto\exp(-2\sqrt{3}\,r/\boldsymbol{a})$, we get

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[*] "rms radius" is the abbreviation of "root-mean-square radius".

[**] For the various types of normalized form functions in terms of the rms radius, refer to the Hofstadter's work[1].

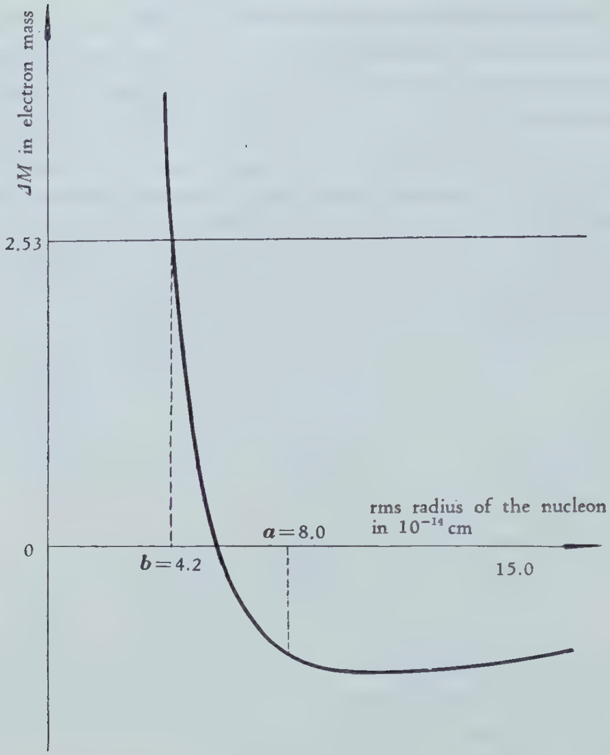

Fig. I. The neutron-proton mass difference, shown as a function of the rms radius of the nucleon, taking the exponential type for the form function. Equal radii are assumed for F_p, G_p and G_n.

$$0.71\,m > \Delta M \quad \text{for any } G_n\,, \tag{2·5}$$

as far as we take the rms radius of the proton to be equal to $\boldsymbol{a}=(8.0\pm0.5)\times10^{-14}$ cm, which fits best to the electron-proton scattering data for the point electron. The situation is entirely the same when we assume other type of functions for $F_p(r)$ and $G_p(r)$.

We may therefore conclude that under the assumptions (A) and (B) the standpoint (I–I) does not reconcile the two experiments, i.e. the neutron-proton mass difference and the electron-proton scattering.

§ 3. A consistent interpretation of the *e–p* scattering and of the *n–p* mass difference

In the previous section there was exhibited a discordance between the rms radii, $\boldsymbol{a}$ and $\boldsymbol{b}$. This leads us to the 2nd picture (I–II). Besides the assumptions (A), (B) and (C), we assume for simplicity's sake that

(D) All the distributions F_p, G_p, G_n and H are of the Gaussian type. Denoting the rms radii of the nucleon and the electron by $\boldsymbol{b}$ and $\boldsymbol{x}$ respectively, the Fourier transforms of the form functions F_p and H are expressed, in terms of them, as follows[1]:

$$\begin{cases} F_p(MR) = \exp(-\boldsymbol{b}^2M^2R^2/6), \\ H(MR) = \exp(-\boldsymbol{x}^2M^2R^2/6). \end{cases} \tag{3·1}$$

As was mentioned in Section 1, the experiment of electron-proton scattering does not give any knowledge about the individual form functions $F_p(=G_p)$ or H, but it does about their product:

$$F_p(MR)\,H(MR) = \exp(-[\boldsymbol{b}^2+\boldsymbol{x}^2]M^2R^2/6) = \exp(-\boldsymbol{a}^2M^2R^2/6). \tag{3·2}$$

From the analysis of the scattering data, we get

$$\boldsymbol{a}^2 = \boldsymbol{b}^2+\boldsymbol{x}^2 = (7.2\pm0.5)^2\times10^{-28}\,\mathrm{cm}^2. \tag{3·3}$$

In order to give the observed neutron-proton mass difference, the rms radius of the nucleon is required to be

$$\boldsymbol{b} = 3.5\times10^{-14}\,\mathrm{cm}. \tag{3·4}$$

From (3·3) and (3·4), we obtain the rms radius of the electron:

$$\boldsymbol{x} = (6.3\pm0.6)\times10^{-14}\,\mathrm{cm}. \tag{3·5}$$

According to the assumption (D), we have chosen the Gaussian type for the form functions. Other types of form functions, which fit well to the scattering data, can also be shown to give similar results. Thus we may conclude that the electron has a finite size as well as the nucleon and its rms radius is of the same order as that of the latter.

§ 4. Discussions

1. *Lamb shifts in hydrogen*

As is well known, the theoretical value for the $2s-2p$ level separation in hydrogen, to be compared with the observed value 1057.77 ± 0.10 *Mc/sec*[9], is 1057.19 ± 0.2 *Mc/sec*[10], including the 4th order radiative corrections. The discrepancy between this value and the observed one is about 0.6 *Mc/sec*. Its source is not known at the present time, but it might well be found in an improved estimate of the many small corrections of this order of magnitude. However the question is still open. Here we shall examine the standpoint (I) with reference to this problem.

The largest correction to the level shift due to the finite size will come from the charge-charge interaction between the two particles. Since the particles have finite sizes, the effective potential between them, V, differs from the Coulomb potential $V_c=-e^2/4\pi\cdot1/r$, by δV. The Schrödinger eqs. for the hydrogen with potentials V_c and $V=V_c+\delta V$ are

$$\begin{cases} \left(-\frac{1}{2\mu}\Delta+V_c\right)\phi_c = E_c\phi_c, \\ \left(-\frac{1}{2\mu}\Delta+V\right)\phi = E\phi, \end{cases} \tag{4·1}$$

respectively, where the suffix c is the abbreviation for Coulomb. μ is the reduced mass of the system. Putting $\phi-\phi_c=\delta\phi$ and $E-E_c=\delta E$, (4·1) leads to an expression for δE

$$\delta E=\frac{1}{1+(\phi_c,\ \delta\phi)}[(\phi_c,\ \delta V\phi_c)+(\phi_c,\ \delta V\delta\phi)]\fallingdotseq(\phi_c,\ \delta V\phi_c). \tag{4·2}$$

Assuming again that the form functions of the electron and the proton are of the Gaussian type, the effective potential V is

$$V=-\frac{e^2}{4\pi}\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\frac{1}{\boldsymbol{k}^2}e^{-\frac{a^2k^2}{6}}e^{i\boldsymbol{kx}} \tag{4·3}$$

where $\boldsymbol{a}^2=\boldsymbol{b}^2+\boldsymbol{x}^2$. Recalling the fact

$$(2s|V_c|2s)=(2p|V_c|2p), \tag{4·4}$$

we can calculate a contribution to the $2s-2p$ level separation due to the finite extent of the particles:

$$\Delta E=(2s|\delta V|2s)-(2p|\delta V|2p)=\frac{e^2}{48\pi}\frac{\boldsymbol{a}^2}{d^3}+0(e^2\boldsymbol{a}^4/d^5), \tag{4·5}$$

where $d=4\pi/\mu e^2$ is the radius of the first Bohr orbit. Inserting the values in Table I for $\boldsymbol{a}$, the correction amounts to

$$\Delta E\fallingdotseq(0.10\pm0.02)M_c/\text{sec}.$$

With use of the other type of form functions we can obtain values of the same order. This is derived independently of the assumptions (A)—(D) which are invoked for the calculations of the neutron-proton mass difference.

Our implication is that the corrections, which arise from the finite size of the particle, are of the same order as the present discrepancy between the theoretical and the observed values, and they reduce the discrepancy to 0.5 Mc/sec, which might be explained if we considered the higher order corrections.

2. *Covariant form functions**)

Up to now we have treated the particles as having their three-dimensional extents, and the discussions so far developed are based on the interaction Hamiltonian (1·3). It might be of interest to investigate what differences occur when we take either, namely, 3- or 4-dimensional form function*) for the neutron-proton mass difference.

Now ΔM is calculated from

(2·1) with $F_p(\boldsymbol{k})$, $G_p(\boldsymbol{k})$ and $G_n(\boldsymbol{k})$,
replaced by the covariant functions (4·6)
$F_p'(k^2)$, $G_p'(k^2)$ and $G_n'(k^2)$, respectively.

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* See the footnote for (1·3).

For the comparison with the three dimensional treatment, it seems reasonable to define the rms radius of the particle in this case as follows:

$$\boldsymbol{b}' = \left[4\pi\int_0^\infty dr\, r^4\, F'_p(r)\right]^{\frac{1}{2}} \quad \text{etc.,} \tag{4·7}$$

where

$$r = \sqrt{\boldsymbol{x}^2}, \quad F'_p(r) = \int_{-\infty}^{\infty} dx_0\, F'_p(x) \quad \text{and} \quad 4\pi\int_0^\infty dr\, r^2\, F'_p(r) = 1.$$

As an illustration we take the Yukawa-type distributions* for the particles:

$$\begin{aligned} F'_p(x) &= \frac{1}{(2\pi)^4}\int dk\left[1+\frac{\boldsymbol{b}'^2k^2}{6}\right]^{-1} e^{ikx}, \\ H'(x) &= \frac{1}{(2\pi)^4}\int dk\left[1+\frac{\boldsymbol{x}'^2k^2}{6}\right]^{-1} e^{ikx} \quad \text{etc.,} \end{aligned} \tag{4·8}$$

where $\boldsymbol{b}'$ and $\boldsymbol{x}'$ are rms radii for the distributions $F'(x)$ and $H'(x)$ respectively. Using (4·6) and $F_p'=G_p'=G_n'$, we can estimate $\boldsymbol{b}'$, the nucleons radius. Fixing $\boldsymbol{b}'$, we can analyse the electron-proton scattering data by varying the parameter $\boldsymbol{x}'$ in H'. $\boldsymbol{x}'$, which gives best fits to the scattering data, is seen in Table II. If we take the point electron and ascribe the finite size effect exclusively to the proton, the rms radius of the proton appears as $\boldsymbol{a}'$:

$$\begin{aligned} &\boldsymbol{a}'^2 = \boldsymbol{b}'^2 + \boldsymbol{x}'^2 && \text{for the covariant form factors} \\ &(\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{x}^2 && \text{for the non-covariant form factors).} \end{aligned} \tag{4·9}$$

Table II. Comparisons of the rms radii of the nucleon and the electron, taking the 3- and 4-dimensional Yukawa type form functions.

	Nucleon radius in 10^{-14} cm	Electron radius in 10^{-14}cm	Nucleon radius (point electron) in 10^{-14}cm
3-dimensional	5.5	5.8±0.7	8.0±0.5
4-dimensional	3.3	7.5±0.7	8.3±0.8

In the Table there are also added the parameters $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{x}$. From comparisons of $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{x}'$ with $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{x}$, it is seen that there is no remarkable difference between them. The situation is entirely the same if we take other type of form functions.

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* Assuming the point electron, distributions which have the singularity at the origin should be excluded from the electron-proton scattering experiment[1]. In the case of the electron with a finite size, the situation is somewhat different. If its size is of the same order as that of the proton, then certain distributions with the singularity at the origin are also allowable.

3. *Anomalous magnetic moment of the electron*

The theoretical value for the a.m.m. of the electron, including the 4th order radiative corrections[11], is 0.0011454 μ_B (μ_B : Bohr magneton). The experimental values are

$$(0.001148 \pm 0.000006)\,\mu_B{}^{12)},$$

$$(0.001165 \pm 0.000011)\,\mu_B{}^{13)},$$

which are therefore in agreement with the theoretical predictions. Then does the attribution of the finite size to the electron of the order of $10^{-13}\,\mathrm{cm} \sim 10^{-14}\,\mathrm{cm}$ change the situation? It will be shown below that this is not the case.

Assume the 4-dimensional Yukawa type distribution for the electron. The contribution from this diffused electron to a.m.m. is obtained, in the e^2-order, from the evaluation of the integral :

$$\int dk \frac{\gamma_\nu [i\gamma(p_2-k)-m](\gamma A)[i\gamma(p_1-k)-m]\gamma_\nu}{[(p_2-k)^2+m^2][(p_1-k)^2+m^2]} \left[1+\frac{x'^2k^2}{6}\right]^{-2} \left[1+\frac{x'^2(p_1-p_2)^2}{6}\right]^{-1}$$

$$\simeq \frac{i\pi}{m^2}(\sigma\cdot[\boldsymbol{A}\times(\boldsymbol{p}_2-\boldsymbol{p}_1)])\left[1-\frac{2}{9}m^2\boldsymbol{x}'^2\right]. \qquad (4\cdot10)$$

The additional contribution due to the finite size amounts only to

$$\Delta\mu_e \doteqdot -\frac{e^2}{8\pi^2}\frac{2}{9}m^2\boldsymbol{x}'^2\mu_B \doteqdot -10^{-9}\,\mu_B\,. \qquad (4\cdot11)$$

This is negligibly small and is not enough for the comparison of the current theory of point model with the experiment. Whether they are 3- or 4-dimensional, form functions of other type give similarly negligibly small contributions to the a.m.m. of the electron.

We have carried out in this paper an analysis of the high-energy electron-proton scattering and of the neutron-proton mass difference from the standpoint (I)—the particle is not a mere point but has a finite size or structure, while the Coulomb law itself holds even at a small distance—under the assumptions (A) and (B). For a consistent explanation it might be required for the electron, as well as the nucleon, to have a finite size, the magnitude of which is of the same order as that of the nucleon. The validity of the present arguments rely on (2·1) and (4·6), invoked in the calculation of mass difference.

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Note added in proof. After the completion of this work the following papers have been published

1. D.R. Yennie, M.M. Levy and D.G. Ravenhall (Rev. Mod. Phys. **29** (1957), 144) have discussed the theoretical implications of various experiments relating to the electromagnetic structure of nucleons, in the light of the current field theory. They concluded that there would be some inconsistency in the present field theory at small distances. Within the approximation where nucleons are rigid bodies with certain electromagnetic structures, their conclusion agrees with ours.
2. W. Aron and A. J. Zuchelli (Phys. Rev. **105** (1957), 1681) have also calculated the contribution to the Lamb shift due to the finite proton size obtained from the Stanford experiments, and given for the correction;

   $$\Delta E^{\text{finite size}} = 0.118 \pm 0.03 \text{ Mc/sec},$$

   which is in agreement with our result.
3. C.M. Sommerfield (Phys. Rev. **107** (1957), 328) has redone in the present instance the calculation of the magnetic moment of the electron to the fourth order and obtained

   $$1.0011596\ \mu_B,$$

   instead of the older one

   $$1.0011454\ \mu_B.$$

   The preference of the former is also suggested by A. Petermann, Nuclear Physics **3** (1957), 689. This new value improves the theoretical value for the Lamb shift and reduces the discrepancy to

   $$0.22 \pm 0.1 \text{ Mc/sec}.$$

   The finite size effects will further reduces it to

   $$\sim 0.1 \text{ Mc/sec}.$$

4. In the present paper, we have analysed the n-p mass difference under the assumption that

   $$G_p = G_n.$$

   One might think that the assumption would have some experimental foundation, R. Hofstadter, Proc. of the 7th Rochester Conference.

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On the Lattice Model of Liquid Helium proposed by Matsubara and Matsuda

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Recently, Matsubara and Matsuda[1)] proposed a lattice model of liquid helium. In their papers, the derivation of the Hamiltonian function for the lattice gas (substance) is not quite clear and subject to some questions. The most important of the questions will be "whether the anti-commutativity of annihilation and creation operators of a lattice site does not destroy the Bose property of the system under consideration?". In this note, the Hamiltonian of the lattice gas, which reduces to that of liquid helium when the lattice constant is taken to be infinitesimal, is derived by the use of the formalism of second quantization. And it is confirmed that the calculations by Matsubara and Matsuda are surely for the Bose lattice gas (substance) when two particles cannot occupy a lattice site at the same time.

Following Matsubara and Matsuda[1)], we consider the lattice gas (substance), in which N molecules are on L lattice sites which constitute a simple cubic lattice of lattice constant d.

Corresponding to the Hamiltonian of liquid helium, of N molecules in volume V:

$$\mathcal{H} = -(\hbar^2/2m)\cdot\sum_{i=1}^{N} \partial^2/\partial \boldsymbol{r}_i^2 + \tfrac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\(i\neq j)}}^{N} V^{(2)}(\boldsymbol{r}_i,\ \boldsymbol{r}_j) \tag{1}$$

where the notations will be clear, we introduce a Hamiltonian of the lattice gas

$$\mathcal{H} = -(\hbar^2/2m)\cdot\sum_{i=1}^{N} \Delta^2/\Delta \boldsymbol{r}_i^2 + \tfrac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\(i\neq j)}}^{N} V^{(2)}(\boldsymbol{r}_i,\ \boldsymbol{r}_j) \tag{2}$$

$$\Delta^2/\Delta \boldsymbol{r}_i^2 = \Delta^2/\Delta x_i^2 + \Delta^2/\Delta y_i^2 + \Delta^2/\Delta z_i^2 \tag{2$'$}$$

where

$$\frac{\Delta^2}{\Delta x_i^2}\psi(\cdots, x_i, \cdots) = \frac{1}{d^2}\left[\psi(\cdots, x_i+d,\ \cdots) - 2\psi(\cdots, x_i, \cdots) + \psi(\cdots x_i - d, \cdots)\right]. \tag{2$''$}$$

Although this Hamiltonian is not the unique one which becomes to (1) when $d\rightarrow 0$, this is adopted here because this is of the simplest form and this can be shown to be equivalent to that used by Matsubara and Matsuda.

The Hamiltonian in the theory of second quantization is[2)]

$$H = -\frac{\hbar^2}{2m}\sum_{r=r_1}^{r_L}\psi^*(\boldsymbol{r})\,\frac{\Delta^2}{\Delta \boldsymbol{r}^2}\psi(\boldsymbol{r}) + \frac{1}{2}\sum_{r=r_1}^{r_L}\sum_{r'=r_1}^{r_L}\psi^*(\boldsymbol{r})\,\psi^*(\boldsymbol{r}')\,V^{(2)}(\boldsymbol{r},\ \boldsymbol{r}')\,\psi(\boldsymbol{r}')\,\psi(\boldsymbol{r}) \tag{3}$$

$$[\psi(\mathbf{r}),\ \psi^*(\mathbf{r}')]=\delta_{\mathbf{r},\mathbf{r}'},\quad [\psi(\mathbf{r}),\ \psi(\mathbf{r}')]=0,\quad (\mathbf{r},\ \mathbf{r}'=\mathbf{r}_1,\cdots,\mathbf{r}_L), \tag{4}$$

where $[A, B]=AB-BA$.

We introduce a complete set of functions for our lattice system:

$$|j> =\delta_{\mathbf{r}_j,\mathbf{r}}=\delta_{x_j,x}\delta_{y_j,y}\delta_{z_j,z}\qquad (j=1,\ 2,\cdots,L) \tag{5}$$

where $\mathbf{r}_j$'s are the coordinates of j-th lattice site and $\delta_{A,B}$ the Kronecker's symbol. We use this complete set for the representation and take

$$\psi(\mathbf{r})=\sum_{j=1}^{L}|j>a_j\,. \tag{6}$$

Noting that (because of (2''))

$$\left\langle k\left|\frac{\Delta^2}{\Delta x^2}\right|j\right\rangle=\sum_{\mathbf{r}=\mathbf{r}_1}^{\mathbf{r}_L}\delta_{\mathbf{r}_k,\mathbf{r}}\frac{\Delta^2}{\Delta x^2}\delta_{\mathbf{r}_j,\mathbf{r}}$$

$$=\frac{1}{d^2}(\delta_{x_j,x_k+d}-2\delta_{x_j,x_k}+\delta_{x_j,x_k-d})\,\delta_{y_j,y_k}\delta_{z_j,z_k}$$

and

$$\sum_{j=1}^{L}\sum_{k=1}^{L}\delta_{x_j,x_k+d}\delta_{y_j,y_k}\delta_{z_j,z_k}a_k{}^*a_j=\sum_{j=1}^{L}a^*_{i(x_j-d,y_j,z_j)}a_j$$

where $i(x_j-d,\ y_j,\ z_j)$ denotes the number of the lattice site whose coordinate is $(x_j-d,\ y_j,\ z_j)$, we obtain, for the Hamiltonian,

$$H=-\frac{\hbar^2}{2m}\sum_{j=1}^{L}\sum_{k=1}^{L}\left\langle k\frac{\Delta^2}{\Delta\mathbf{r}^2}j\right\rangle a_k^*a_j+\frac{1}{2}\sum_{l}\sum_{m}\sum_{k}\sum_{j}\langle lm|V^{(2)}(\mathbf{r},\mathbf{r}')|k\,j\rangle a_m^*a_l^*a_ka_j$$

$$=\frac{\hbar^2}{2md^2}\sum_{\langle jk\rangle}(a_j^*-a_k^*)(a_j-a_k)+\frac{1}{2}\sum_{j=1}^{L}\sum_{k=1}^{L}V^{(2)}(\mathbf{r}_j,\mathbf{r}_k)a_j^*a_k^*a_ka_j\,, \tag{7}$$

the summation in the first term being over nearest neighbor lattice sites; with

$$[a_j,\ a_k^*]=\delta_{j,k},\quad [a_j,\ a_k]=0, \tag{8}$$

or, explicitly in the representation where the numbers of molecules on the L lattice sites are diagonal,

$$a_i=\overset{1}{1}\times\cdots\times\overset{i-1}{1}\times\overset{i}{a}\times\overset{i+1}{1}\times\cdots\times\overset{L}{1} \tag{9}$$

$$a_i^*=1\times\cdots\times1\times a^*\times1\times\cdots\times1$$

$$a=\begin{pmatrix}0&\sqrt{1}&0&0&\cdots\\0&0&\sqrt{2}&0&\cdots\\0&0&0&\sqrt{3}&\cdots\\\cdots\cdots\end{pmatrix},\quad a^*=\begin{pmatrix}0&0&0&\cdots\\\sqrt{1}&0&0&\cdots\\0&\sqrt{2}&0&\cdots\\\cdots\cdots\end{pmatrix}. \tag{10}$$

Because of (9),

$$\prod_i f_i(a_i, a_i^*) = f_1(a, a^*) \times f_2(a, a^*) \times \cdots . \tag{11}$$

The Hamiltonian of the Bose lattice gas has been obtained in the second quantized form.*

We write the μ-th eigenfunction and eigenvalue of the Hamiltonian H as Φ_μ and E_μ :

$$H\Phi_\mu = E_\mu \Phi_\mu , \tag{12}$$

then Φ_μ can be expressed as

$$\Phi_\mu = \sum_{\substack{n_1=0 \\ \Sigma_{i=1}^{L} n_i = N}}^{\infty} \cdots \sum_{n_L=0}^{\infty} \psi_\mu(n_1, \cdots, n_L)\, a_1^{*n_1} \cdots a_L^{*n_L} \Phi_0 \tag{13}$$

where Φ_0 is the state when no molecules are present.

Now, we proceed to the case where $V^{(2)}(\mathbf{r}, \mathbf{r}) = \infty$. Then ψ_μ has the property that

$$\psi_\mu(n_1, \cdots n_L) = 0 \tag{14}$$

if there exists a number i for which $n_i \geqq 2$. (If otherwise, E_μ would be necessarily infinity.) Consequently, the density matrix

$$\rho(n_1, \cdots, n_L ; n_1', \cdots, n_L') = \sum_\mu e^{-\beta E_\mu} \psi_\mu^*(n_1', \cdots, n_L')\, \psi_\mu(n_1, \cdots, n_L) \tag{15}$$

is necessarily equal to zero if there appears at least one number i for which $n_i \geqq 2$ or $n_i' \geqq 2$. And this statement is valid also for the reduced density matrices. For instance, the point reduced density matrix is necessarily of the form :

$$\rho_i^{(1)} = \begin{array}{r} \\ n_i'=0 \\ 1 \\ 2 \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{cccc} n_i=0 & 1 & 2 & \end{array} \\ \left(\begin{array}{cccc} \rho & \xi & 0 & \cdots \\ \xi^* & 1-\rho & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots\cdots & & & \end{array} \right) \end{array} . \tag{16}$$

The consequences of this property of our eigenfunctions and density matrices are two. First, the expectation value for a matrix in the form of (11) should be dependent only on 4 elements of $f_i(a, a^*)$ for $n_i=0$ and 1 and $n_i'=0$ and 1 $(i=1, 2, \cdots, L)$, and so it is sufficient to consider only two-two matrices instead of ∞-∞ matrices. For instance when the expectation of the Hamiltonian (7) is to be calculated, we may use

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{17}$$

$$a^* a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a^* a^* a a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{18}$$

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* It should be noted that we are not dealing with the problem of approximation because we have resorted to a model on the outset ; this situation is quite the contrary to the development by Koide[3]

The two matrices of (18) are just the form which is calculated by using (17) in the left hand sides. However, this is not always the case. For instance, the expectation of a matrix aa^* should be calculated by using

$$aa^* = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{19}$$

but not by using

$$aa^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{20}$$

the latter being obtained if we adopt (17) for a and a^*.

Secondly, when we calculate the wave functions of the system or the density matrices, we may drop $j=k$ in the summation of the second term of (7), if we do not consider the state where $n_i \geqq 2$; (7) with the restriction $j \neq k$ on the second summation will be referred to as (7′) hereafter.

From these considerations, it follows that, when $V^{(2)}(\boldsymbol{r}, \boldsymbol{r}) = \infty$, to solve the eigenvalue problem it is sufficient to calculate the eigenvalue of (7′) with (17) and (18) if we are not to consider the states with $n_i \geqq 2$.

The products of a and a^* appeared in our Hamiltonian are (17) and (18) and no products such as aa^* appear. And, whether aa^* is of the form (19) or (20) does not matter for the eigenvalue problem of the Hamiltonian and so it is allowed to replace (19) by (20). This replacement is the replacement of (10) by (17) for a and a^*.* Here, we notice that this replacement is accompanied by the operational relation

$$a^* a^* = 0, \tag{21}$$

and this automatically drops the states with $n_i \geqq 2$ as seen from (13). As the result, we have reached the result that, *when* $V^{(2)}(\boldsymbol{r}, \boldsymbol{r}) = \infty$, *the eigenvalue problem of* (7) *with* (9) *and* (10) *reduces to that of* (7′) *with* (9) *and* (17). And the latter is just the Hamiltonian from which Matsubara and Matsuda started their calculations. And it has been confirmed that their calculations are surely for the Bose lattice gas for which $V^{(2)}(\boldsymbol{r}, \boldsymbol{r}) = \infty$.

The identity of the results using (7) with (9) and (10) and those using (7′) with (9) and (17) is also confirmed, when we consider that the free energy F of the system is obtained by the variational principle:

$$F = \text{Min.} \{ \text{Tr}(H\rho) + kT\,\text{Tr}(\rho \ln \rho) \}. \tag{22}$$

According to the discussion following the equation (15), as the variational function ρ, it is sufficient to consider the ones which have values only for $n_i = 0$ and 1, and the values of $\{\cdots\}$ in (22) calculated by using such a ρ are equal for these two Hamiltonians. Consequently, the free energies obtained should coincide automatically.

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* If there have appeared products such as aa^*, we would have to rewrite them in the form in which the two-two matrices do not change by the replacement of (10) by (17). For instance aa^* should be replaced by a^*a+1 before (10) is replaced by (17).

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Pion-Nucleon Scattering and the Structure of the Pion

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Effects of the "Structure" of the pion to the pion-nucleon scattering are discussed. A formal expression of the interaction kernel for the nucleon-antinucleon propagator is given, from which the "effective Hamiltonian" for the emission and absorption of one pion and that for two pions are derived respectively. Effects of two pion vertex, which is characteristic of the structure of pion, seem to be not necessarily negligible compared with the effects of one pion vertex, according to a very rough estimate of the order of magnitude of the "coupling constant" appearing in the interaction kernel for the nucleon-antinucleon propagator.

At present two standpoints are recognized with respect to the theory of pion, i.e., the compound theory of the pion and the conventional local pion field theory. However, we intended to discuss some important properties burried under both theories by using the above method, and obtained the qualitative properties concerning the isospin dependence of the pion-nucleon scattering matrix elements.

Quantitative evaluation of the possible effects are not made, since for that purpose it is necessary to attack the details of the structure of the pion. Such an attempt is not made in the present paper.

It is shown, in the Appendix, that the crossing symmetry for the pion-nucleon scattering holds even if the pion is considered to be composed of arbitrary number of fermion pairs and bosons, provided that the primary interaction Lagrangian satisfies the integrability condition to assure the existence of propagators.

§ 1. Introduction

It is well known that the main features of low energy P-wave pion-nucleon scattering are well reproduced by the charge independent meson theory with fixed extended source. Concerning the S-wave scattering, however, we cannot explain the experiments even qualitatively[1].

It seems doubtless that the effects of virtual nucleon-antinucleon pairs are important for the understanding of the behaviors of the S-wave pion-nucleon scattering as well as the electromagnetic properties of the nucleons, including the anomalous magnetic moments, neutron-electron interaction, and the high energy electron-nucleon scattering.

It is the purpose of this paper to discuss one of the effects of the nucleon-antinucleon pairs, i.e. the effects of the structure of the pion, to the low energy pion-nucleon scattering. We regard the physical pion to be a "compound" state of nucleon pairs. However, the following discussions are not restricted to the compound theory of the pion, but may be applicable to the pseudoscalar meson theory which seems now standard, as an approximation with an appropriate consideration.

In most of the compound theories of "elementary" particles[2], pions are identified with suitable bound states composed of a nucleon pair. However, more exactly speaking, they should be considered to be composed of equal numbers of particles and antiparticles belonging to the nucleon family. We will not go into the details of the primary interaction between constituent particles, which should cause the desired bound states, but only assume the existence of such an interaction. It might be worthwhile to point out in this connection that it is very difficult to draw some conclusion on the primary interaction from the binding energy data, since it is needed to treat the interaction very rigorously even at very small distances for the calculation of the binding energy.

The following formalism is also applicable to the amplitudes of nucleon pair components in the case of the conventional local theory of field, where both nucleons and pions are regarded as structureless particles. Such a treatment should be a good approximation to phenomena where main contributions are given by states where only nucleon pairs exist. Interactions between nucleons and antinucleons are due, in this case, to the intermediary of pion field.

In principle, there must be differences between the consequences of the " compound " pion theory, where pions are assumed to be appropriate bound states of more fundamental particles, and the " elementary " pion theory, where pions are assumed to be fundamental particles. It seems, however, very difficult to show it, since the differences between the consequences of the two standpoints appear always quantitatively but not qualitatively. This fact is most easily understood by seeing that the primary interaction in one theory may be derived by the other, although these detailed functional forms may be different. Therefore, in the following, our standpoint need not be restricted to one of the above mentioned two theories.

Experimental values of the low energy S-wave scattering phase shifts* are characterized by their small magnitudes and the large isospin dependence, to both of which any reasonable explanations are not yet given from the pion field theory. The large isospin dependence seems, at first sight, hard to be explained in connection with the crossing symmetry[4], which is expressed as

$$T_{ji}(p', p\,;\, q'\, q) = T_{ij}(p', p\,;\, -q, -q'), \qquad (1\cdot1)$$

where $T_{ji}(p', p; q', q)$ is the scattering amplitude for the scattering of an i-th meson in momentum state q to a j-th meson in state q', the scattering nucleon going from p to p'. It is easily seen from eq. (1·1) that the isospin dependent term of the scattering amplitude contains a factor μ/M(μ and M are masses of a pion and a nucleon respectively) since for $q'=-q=0$ (which is only possible for $\mu=0$) $T_{ji}=T_{ij}$ and the scattering is independent of isospin. In fact, none of calculations, performed up to present, starting from the charge-independent pseudoscalar meson theory have succeeded in obtaining the adequate isospin dependence[5], although it may be well reproduced by a model proposed by Bosco and Stroffolini[6] which manifestly destroys the crossing symmetry as well as by semiphenomenological effective Hamiltonian theory with two coupling constants[7]. In connection with the above mentioned situation, it seems important to investigate whether or not the crossing symmetry holds in the case of scattering of compound particles. This problem is studied in the Appendix, and it is proved that the crossing symmetry does also hold for the scattering of compound particles if the primary interaction between constituent particles satisfies the integrability condition.

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* $\delta_3 \cong -0.11k/\mu$, $\delta_1 \cong 0.16k/\mu$[3]. δ_i's are the phase shifts of the isospin $i/2$ states, k is the momentum of the incident pion, and μ is the mass of a pion.

In the present paper, we will investigate the relation between the effective pion-nucleon interaction Hamiltonian and the nucleon-antinucleon interaction kernel. In § 2, a formal representation is given for the nucleon-antinucleon interaction kernel, in § 3 the effective Hamiltonian for the emission or absorption of a pion is derived, and in § 4 the effective Hamiltonian for the emission or absorption of two pions is derived.

§ 2. Interaction Kernel

Let us suppose the physical pion to be composed of the same number of nucleons and antinucleons. We shall illustrate by making use of the shaded area in Fig. 1 the interaction kernel of the relativistic integral equation satisfied by one nucleon and one antinucleon amplitude, $\langle 0|T(\psi(x)\bar{\psi}(y))|\pi\rangle$, where $|\pi\rangle$ and $|0\rangle$ means physical one pion and vacuum state vector respectively, ψ is the nucleon field operator in the Heisenberg representation, and $T(\quad)$ means T-product defined by Wick[8]. The interaction kernel may be given, in principle, once the Lagrangian is given explicitly[9]. It is given by the sum of contributions from the Feynman diagrams shown in Fig. 2 in the compound theory of the pion and by that in Fig. 3 in the ordinary " elementary " theory of the pion. We will not, however, restrict our standpoint to one of the above mentioned two, but start from the formal expression of the " interaction kernel " corresponding to Fig. 1, i.e.,

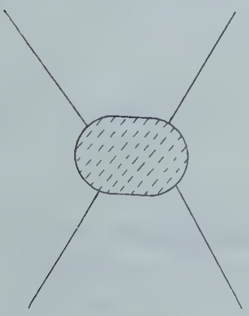


Fig. 1

—— nucleon or antinucleon

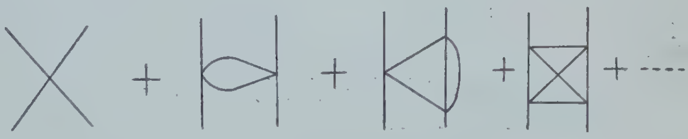


Fig. 2.

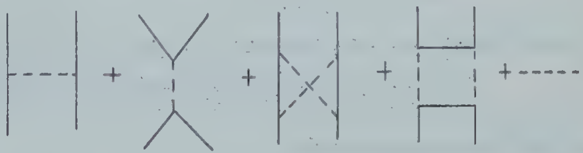


Fig. 3 —— nucleon or antinucleon ------ pion

$$\int K d^4\bar{X} = g_i \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, \Phi(x_1, x_2, x_3, x_4)\,\cdot$$

$$\cdot\,\bar{\psi}(x_1)\,\Omega_i I \psi(x_3)\cdot\bar{\psi}'(x_2)\,\Omega_i I' \psi(x_4)\ ; \qquad (2\cdot 1)$$

the interaction kernel is characterized by g_i's, Ω_i's, I, I', and $\Phi(x_1, x_2, x_3, x_4)$, where g_i's are effective coupling constant, Ω_i's are operators involving Dirac and differential operators,

I and I' are operators in isospin space, and $\Phi(x_1, x_2, x_3, x_4)$ is the invariant form factor. Definitions of other notations are as follows:

$$\bar{X} = (x_1 + x_2 + x_3 + x_4)/4,$$

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \quad \psi' = C\bar{\psi} = \begin{pmatrix} -i\bar{\psi}_n \\ i\bar{\psi}_p \end{pmatrix},$$

$$C = \gamma_4 \gamma_2 \tau_2,$$

and ψ's represent symbolically the outgoing nucleon lines in Fig. 1, and anticommute with each other. Contributions to the interaction between a nucleon and an antinucleon from the other amplitudes, e.g., that from the states where many nucleon pairs exist, are included in eq. $(2\cdot1)$.

The form factor, $\Phi(x_1, x_2, x_3, x_4)$, must satisfy following general requirements. In the first place, it must be invariant with respect to arbitrary displacements of the origin of the system of space-time coordinates. This condition imposes a factor $\delta^4(l^1 + l^2 + l^3 + l^4)$ upon the Fourier transform $F(l^1, l^2, l^3, l^4)$ of the function $\Phi(x_1, x_2, x_3, x_4)$. Accordingly, putting $F(l^1, l^2, l^3, l^4) = \delta^4(l^1 + l^2 + l^3 + l^4) \cdot G$, we get

$$\begin{aligned} \Phi(x_1, x_2, x_3, x_4) &= \Phi(X, x_{12}, x_{34}) \\ &= (2\pi)^{-12} \int G(L, l^{12}, l^{34}) \exp i(LX + l^{12}x_{12} + l^{34}x_{34})\, d^4L\, d^4l^{12}\, d^4l^{34}, \end{aligned} \tag{2·2}$$

where

$$\begin{aligned} X &= (x_1 + x_2)/2 - (x_3 + x_4)/2, \\ x_{12} &= x_1 - x_2, \\ x_{34} &= x_3 - x_4, \end{aligned}$$

and

$$\begin{aligned} L &= (l^1 + l^2)/2 - (l^3 + l^4)/2, \\ l^{12} &= (l^1 - l^2)/2, \\ l^{34} &= (l^3 - l^4)/2. \end{aligned}$$

Next, since $(2\cdot1)$ must be Hermitian, a further restriction can be imposed:

$$\Phi(X, x_{12}, x_{34}) = \Phi^*(-X, -x_{34}, -x_{12}), \tag{2·3}$$

or, in the Fourier representation,

$$G(L, l^{12}, l^{34}) = G^*(L, l^{34}, l^{12}). \tag{2·3'}$$

Finally, Φ and G should be invariant with respect to Lorentz transformations.* We will conveniently normalize Φ by

---

* These conditions have been discussed by Kristensen and Møller[10] for the case of three point nonlocal interaction.

$$\int \Phi(X,\ x_{12},\ x_{34})\, d^4X\, d^4x_{12}\, d^4x_{34}=1$$

or in the Fourier representation by

$$G(0,\ 0,\ 0)=1.$$

For simplicity, we will hereafter not consider the case where Ω_i's include differential operators, but assume only five Ω_i's,

$$\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5=1,\ \gamma_\lambda,\ \sigma_{\lambda\rho},\ i\gamma_5\gamma_\lambda,\ \gamma_5\,.$$

Moreover, assuming charge independence, we take general forms for I and I' as

$$I=(\tau,\ \eta_i\ \tau_4)\quad\text{and}\quad I'=(-\tau,\ \tau_4),$$

where η_i's are arbitrary real constants corresponding to five Ω_i, and τ and τ_4 are ordinary Pauli-type vector operator and unit operator in the nucleon isospin space, respectively.

In order to proceed with our calculation it seems convenient to express $\bar{\psi}'$ and ψ' in (2·1) by ψ and $\bar{\psi}$ as follows:

$$\bar{\psi}'(x_2)\,\Omega_i\begin{pmatrix}-\tau\\ \tau_4\end{pmatrix}\psi'(x_4)=\epsilon_i\bar{\psi}(x_4)\,\Omega_i\begin{pmatrix}\tau\\ \tau_4\end{pmatrix}\psi(x_2),$$

where

$$\epsilon_i=\begin{cases}\ \ 1 & \text{for}\quad i=1,\ 4,\ 5,\\ -1 & \text{for}\quad i=2,\ 3.\end{cases}$$

Eq. (2·1) may, then, be witten very explicitly as

$$\int K\, d^4\,\bar{X}=g_i\epsilon_i\int d^4\bar{X}\, d^4X\, d^4x_{12}\, d^4x_{34}\,\Phi(X,\ x_{12},\ x_{34})\cdot$$
$$\cdot[\bar{\psi}^k_\alpha(x_1)\,\Omega_{i\alpha\beta}\,\tau^{kl}\psi^l_\beta(x_3)\cdot\bar{\psi}^m_\gamma(x_4)\,\Omega_{i\gamma\delta}\tau^{mn}\psi^n_\delta(x_2)$$
$$+\eta_i\bar{\psi}^k_\alpha(x_1)\,\Omega_{i\alpha\beta}\delta^{kl}\psi^l_\beta(x_3)\cdot\bar{\psi}^m_\gamma(x_4)\,\Omega_{i\gamma\delta}\delta^{mn}\psi^n_\delta(x_2)\,], \tag{2·4}$$

where indices of ψ and $\bar{\psi}$, k, l, m and n vary as 1 for proton and 2 for neutron.

§ 3. One pion vertex

We consider the contribution of the interaction kernel, (2·4), to the one pion vertex (Fig. 4). One obtains from the first term of the right side of eq. (2·4),

$$g_i\epsilon_i\int d^4x_1d^4x_2dx^4_3d^4x_4\,\Phi(X,\ x_{12},\ x_{34})\,.$$
$$\cdot[-\bar{\psi}^k_\alpha(x_1)\,\Omega_{i\alpha\beta}\tau^{kl}\psi^l_\beta(x_3)\cdot\Omega_{i\gamma\delta}\tau^{mn}\varphi^{nm}_{\delta\gamma}(x_2|x_4)$$
$$-\bar{\psi}^m_\gamma(x_4)\,\Omega_{i\gamma\delta}\tau^{mn}\psi^n_\delta(x_2)\,\Omega_{i\alpha\beta}\tau^{kl}\varphi^{lk}_{\beta\alpha}(x_3|x_1)$$
$$+\bar{\psi}^k_\alpha(x_1)\,\Omega_{i\alpha\beta}\tau^{kl}\Omega_{i\gamma\delta}\tau^{mn}\psi^n_\delta(x_2)\,\varphi^{lm}_{\beta\gamma}(x_3|x_4)$$
$$+\bar{\psi}^m_\gamma(x_4)\,\Omega_{i\gamma\delta}\tau^{mn}\Omega_{i\alpha\beta}\tau^{kl}\psi^l_\beta(x_3)\,\varphi^{nk}_{\delta\alpha}(x_2|x_1)\,], \tag{3·1}$$

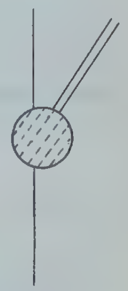


Fig. 4

═ compound pion

where

$$\varphi^{lm}_{\beta\gamma}(x_3|x_4) \equiv \langle 0|T(\psi^l_\beta(x_3)\bar{\psi}^m_\gamma(x_4))|\pi\rangle \tag{3·2}$$

is the Bethe-Salpeter amplitude for the existence of a nucleon at x_3 and an antinucleon at x_4 respectively in the physical pion.* Interchanging the notations in the second and fourth terms in (3·1) as follows,

$$x_1 \longleftrightarrow x_4,\ x_2 \longleftrightarrow x_3\ ;\ \alpha \longleftrightarrow \gamma,\ \beta \longleftrightarrow \delta\ ;\ k \longleftrightarrow m,\ l \longleftrightarrow n,$$

(3·1) becomes

$$\begin{aligned} &2g_i\epsilon_i\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, \bar{\Phi}(X, x_{12}, x_{34})\,. \\ &\quad\cdot[-\bar{\psi}^k_\alpha(x_1)\,\Omega_{i\alpha\beta}\tau^{kl}\psi^l_\beta(x_3)\,\Omega_{i\gamma\delta}\tau^{mn}\varphi^{nm}_{\delta\gamma}(x_2|x_4) \\ &\qquad+\bar{\psi}^k_\alpha(x_1)\,\Omega_{i\alpha\beta}\tau^{kl}\,\Omega_{i\gamma\delta}\tau^{mn}\psi^n_\delta(x_2)\,\varphi^{lm}_{\beta\gamma}(x_3|x_4)], \end{aligned} \tag{3·3}$$

where

$$\begin{aligned} 2\bar{\Phi}(X, x_{12}, x_{34}) &\equiv \Phi(X, x_{12}, x_{34}) + \Phi(-X, -x_{34}, -x_{12}) \\ &= 2\mathcal{R}\mathrm{e}\ \Phi(X, x_{12}, x_{34})\ ; \end{aligned}$$

the last equation comes from eq. (2·3).

We may expand $\varphi^{lm}_{\beta\gamma}(x_3|x_4)$ in the following way:

$$\varphi^{lm}_{\beta\gamma}(x_3|x_4) = \sum_{S=1}^{16}\sum_{j=1}^{4}\phi^j_s(x_3|x_4)\gamma^{(s)}_{\beta\gamma}\tau^{lm}_j, \tag{3·4}$$

where $\gamma^{(s)}$'s are linearly independent sixteen Dirac operators, and τ_j's are linearly independent four isospin operators. Substituting (3·4) in (3·3), the bracket in (3·3) becomes

$$\begin{aligned} &\sum_s\sum_j[-\bar{\psi}(x_1)\,\Omega_i\tau\psi(x_3)\,\phi^j_s(x_2|x_4)\cdot Tr(\Omega_i\gamma^{(s)})\,Tr(\tau\tau_j) \\ &\quad+\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\Omega_i\tau\tau_j\tau\psi(x_2)\,\phi^j_s(x_3|x_4)]\,. \end{aligned} \tag{3·3'}$$

Using

$$\tau Tr(\tau\tau_j) = \begin{cases} 2\tau_j & \text{for}\quad j=1,\ 2,\ 3, \\ 0 & \text{for}\quad j=4, \end{cases}$$

$$\tau\tau_j\tau = \begin{cases} -\tau_j & \text{for}\quad j=1,\ 2,\ 3, \\ 3 & \text{for}\quad j=4, \end{cases}$$

we can write (3·1) as

$$\begin{aligned} &2g_i\epsilon_i\int d^4\bar{X}\, d^4X d^4x_{12} d^4x_{34}\bar{\Phi}(X, x_{12}, x_{34})\,\cdot \\ &\quad\cdot\sum_s[-2\bar{\psi}(x_1)\,\Omega_i\tau\psi(x_3)\,\phi_s(x_2|x_4)\,Tr(\Omega_i\gamma^{(s)}) \end{aligned}$$

---

* Although we use here the Bethe-Salpeter amplitude for its familiarity, it may be more appropriate to use the connected Feynman amplitude as has been discussed in detail by Maki.[11] However, our discussions would also be meaningful in that case, if appropriately interpreted.

$$-\bar{\psi}(x_1)\,\Omega i\gamma^{(s)}\Omega_i\boldsymbol{\tau}\psi(x_2)\,\boldsymbol{\phi}_s(x_3|x_4)$$
$$+3\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\Omega_i\psi(x_2)\,\phi_s{}^4(x_3|x_4)\,], \tag{3·5}$$

where the bold-faced letter $\boldsymbol{\phi}_s$ means a vector in three dimensional isospin space.

Contributions from the second term of the right side of eq. (2·4) are given by (3·4) if one replaces $\boldsymbol{\tau}$ by $\tau_4=1$ and multiplies by η_i, resulting in

$$2g_i\epsilon_i\eta_i\int d^4\bar{X}d^4X\,d^4x_{12}\,d^4x_{34}\,\bar{\Phi}(X,\ x_{12},\ x_{34})\cdot$$
$$\cdot\sum_s[-2\bar{\psi}(x_1)\,\Omega_i\psi(x_3)\,\phi_s^4(x_2|x_4)\,Tr(\Omega_i\gamma^{(\varepsilon)})$$
$$+\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\Omega_i\boldsymbol{\tau}\psi(x_2)\,\boldsymbol{\phi}_s(x_3|x_4)$$
$$+\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\Omega_i\psi(x_2)\,\phi_s^4(x_3|x_4)\,]. \tag{3·6}$$

Adding (3·5) and (3·6), one obtains

$$\int H_{\mathrm{eff}}^{(1)}d^4\bar{X}=2g_i\epsilon_i\int d^4\bar{X}\,d^4X\,d^4x_{12}\,d^4x_{34}\,\bar{\Phi}(X,\ x_{12},\ x_{34})\cdot$$
$$\cdot\sum_s[-2\bar{\psi}(x_1)\,\Omega_i\{\boldsymbol{\tau}\boldsymbol{\phi}_s(x_2|x_4)+\eta_i\phi_s^4(x_2|x_4)\}\,\psi(x_3)\,Tr(\Omega_i\gamma^{(s)})$$
$$+\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\Omega_i\{-(1-\eta_i)\,\boldsymbol{\tau}\boldsymbol{\phi}_s(x_3|x_4)+(3+\eta_i)\,\phi_s^4(x_3|x_4)\}\,\psi(x_2)\,], \tag{3·7}$$

where one may call $H_{\mathrm{eff}}^{(1)}$ the "effective Hamiltonian density" for emission or absorption of a pion.

We have, as yet, put no restriction on the transformation property of ϕ_s^j. We will retain, in the following, only the charge triplet function $\boldsymbol{\phi}_s$ and discard $\phi_s{}^4$, considering it not to correspond to the pion. Due to the requirement that the Lagrangian be a scalar, the neutral pion field should transform as a "first kind pseudoscalar" in terms of Watanabe[12], that is, the transformation rule is given by

$$\phi\rightarrow\phi'=\sigma\phi \qquad (x_\mu\rightarrow x_\mu'), \tag{3·8}$$

where

$$\sigma=\partial(x',\ y',\ z',\ t')/\partial(x,\ y,\ z,\ t).$$

Contrary to the case of the neutral pion field, we cannot infer any definite specific "kind" of the charged pion fields, because of the unknown difference of the phases which appear in the transformation of the neutron and the proton fields. We assume, however, that the charged pion fields are also first kind pseudoscalars.

Then, if φ is a pion field wave function itself, it should have the following transformation properties:

$$\varphi\rightarrow\varphi'=\gamma_4\varphi\gamma_4=-\varphi \quad \text{for space reflection,} \tag{3·9a}$$

and

$$\varphi\rightarrow\varphi'=\gamma_5\gamma_4\varphi\gamma_4\gamma_5=-\varphi \quad \text{for time reflection.} \tag{3·9b}$$

Therefore, we need to retain only γ_5 among sixteen $\gamma^{(s)}$'s. The ϕ_s^j corresponding to γ_5 and $\boldsymbol{\tau}$ in eq. (3·4) will hereafter be denoted as $\boldsymbol{\phi}$.

φ could, in general, be derivatives of the wave function of the pion. We will consider the case of φ being the first derivative of the wave function of the pion, discarding the case of higher derivatives for simplicity. φ should, then, be a "first kind pseudovector" for the neutral pion field. We assume, as before, that φ has the same transformation property both for charged pion and for neutral pion. Then, only $i\gamma_5\gamma_\mu$ of $\gamma^{(s)}$ survives in eq. (3·4), and we denote the ϕ_s in this case as ϕ_μ. ϕ_μ is a first derivative with respect to certain space-time coordinate of ϕ times a quantity with dimension of length, since ϕ_μ has the same dimension with ϕ.

Consequently, eq. (3·7) becomes

$$\begin{aligned}\int H_{\text{eff}}^{(1)} d^4\bar{X} = 2\int & d^4\bar{X} d^4X d^4x_{12} d^4x_{34}\, \bar{\Phi}(X, x_{12}, x_{34}) \cdot \\ & \cdot[-2^3 g_5 \bar{\psi}(x_1)\gamma_5 \boldsymbol{\tau}\phi(x_2|x_4)\psi(x_3) \\ & -2^3 g_4 \bar{\psi}(x_1) i\gamma_5\gamma_\mu \boldsymbol{\tau}\boldsymbol{\phi}_\mu(x_2|x_4)\psi(x_3) \\ & -g_i\epsilon_i(1-\eta_i)\xi_i\bar{\psi}(x_1)\gamma_5\boldsymbol{\tau}\phi(x_3|x_4)\psi(x_2) \\ & -g_i\epsilon_i(1-\eta_i)\xi_i'\bar{\psi}(x_1) i\gamma_5\gamma_\mu\boldsymbol{\tau}\boldsymbol{\phi}_\mu(x_3|x_4)\psi(x_2)],\end{aligned} \tag{3·10}$$

where ξ_i, ξ_i' are defined by

$$\Omega_i\gamma_5\Omega_i = \xi_i\gamma_5, \qquad \Omega_i i\gamma_5\gamma_\mu\Omega_i = \xi_i' i\gamma_5\gamma_\mu,$$

their numerical values being

$$\left.\begin{aligned}\xi_i &= 1,\ -4,\ 6,\ -4,\ 1 \\ \xi_i' &= 1,\ 2,\ 1,\ -2,\ -1\end{aligned}\right\} \text{ for } i=1,\ 2,\ 3,\ 4,\ 5.$$

Eq. (3·10) is the desired "effective Hamiltonian" for emission or absorption of a pion, where the terms including ϕ correspond to the pseudoscalar coupling in the local pion theory, and theose including ϕ_μ to the pseudovector coupling. It should, however, be noted that ϕ_μ may be a derivative of ϕ with respect to the relative coordinate of nucleon pair composing the pion as well as a derivative with respect to the center of mass coordinate.

To compare eq. (3·10) with the local pion theory, we consider the case of $\eta_i = 1$, $g_4 = 0$ and $g_5 \neq 0$ in a little more detail. In such a case eq. (3·10) takes a simple form

$$\begin{aligned}\int H_{\text{eff}}^{(1)} d^4\bar{X} = -2^4 g_5 \int & d^4\bar{X} d^4X d^4x_{12} d^4x_{34}\, \bar{\Phi}(X, x_{12}, x_{34}) \cdot \\ & \cdot\bar{\psi}(x_1)\gamma_5\boldsymbol{\tau}\phi(x_2|x_4)\psi(x_3).\end{aligned} \tag{3·11}$$

Substituting the following expressions in eq. (3·11) :

$$\begin{aligned}\bar{\Phi}(X, x_{12}, x_{34}) = (2\pi)^{-12}\int & d^4L d^4l^{12} d^4l^{34} \cdot \\ & \cdot \exp i(LX + l^{12}x_{12} + l^{34}x_{34}) \cdot \bar{G}(L, l^{12}, l^{34}),\end{aligned}$$

$$\bar{\psi}(x_1) = (2\pi)^{-4}\int d^4 p_1 \cdot \exp(-ip'x_1)\bar{\psi}(p'), \tag{3·12}$$

$$\psi(x_3) = (2\pi)^{-4}\int d^4p \cdot \exp(ipx_3)\psi(p),$$

$$\boldsymbol{\phi}(x_2|x_4) = (2\pi)^{-8}\int d^4k\, d^4q \cdot \exp(ikX_{24} - iqx_{24})\boldsymbol{\phi}(k, q),$$

where $X_{24} = (x_2+x_4)/2$, $x_{24} = x_2 - x_4$, the " effective Hamiltonian " is given in the momentum representation by

$$\int H_{\text{eff}}^{(1)} d^4\bar{X} = -2^4(2\pi)^{-12}g_5 \int d^4p'\, d^4p\, d^4k\, \delta^4(p+k-p') \cdot$$
$$\cdot \int d^4q \bar{G}\left(p + \frac{k}{2} + q,\ \frac{p'}{2} + \frac{k}{4} - \frac{q}{2},\ -\frac{p}{2} + \frac{k}{4} + \frac{q}{2}\right) \cdot$$
$$\cdot \bar{\psi}(p')\gamma_5 \boldsymbol{\tau\phi}(k, q)\psi(p). \tag{3·13}$$

In the local limit, $\Phi(X, x_{12}, x_{34}) \to \delta^4(X)\delta^4(x_{12})\delta^4(x_{34})$ and $\bar{G}(L, l^{12}, l^{34}) \to 1$, one obtains

$$\int H_{\text{eff}}^{(1)} d^4\bar{X} \to -2^4(2\pi)^{-8}g_5 \int d^4p'\, d^4p\, d^4k \delta^4(p+k-p') \cdot \bar{\psi}(p')\gamma_5 \boldsymbol{\tau}\bar{\boldsymbol{\phi}}(k)\psi(p), \tag{3·14}$$

where

$$\bar{\boldsymbol{\phi}}(k) = (2\pi)^{-4}\int d^4q\, \phi(k, q)$$
$$= (2\pi)^{-4}\int d^4q \int d^4r \exp(iqr)\phi(k, r) = \phi(k, r=0) \tag{3·15}$$

is the amplitude of the nucleon pair existing at the same space-time point. $\bar{\phi}(k)$ is related to the ordinary local pion field operator, denoted as $\boldsymbol{u}(k)$, by the following equation

$$\bar{\phi}(k) = iN^{-1}\boldsymbol{u}(k), \tag{3·16}$$

where N is a quantity with dimension of the square of length and may depend on the detailed structure of the pion. We will not try to determine the value of N in this paper and only mention that Maki[2)] has determined a similar quantity to the above N by solving the covariant integral equation for the composite pion in a very rough approximation. Ordinary dimensionless coupling constant, f, is given by

$$f = 2^4 N^{-1} g_5. \tag{3·17}$$

§ 4. Two pion vertex

The interaction kernel for the pion-nucleon propagator may be illustrated by a diagram in Fig. 5, if the pion is represented by the nucleon pair amplitude. In this section, we shall consider the processes given by Fig. 5 except the " three body " interaction. They are illustrated by the diagrams given by Fig. 6, and give rise to pion-nucleon

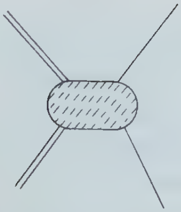
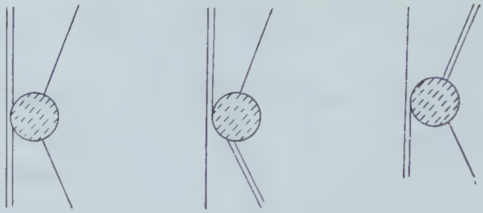


Fig. 5



(a) (b) (c)

Fig. 6

scattering matrix elements which include terms of first order in g_i, contrary to the contribution from the one pion vertex, eq. $(3\cdot 10)$, which are the second and higher orders in g_i[*]. The processes shown in Fig. 6 are similar to the deuteron stripping or pick up process, and one may treat the former processes in the same way as the latter[14)].

Although the two body interaction kernel, $(2\cdot 1)$, is rather complex, the isospin dependence of the " effective Hamiltonian " for the processes in Fig. 6 is determined rather definitely by the charge independence hypothesis.

We will calculate the pion-nucleon scattering matrix elements of

$$\int K d^4\bar{X}\cdot\int\bar{\psi}(x)\,\psi(x)\,d^4x,$$

which is the interaction kernel of the three particle integral equation shown in Fig. 6, where K is the two-body kernel given by eq. $(2\cdot 1)$, and $\int\bar{\psi}(x)\,\psi(x)\,d^4x$ denotes symbolically the change of the state of a nucleon as shown by a straight line from bottom to top in Fig. 6. For example, in Fig. 6 (b) it represents a transition from a free nucleon state to a bound state.

By straightforward calculations similar to that in § 3, one obtains the " effective Hamiltonian density " for emission or absorption of two pions, $H_{\text{eff}}^{(2)}$, as follows:

$$\begin{aligned}\int H_{\text{eff}}^{(2)}d^4\bar{X}=g_i\epsilon_i\int d^4\bar{X}d^4X\,d^4x_{12}d^4x_{34}\Phi(X,\,x_{12},\,x_{34})\,\cdot&\\ \cdot[(A_1)+(A_2)+(B_1)+(B_2)+(C_1)+(C_2)+(C_3)+(C_4)],&\qquad(4\cdot 1)\end{aligned}$$

where

$$\begin{aligned}(A_1)=&-(1-\eta_i)\,\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\Omega\,\gamma^{(s')}\{\tau\phi_s^*(x_4|x_3)\,\tau\phi_{s'}(x_2|x)\\ &\qquad+\tau\phi_s(x_3|x_4)\,\tau\phi_{s'}^*(x_2|x)\}\,\psi(x)\\ &-2\bar{\psi}(x_1)\,\Omega_i\gamma^{(s)}\{\tau\phi_{s'}(x_2|x_4)\,\tau\phi_s^*(x|x_3)\\ &\qquad+\tau\phi_{s'}^*(x_4|x_2)\,\tau\phi_s(x_3|x)\}\,\psi(x)\cdot Tr(\Omega_i\gamma^{(s')}),\\ (A_2)=&-(1-\eta_i)\,\bar{\psi}(x_4)\,\Omega_i\gamma^{(s)}\Omega_i\gamma^{(s')}\{\tau\phi_s^*(x_1|x_2)\,\tau\phi_{s'}(x_3|x)\\ &\qquad+\tau\phi_s(x_2|x_1)\,\tau\phi_{s'}^*(x|x_3)\}\,\psi(x)\end{aligned}$$

---

[*] The fact that the contributions from Fig. 6 might not be so small even in low energy pion-nucleon scattering was first noted by Hayakawa[13)].

$$
\begin{aligned}
&-2\overline{\psi}(x_4)\,\Omega_i\gamma^{(s)}\{\tau\phi_{s'}^*(x_1|x_3)\,\tau\phi_s(x_2|x) \\
&\qquad+\tau\phi_{s'}(x_3|x_1)\,\tau\phi_s^*(x|x_2)\}\,\psi(x)\,Tr(\Omega_i\gamma^{(s')}), \\
(B_1)=&-(1-\eta_i)\overline{\psi}(x)\,\gamma^{(s)}\Omega_i\gamma^{(s')}\Omega_i\{\tau\phi_s(x|x_4)\,\tau\phi_{s'}^*(x_1|x_2) \\
&\qquad+\tau\phi_s^*(x_4|x)\,\tau\phi_{s'}(x_2|x_1)\}\,\psi(x_3) \\
&-2\overline{\psi}(x)\,\gamma^{(s)}\Omega_i\{\tau\phi_s^*(x_1|x)\,\tau\phi_{s'}(x_2|x_4) \\
&\qquad+\tau\phi_s(x_2|x_1)\,\tau\phi_{s'}^*(x_4|x_2)\}\,\psi(x_3)\cdot Tr(\Omega_i\gamma^{(s')}), \\
(B_2)=&-(1-\eta_i)\overline{\psi}(x)\,\gamma^{(s)}\Omega_i\gamma^{(s')}\Omega_i\{\tau\phi_s^*(x_1|x)\,\tau\phi_{s'}(x_3|x_4) \\
&\qquad+\tau\phi_s(x|x_1)\,\tau\phi_{s'}^*(x_4|x_3)\}\,\psi(x_2). \\
&-2\overline{\psi}(x)\,\gamma^{(s)}\Omega_i\{\tau\phi_s(x|x_4)\,\tau\phi_{s'}^*(x_1|x_3) \\
&\qquad+\tau\phi_s^*(x_4|x)\,\tau\phi_{s'}(x_3|x_1)\}\,\psi(x_2)\,Tr(\Omega_i\gamma^{(s')}), \\
(C_1)=&-(1-\eta_i)\overline{\psi}(x_1)\,\Omega_i\gamma^{(s)}\gamma^{(s')}\Omega_i\{\tau\phi_s(x_3|x)\,\tau\phi_{s'}^*(x_4|x) \\
&\qquad+\tau\phi_s^*(x|x_3)\,\tau\phi_{s'}(x|x_4)\}\,\psi(x_2) \\
&+4\overline{\psi}(x_1)\,\Omega_i\gamma^{(s)}\gamma^{(s')}\Omega_i\psi(x_2)\,\{\phi_{s'}^*(x_4|x)\,\phi_s(x_3|x) \\
&\qquad+\phi_s^*(x|x_3)\,\phi_{s'}(x|x_4)\}, \\
(C_2)=&-2i\overline{\psi}(x_1)\,\Omega_i\tau[\phi_s(x_2|x)\times\phi_{s'}^*(x_4|x) \\
&\qquad+\phi_s^*(x|x_2)\times\phi_{s'}(x|x_4)]\psi(x_3)\,Tr(\Omega_i\gamma^{(s)}\gamma^{(s')}) \\
&-2\eta_i\overline{\psi}(x_1)\,\Omega_i\psi(x_3)\,Tr(\Omega_i\gamma^{(s)}\gamma^{(s')})\,\{\phi_{s'}^*(x_4|x)\,\phi_s(x_2|x) \\
&\qquad+\phi_s^*(x|x_2)\,\phi_{s'}(x|x_4)\}, \\
(C_3)=&-2i\overline{\psi}(x_4)\,\Omega_i\tau[\phi_s(x_3|x)\times\phi_{s'}^*(x_1|x) \\
&\qquad+\phi_s^*(x|x_3)\times\phi_{s'}(x|x_1)\}\,\psi(x_2)\,Tr(\Omega_i\gamma^{(s)}\gamma^{(s')}) \\
&-2\eta_i\overline{\psi}(x_4)\,\Omega_i\psi(x_2)\,Tr(\Omega_i\gamma^{(s)}\gamma^{(s')})\,\{\phi_{s'}^*(x_1|x)\,\phi_s(x_3|x) \\
&\qquad+\phi_s^*(x|x_3)\,\phi_{s'}(x|x_1)\}, \\
(C_4)=&-(1-\eta_i)\overline{\psi}(x_4)\,\Omega_i\gamma^{(s)}\gamma^{(s')}\Omega_i\{\tau\phi_s(x_2|x)\,\tau\phi_{s'}^*(x_1|x) \\
&\qquad+\tau\phi_s^*(x|x_2)\,\tau\phi_{s'}(x|x_1)\}\,\psi(x_3) \\
&+4\overline{\psi}(x_4)\,\Omega_i\gamma^{(s)}\gamma^{(s')}\Omega_i\psi(x_3)\,\{\phi_{s'}^*(x_1|x)\,\phi_s(x_2|x) \\
&\qquad+\phi_s^*(x|x_2)\,\phi_{s'}(x|x_1)\},
\end{aligned}
\tag{4·2}
$$

and γ's are γ_5 or $i\gamma_5\gamma_\mu$. Since ϕ^4 does not correspond to the pion, it has been omitted in the above equations, and the same assumption on the properties of φ as in § 3 is made.

In the case of treating the low energy pion phenomena, we shall be allowed to approximate the form factor Φ by a δ-function, and replace the wave function, $\phi_s(x|x')$, of the pion by $\phi_s(x|x)$, since the spatial extention of the internal wave function of the pion would be very small compared with the de Broglie wave length of the pion. All terms with $s=s'$, then, give isospin independent contributions, and terms with $s \fallingdotseq s'$ and containing τ explicitly give isospin dependent contributions. Putting $\eta_i=1$ for simplicity, and retaining terms with $s \fallingdotseq s'$, one has

$$\begin{aligned} g_i\epsilon_i \times (A_1) \cong & -2^3 g_5 \bar{\psi}(x) i\gamma_\mu \{\tau\phi(x|x)\,\tau\phi^*_\mu(x|x) \\ & \qquad + \tau\phi^*(x|x)\,\tau\phi_\mu(x|x)\}\psi(x) \\ & + 2^3 g_4 \bar{\psi}(x) i\gamma_\mu \{\tau\phi_\mu(x|x)\,\tau\phi^*(x|x) \\ & \qquad + \tau\phi^*_\mu(x|x)\,\tau\phi(x|x)\}\psi(x), \end{aligned} \tag{4·3}$$

where space-time extention of the internal wave function of the pion and also that of the interaction kernel were neglected. Discarding the odd Dirac operators, the above equation turns to

$$\begin{aligned} g_i\epsilon_i \times (A_1) \cong & -2^4 g_5 \psi^*(x)\,(\phi\cdot\pi + i\tau\cdot[\phi\times\pi])\,\psi(x) \\ & + 2^4 g_4 \psi^*(x)\,(\pi\cdot\phi + i\tau\cdot[\pi\times\phi])\,\psi(x) \end{aligned} \tag{4·4}$$

where π is the conjugate momentum to ϕ. The $\tau\cdot[\pi\times\phi]$ terms give rise to the desired isospin dependence of low energy S-wave pion-nucleon scattering phase shifts. Similar results are obtained from the remaining terms in eq. (4·2). These terms arises from the terms with $i=2$, 4, and 5 in the interaction kernel (2·1).

To calculate the low energy S-wave scattering phase shifts, one also need to know the magnitude of isospin independent parts. These terms have about the same magnitude as that in eq. (4·2). Moreover, contributions from the one pion vertex, eq. (3·10), must be calculated, which are at least of second order in g_i, contrary to the two pion vertex obtained in this section. Relative magnitudes of the contributions from these two processes are not so easy to be evaluated, since it needs the more detailed information about the structure of the pion as was briefly mentioned at the end of § 3. According to a very rough estimation such as given by Finkelstein[2], i.e., the order of magnitudes of g_i's being $10^{-42} \sim 10^{-44}$ erg·cm^3, contributions from the two pion vertex derived in this section may not necessarily be neglected compared with that from the repetition of the one pion vertex.

Next, we give Fourier transform of eq. (4·1). It is sufficient to consider only one term in eq. (4·1) explicitly. Putting $\eta_i=1$, $\gamma^{(s)}=\gamma^{(s')}=\gamma_5$, the contribution to $\int H^{(2)}_{\text{eff}} d^4\bar{X}$ from (A_1) is

$$\begin{aligned} -2^3 g_5 \int & d^4\bar{X}\, d^4X\, d^4x_{12}\, d^4x_{34}\, d^4x\, \Phi(X, x_{12}, x_{34}) \cdot \\ & \cdot \bar{\psi}(x_1)\{\tau\phi(x_2|x_4)\,\tau\phi^*(x|x_3) + \tau\phi^*(x_4|x_2)\,\tau\phi(x_3|x))\}\psi(x). \end{aligned} \tag{4·5}$$

Substituting eq. (3·12) in the above expression, one obtains

$$
\begin{aligned}
&-2^5 g_5 (2\pi)^{-16}\int d^4p\, d^4p'\, d^4k\, d^4k'\, \delta^4(p+k-p'-k') \cdot \\
&\quad \cdot \bar{\psi}(p') \left\{\left[\int d^4q \cdot G_1 \tau\phi(k,\ q)\right] \cdot \tau\phi^*\left(k',\ p-\frac{k'}{2}\right)\right. \\
&\quad \left. +\left[\int d^4q' G_2 \tau\phi^*(k',\ q')\right] \cdot \tau\phi\left(k,\ -p-\frac{k}{2}\right)\right\}\psi(p), \qquad (4\cdot 6)
\end{aligned}
$$

where

$$
G_1 = G\left(p'-\frac{k}{2}+q,\ \frac{p'}{2}+\frac{k}{4}-\frac{q}{2},\ -\frac{p}{2}+\frac{k}{4}+\frac{k'}{2}+\frac{q}{2}\right),
$$

$$
G_2 = G\left(p'+\frac{k'}{2}-q',\ \frac{p'}{2}-\frac{k'}{4}+\frac{q'}{2},\ -\frac{p}{2}-\frac{k}{2}-\frac{k'}{4}-\frac{q}{2}\right).
$$

In the local limit, eq. (4·6) turns to

$$
\begin{aligned}
&-2^3 g_5 (2\pi)^{-12}\int d^4p\, d^4p'\, d^4k\, d^4k'\, \delta^4(p+k-p'-k') \cdot \\
&\quad \cdot \bar{\psi}(p') \left\{\bar{\phi}(k)\phi^*\left(k',\ p-\frac{k'}{2}\right) + \bar{\phi}^*(k')\phi\left(k,\ -p-\frac{k}{2}\right)\right. \\
&\quad \left. + i\tau \cdot \left[\bar{\phi}(k) \times \phi^*\left(k',\ p-\frac{k'}{2}\right) - \phi\left(k,\ -p-\frac{k}{2}\right) \times \bar{\phi}^*(k')\right]\right\}\psi(p), \qquad (4\cdot 7)
\end{aligned}
$$

where $\bar{\phi}(k)$, being given by eq. (3·15), represents the amplitude for a nucleon and an antinucleon to be at the same space-time point.

If one is considering the low energy pion phenomena, the momentum of the center of mass and the relative momentum will be approximately separable, since effects of the Lorentz contraction will be small. We, then, have

$$
\phi(k,\ q) \simeq \bar{\phi}(k)\, v(q),
$$

where $v(q)$ is the internal wave function of the pion at rest in the momentum representation, and is normalized according to

$$
(2\pi)^{-4}\int d^4q\, v(q) = 1.
$$

Apparently there is no isospin dependent terms in eq. (4·7) in this approximation, and this fact corresponds to that in x-representation shown below eq. (4·2).

§ 5. Conclusions

We have discussed the effects of the nucleon-antinucleon amplitude in the physical pion on the pion-nucleon scattering, having started from the interaction kernel. Usually they have started their discussions from the Lagrangian, which determines the "dynamics" of the system, when they want to attack such a problem as a pair effect. However, it seems very hard to get a lucid insight if we start from the Lagrangian and does not separate the "kinematical" and "dynamical" aspects of the problem as in the conventional way. Therefore we assumed a formal expression of the interaction kernel for the relativistic integral equation of the nucleon-antinucleon system as given in § 2, which is general enough to cover both the compound pion theory and the conventional local pion theory. We have derived the "effective Hamiltonian" for the emission or absorption of one or two pions in § 3 and in § 4 respectively, neglecting for the latter the "three body" interaction. One can evaluate immediately their effects to the low energy pion-nucleon scatterings quantitatively, if the interaction kernel is given explicitly.

A very rough estimate shows that the "effective Hamiltonian" for the two pion vertex, which is characteristic of the structure of the pion, may not necessarily be neglected even in the low energy pion-nucleon scattering and may give a rather large isospin dependent phase shifts. It is qualitatively in agreement with experiment. We are, however, not able to discuss these points quantitatively, unless we go into the dynamics. Such a problem has not been treated but to be postponed to a future paper.

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Appendix. Crossing symmetry for the scattering of a particle with internal structure

We consider the pion to be composed of the same number of nucleons and antinucleons. In a relativistically invariant local field theory, the propagator for the two nucleon and one antinucleon system may be written as[9)]

$$\begin{aligned}
&G(x_1, x_2, x_3\,;\, x_1', x_2', x_3')\\
&=i\{\langle T(\psi(x_1)\psi(x_2)\psi'(x_3)\bar{\psi}(x_1')\bar{\psi}(x_2')\bar{\psi}'(x_3'))\rangle\\
&\quad-\langle T(\psi(x_1)\psi(x_2)\psi'(x_3)\bar{\psi}(x_1'))\rangle\langle T(\bar{\psi}(x_2')\bar{\psi}'(x_3'))\rangle\\
&\quad+\langle T(\psi(x_1)\psi(x_2)\psi'(x_3)\bar{\psi}(x_2'))\rangle\langle T(\bar{\psi}(x_1')\bar{\psi}'(x_3'))\rangle\},
\end{aligned}\tag{A·1}$$

where $\psi(x)$ is the Heisenberg operator for the nucleon field, $\psi'(x)$ is its charge conjugate, and $\langle T(\ )\rangle$ means the true vacuum expectation value of the T-product; the second and third terms in the right side of eq. (A·1) merely correct for the fact that the uncoupled nucleon-antinucleon system cannot undergo a virtual annihilation process.

Let us suppose that the nucleon pair ($\psi(x_2)$ and $\psi'(x_3)$, $\bar{\psi}(x_2')$ and $\bar{\psi}'(x_3')$) compose a pion, then the pion-nucleon propagator may be written as

$$\begin{aligned}
&T_{ij}(p, p'\,;\, q, q')\\
&=i\int d^4x_1\, d^4x_1'\, d^4y\, d^4y' \exp i(px_1-p'x_1'+qy-q'y')\cdot\\
&\cdot\int d^4z\int d^4z'\cdot\chi_i(y, z)\,G(x_1, x_2, x_3\,;\, x_1', x_2', x_3')\,\chi_j(y', z'),
\end{aligned}\tag{A·2}$$

where p', q' are the nucleon and pion incoming four momenta, respectively, and p, q are the outgoing momenta, $y=(x_2+x_3)/2$, $z=x_2-x_3$, $y'=(x_2'+x_3')/2$, $z'=x_2'-x_3'$, and $\chi_i(y, z)$, $\chi_j(y', z')$ are the wave functions of the incoming and outgoing pions, respectively.

Substituting (A·1) in eq. (A·2) we obtain

$$\begin{aligned} T_{ij}(p, p'; q, q') = \int & d^4x_1\, d^4x_1'\, d^4y\, d^4y' . \\ & \exp i(px_1 - p'x_1' + qy - q'y') \cdot \\ & \cdot \int d^4z\, d^4z' \langle T(\psi(x_1)\bar{\psi}(x_1')\phi_i(y, z)\phi_j(y', z'))\rangle, \end{aligned} \qquad (\mathrm{A}\cdot 3)$$

where

$$\phi_i(y, z) = \chi_i(y, z)\psi(x_2)\psi'(x_3),$$
$$\phi_j(y', z') = \chi_j(y', z')\bar{\psi}(x_2')\bar{\psi}'(x_3').$$

In eq. (A·3) we write only the first term of eq. (A·1) for simplicity. However, it will be seen easily that the following discussions hold also in the case that the omitted terms are included.

Since $\phi_i(y, z)$ and $\phi_j(y', z')$ commute inside a T-product, and since y and y' are dummy variables, the right side of eq. (A·3) is equal to

$$\begin{aligned} T_{ij}(p, p'; q, q') = \int & d^4x_1\, d^4x_1'\, d^4y\, d^4y' . \\ & \cdot \exp i(px_1 - p'x_1' - q'y + qy') \cdot \int d^4z\, d^4z' . \\ & \cdot \langle T(\psi(x_1)\bar{\psi}(x_1')\phi_j(y, z)\phi_i(y', z'))\rangle \\ = & T_{ji}(p, p'; -q', -q). \end{aligned} \qquad (\mathrm{A}\cdot 4)$$

Consequently, the crossing symmetry holds also for the case of the scattering of a compound particle, provided that the propagator, (A·1), exists. Clearly, the proof given above, which is essentially the same as that given by Feldman and Matthews[15)], can be valid, even if the incoming and outgoing particles are composed of arbitrary number of fermion pairs and bosons. If the propagators such as eq. (A·1) do not exist, we cannot say whether the crossing symmetry holds or not.

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Pairing Energy of Nuclear Particles

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From the nuclear binding energies given by Wapstra, the pairing energies of identical nucleons have been computed and found to be expressed in the form $(22.3\pm5.5)(2j+1)/A$ Mev for the even-even nuclei with mass number A, and pairs in a definite j-shell. By using the computed values of the pairing energy for the even-even and odd A nuclei and assuming the zero range force, we have found the inter-nucleon potential to be $V_{ij}=-570r_0^3[1+(0.16\pm0.15)\sigma_i\cdot\sigma_j]\delta(\boldsymbol{r}_i-\boldsymbol{r}_j)$ Mev, where r_0 is the nucleon radius.

§ 1. Introduction

The nuclear shell model has had a considerable success in the explanation of the spin values and parities of the medium and heavy weight nuclei by introducing the pairing energy.[1] Furthermore, this pairing energy leads to a satisfactory understanding for the difference of the β-stability looked between the even and odd A nuclei. The pairing energy is estimated as the difference: the binding energy of the second neutron (or proton, excepting the contribution from the Coulomb repulsion) minus that of the first neutron (or proton) in the case when two neutrons (or protons) are successively added to the nucleus with even neutron (or proton) number.[2]

Recently, Wapstra[3] and Huizenga[4] have computed the binding energies of many nuclei from the Q values of nuclear reaction, the nuclear disintegration energies and the mass spectroscopic data. It is possible, therefore, to calculate the pairing energies for more nuclei than such calculations have been made in the past,[1],[5] by using these values of the binding energy, and to find the dependence of the pairing energy on the nuclear properties and the state occupied by the pair.

For the even-even nuclei, assuming the nuclear force of zero range, Mayer[1] has theoretically estimated the pairing energy according to the $j-j$ coupling scheme and has found it proportional to $(2j+1)/A$ when the nuclear mass number is A and the pair is in a definite j-shell, and the proportional coefficient was suggested to be about 12.5 Mev. In § 2, we examine the dependence of the pairing energy for the even-even nucleus on the nuclear mass, the kind of the particles which compose the pair, the state occupied by the pair and also examine Mayer's form critically, by using the pairing energies computed form Wapstra's data. From the pairing energies for the even-even nuclei, we are able to estimate the strength of the singlet nuclear force only; such an estimation is made in § 2.

From the results calculated by using Wapstra's binding energy data, it is found that the pairing energy for the odd A nucleus is systematically smaller than that for the even-even nucleus by factor $1/2$ to $2/3$ when the pair is in the same shell and their masses are close to one another. This difference between two kinds of the pairing energy may depend on the character of the neutron-proton force. In § 3, we estimate the strength of the neutron-proton interaction from this difference and the strength of the nuclear force for the triplet spin state on the assumption of charge independence. Furthermore, in § 3, the internucleon potential is formed in terms of the obtained force strength for the singlet and triplet spin states and is compared with the potential given by Schwartz.[6] It is shown from this comparison that the ratio of the triplet and singlet force strength of the potential obtained here from the pairing energies is in good agreement with that of Schwartz's potential which is obtained from the consideration for the ground state spin of the odd nuclei.

§ 2. Pairing energy for even-even nucleus

Let us assume that the nucleus can be represented by the shell model. We then denote shortly the state characterized by the quantum numbers n, l and j as the shell j. If $S_{n(p)}(p, n)$ stands for the separation energy of a neutron (proton) in the last neutron (proton) shell of the nucleus in which n neutrons and p protons occupy the last neutron and proton shells j_n and j_p, respectively, then the pairing energy $E^e_{\text{pair}(n)}$ of the next neutron pair to the even-even nucleus becomes, according to the estimates of de-Shalit[7] for $S_{n(p)}(p, n)$,

$$\begin{aligned} E^e_{\text{pair}(n)} &= S_n(p, n+2) - S_n(p, n+1) \\ &= E_{\text{int}}(j_n^{n+2}) + E_{\text{int}}(j_n^{n}) - 2E_{\text{int}}(j_n^{n+1}) + [p, n+2] + [p, n] - 2[p, n+1] \qquad (2\cdot1) \end{aligned}$$

p and n even,

where $E_{\text{int}}(j^k)$ is the interaction energy of k identical nucleons in the shell j with each other, and $[p, n]$ is the interaction energy of p protons and n neutrons in the shells j_p and j_n, respectively. Since we have $[p, k] = pkF(j_p, j_n)$ (Eq. (7) in reference 7), assuming that the identical nucleons of even number couple to a zero total angular momentum, the last three terms in (2·1) cancel each other. Thus

$$E^e_{\text{pair}(n)} = E_{\text{int}}(j_n^{n+2}) + E_{\text{int}}(j_n^{n}) - 2E_{\text{int}}(j_n^{n+1}). \qquad (2\cdot2)$$

We get similarly for the pairing energy of the proton pair, $E^e_{\text{pair}(p)}$,

$$E^e_{\text{pair}(p)} = E_{\text{int}}(j_p^{p+2}) + E_{\text{int}}(j_p^{p}) - 2E_{\text{int}}(j_p^{p+1}). \qquad (2\cdot2\text{a})$$

Now, we assume the nuclear force of the zero range after Mayer. The internucleon potential between the ith and jth nucleons is then expressible by a δ function

$$V_{ij} = \left(\frac{1+p_\sigma}{2} V_t + \frac{1-p_\sigma}{2} V_s\right)\delta(\boldsymbol{r}_i - \boldsymbol{r}_j), \qquad (2\cdot3)$$

where P_σ represents the spin exchange operator $(1+\sigma_i \cdot \sigma_j)/2$, and V_t and V_s are the force strength for the triplet and singlet spin states, respectively. When even number of the identical particles couple to a zero total spin and odd number of the particles couple to j of the shell occupied by these particles, the interaction energy of such a group of the identical nucleons gives for both $E^e_{\text{pair}(n)}$ and $E^e_{\text{pair}(p)}$ 8)

Table I. The pairing energies of the next neutron or proton pair to the even-even nuclei, calculated from Wapstra's binding energy data. The elements in the first and 4th columns show the original even-even nuclei. The configurations in the third and last columns show the shells occupied by the pair.

Original nucleus	$E^e_{\text{pair}(n)}$ (Mev)	Shell	Original nucleus	$E^e_{\text{pair}(p)}$ (Mev)	Shell
Be^8	5.15±0.04	1*p* 3/2	C^{12}	3.06±0.08	1*p* 1/2
C^{12}	3.23±0.08	1*p* 1/2	C^{14}	2.27±0.09	,,
O^{16}	3.93±0.10	1*d* 5/2	O^{16}	3.89±0.27	1*d* 5/2
Ne^{20}	3.61±0.14	,,	Ne^{22}	3.22±0.16	,,
Mg^{24}	3.79±0.16	,,	Mg^{26}	3.64±0.18	,,
Mg^{26}	2.07±0.18	2*s* 1/2	Si^{30}	1.95±0.19	2*s* 1/2
Si^{28}	2.14±0.18	,,	S^{34}	2.37±0.23	1*d* 3/2
Si^{30}	2.72±0.22	1*d* 3/2	S^{36}	2.00±0.32	,,
S^{32}	2.74±0.21	,,	A^{38}	2.18±0.33	,,
S^{34}	2.88±0.24	,,	A^{40}	2.66±0.31	,,
A^{38}	3.21±0.32	1*f* 7/2	Ca^{44}	3.84±0.40	1*f* 7/2
Ca^{40}	3.11±0.32	,,	Ti^{48}	3.14±0.46	,,
Ca^{42}	3.23±0.38	,,	Ti^{50}	2.58±0.50	,,
Ti^{46}	2.73±0.40	,,	Cr^{50}	2.80±0.62	,,
Ti^{48}	2.87±0.40	,,	Cr^{52}	2.63±0.58	,,
Cr^{50}	2.85±0.56	,,	Cr^{54}	2.23±0.62	,,
Cr^{52}	1.78±0.56	2*p* 3/2	Fe^{58}	2.62±0.92	,,
Fe^{54}	1.87±0.64	,,	Ni^{62}	2.08±0.88	2*p* 3/2
Zn^{66}	3.30±0.84	1*f* 5/2	Ni^{64}	1.84±0.84	,,
Zn^{68}	2.90±0.84	1*p* 1/2	Kr^{84}	3.07±0.84	1*f* 5/2
Ge^{72}	3.46±0.64	1*g* 9/2	Sr^{88}	1.66±0.91	2*p* 1/2
Se^{78}	2.88±0.71	,,	Sr^{90}	2.08±0.90	,,
Se^{80}	2.39±0.86	,,	Zr^{90}	3.19±1.17	1*g* 9/2
Kr^{80}	3.02±1.00	,,	Zr^{92}	2.41±1.05	,,
Kr^{82}	3.00±0.88	,,	Zr^{94}	2.71±1.59	,,
Kr^{84}	2.73±0.88	,,	Mo^{94}	2.67±1.60	,,
Sr^{86}	2.58±0.82	,,	Cd^{110}	2.28±2.41	,,
Sr^{88}	1.65±0.91	2*d* 5/2	Cd^{112}	2.56±2.10	,,
Sr^{90}	1.78±0.93	,,	Cd^{114}	2.50±2.00	,,
Zr^{90}	1.51±0.89	,,	—	——	—
Mo^{92}	1.79±1.20	,,	—	——	—
Mo^{94}	2.02±1.53	,,	—	——	—
Mo^{96}	1.38±1.93	,,	—	——	—

$$E^e_{\text{pair}} = -\frac{2j+1}{8\pi} V_s \int R^4(n,\ l,\ j;\ \boldsymbol{r}) r^2 dr, \tag{2·4}$$

where $R(n,\ l,\ j;\ \boldsymbol{r})$ is the radial wave function of the particle in the shell j.

For the nuclei in the region of $A < 120$, the values of E^e_{pair} calculated from the Wapstra's binding energy data are listed in Table I, and the values of $E^e_{\text{pair}}/(2j+1)$ are graphically shown as a function of A in Fig. 1 by solid and open circles, corresponding

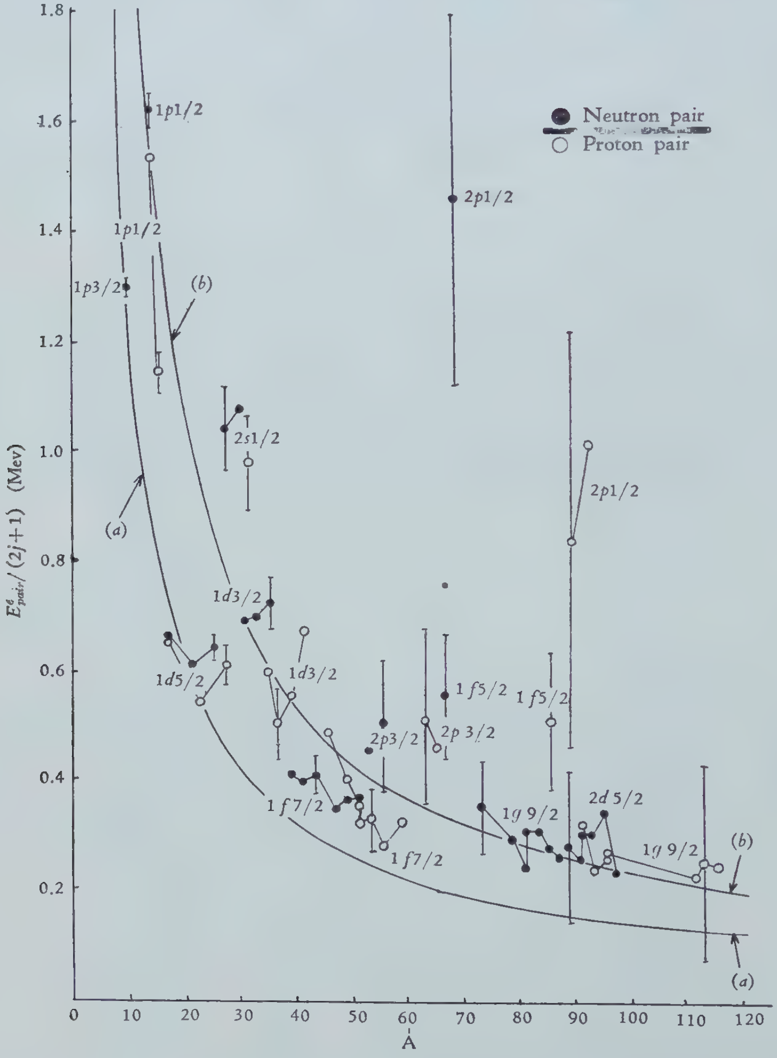

Fig 1. $E^e_{\text{pair}}/(2j+1)$ values calculated from the data listed in Table I, as a function of A. The points corresponding to the pairs in the same j shell are connected by straight lines, and the errors are indicated only on a single point for each of the shells. The configurations show the shells occupied by the pair. (a) shows $25/2A$ Mev. (b) shows $22.3/A$ Mev.

to the neutron and proton pairs, respectively. For the nuclei with mass number A larger than 120, the calculations of the pairing energies are not carried out, because there are considerably large errors in their binding energies and hence it is not possible to estimate the pairing energies with the satisfactory accuracy for the present investigation. In Fig. 1, the points corresponding to the pairs in the same j shell are connected by straight lines; the errors are indicated only on a single point for each of the shells, since the errors belonging to the same shell are of the same order. The curve (a) shows $25/2A$ Mev suggested by Mayer.

It is seen from Fig. 1 that a) there is not much difference between the pairing energies of proton and neutron pairs; b) the pairing energies of the pairs in the shell $j=l-1/2$ are systematically higher than those of the pairs in the shell $j'=l+1/2$*; c) bulk of the points assemble around a curve roughly proportional to $1/A$ but curve (a) seems to be too low for these points. Corresponding to each point in Fig. 1, the values of $AE^e_{\text{pair}}/(2j+1)$ are also shown (omitting the errors) as a function of A in Fig. 2. Their mean value is 22.3 Mev** and the mean error is estimated to be 5.5

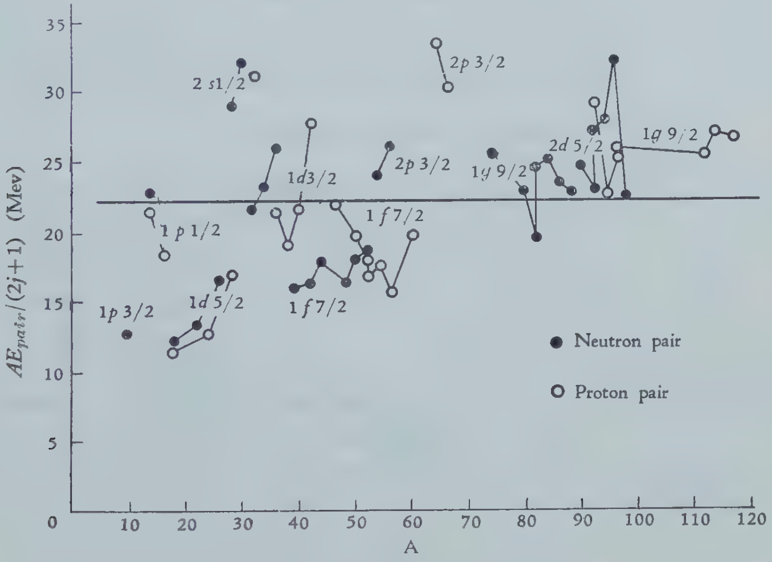

Fig. 2. $AE^e_{\text{pair}}/(2j+1)$ values calculated from the data listed in Table I, as a function of A. The horizontal line indicates the mean of these values, 22.3 Mev. The configurations show the shells occupied by the pair.

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* The integral factor in (2·4) may decrease with increasing nuclear mass for any familiar nuclear potential. However, though the masses of the nuclei which have the last pair in the shell $1d\,3/2$ or $1f\,5/2$, etc., are larger than those of the nuclei which have the last pair in the shell $1d\,5/2$ or $1f\,7/2$, etc., the pairing energies for the former are generally higher than those for the latter as seen from Fig. 1. Therefore, the pairing energy of the pair in the shell $j=l-1/2$ may be appreciably higher than that of the pair in the shell $j'=l+1/2$ for the nuclei with the same mass.

** The 5 points belonging to $2p\,1/2$ and $1f\,5/2$ are so far away from the other points that they cannot be plotted in Fig. 2, and hence these points are omitted in the estimation of the mean value.

Mev. Thus we get 22.3 ± 5.5 Mev for the most suitable proportional coefficient of E^e_{pair} to $(2j+1)/A$. The curve corresponding to this proportional coefficient is shown as (b) in Fig. 1.

To estimate the variance of the integral factor in (2·4) for the various nuclei, we calculate this integral in the case when the nuclear potential is a central harmonic oscillator well. A detailed account of the radial oscillator well function is to be found in the thesis of Talmi.[9)] In the manner[2)] that the mean value $\langle r^2\rangle_{nl}$ of r^2 for the particle in the last shell j is approximately replaced by the square of the nuclear radius, R^2, we calculated this integral for the various n, l values, referring to Table 2 in reference 10, and got

$$\int R^4(n,\ l,\ j;\ \mathbf{r})r^2dr = C_{nl}/R^3 = C_{nl}/Ar_0^3, \tag{2·5}$$

where r_0 is the nucleon radius $r_0 = 1.4\times10^{-13}$ cm. The values of C_{nl} are numerically given in Table II. Thus we have for E^e_{pair}

$$E^e_{\text{pair}} = -\frac{C_{nl}}{8\pi r_0^3 A}V_s(2j+1). \tag{2·6}$$

Table II. Values of C_{nl} which enter into (2·5).

n \ l	0	1	2	3	4
1	1.47	1.32	1.38	1.47	1.56
2	3.36	2.17	1.90	—	—

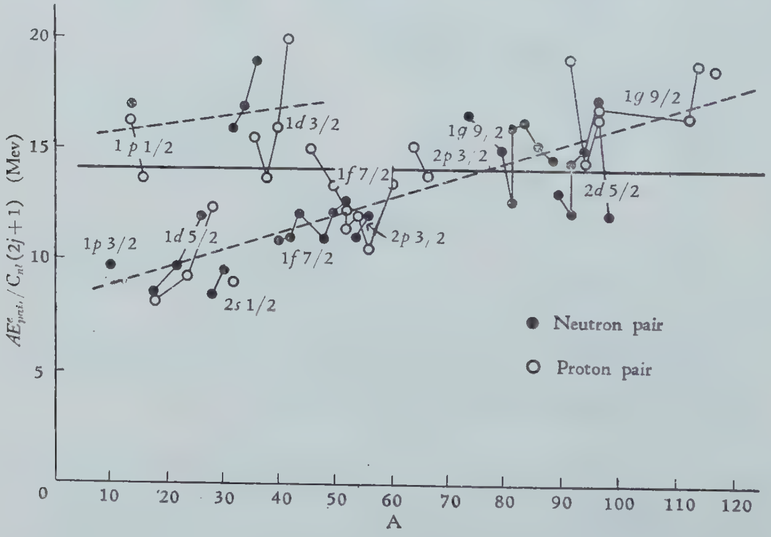

Fig. 3. $AE^e_{\text{pair}}/C_{nl}(2j+1)$ values calculated from the data listed in Table I by using the C_{nl} values in Table II, as a function of A. The horizontal line indicates the mean of these values, 14.2 Mev. The upper and lower dashed lines represent the distribution of the points belonging to the shells $j=l-1/2$ and $j'=l+1/2$, respectively.

The values of $AE^e_{\text{pair}}/C_{nl}(2j+1)$ are shown against A in Fig. 3, corresponding to each point in Fig. 2 and using the values of C_{nl} in Table II. From Fig. 3, it is seen that a) bulk of the points distribute in a region from 9 Mev to 20 Mev, and the mean value, with the mean error, is estimated to be 14.2±3.1 Mev; b) there is a tendency that $AE^e_{\text{pair}}/C_{nl}(2j+1)$ values increase slowly with increasing A; c) the points belonging to the shell $j=l-1/2$ distribute in the higher place than those belonging to the shell $j'=l+1/2$, separating from one another.

Thus we get for the most probable value of the strength V_s of the force for the singlet spin state

$$V_s=-(14.2\pm3.1)\times8\pi r_0^3 \text{ Mev} \tag{2·7}$$

from the pairing energies for the even-even nuclei. The fact mentioned in b) may show that the internucleon potential and the nuclear potential deviate from the δ function type and the oscillator well, respectively. From the fact mentioned in c), it seems that the spin-orbit coupling for the nuclear particle makes some contribution to the integral in (2·4), and hence that C_{nl} depends not only on n and l but also on j, for the singlet force strength V_s which is proportional to $AE_{\text{pair}}/C_{nl}(2j+1)$ with the constant factor probably does not depend on j.

§ 3. Internucleon potential

Let us consider now the difference between the pairing energies for the even-even and odd A nuclei. The pairing energy $E^o_{\text{pair}(n)}$ of the next neutron pair to the odd-even nucleus with $n-1$ neutrons in the shell j_n is given, as in (2·1),

$$E^o_{\text{pair}(n)}=E_{\text{int}}(j_n^{n+1})+E_{\text{int}}(j_n^{n-1})-2E_{\text{int}}(j_n^{n})+[p,\ n+1]+[p,\ n-1]-2[p,\ n] \tag{3·1}$$

$$p \text{ and } n \text{ odd.}$$

Thus, subtracting (3·1) from the pairing energy (2·1) for the even-even nucleus with $n-1$ neutrons and $p\pm1$ protons (replacing $n\to n-1$, $p\to p\pm1$ in (2·1)), the difference $\Delta_n(n-1,\ p\pm1)$ between these two pairing energies becomes

$$\begin{aligned}\Delta_n(n-1,\ p\pm1)&=E^e_{\text{pair}(n)}-E^o_{\text{pair}(n)}\\&=2[p,\ n]-[p,\ n+1]-[p,\ n-1].\end{aligned} \tag{3·2}$$

This expression is also concluded in the case when the neutron number of the odd-even nucleus is not equal to that of the even-even nucleus and the proton numbers differ by two or more from one another, since the former three terms in (2·1) and (3·1) are reduced to the same expression indifferently to the neutron number in the shell j_n, as seen from (2·4), and $E^e_{\text{pair}(n)}$ does not depend on the proton number of the even-even nucleus.

Using the expression for $[p,\ n]$ given by Schwartz[6] and denoting $\Delta_n(n-1,\ p\pm1)$ as Δ_n in short by means of the above reason, we have, when the interaction between proton and neutron is of the type (2·3),

$$\Delta_n = \frac{3V_t + V_s}{2}\left[pnF_0 - E_W(j_p^p J_p = j_p,\ j_n^n J_n = j_n,\ J)\right]$$

$$-\frac{V_t - V_s}{2} E_\sigma(j_p^p J_p = j_p,\ j_n^n J_n = j_n,\ J), \tag{3·3}$$

where

$$E_W(j_p^p J_p = j_p,\ j_n^n J_n = j_n,\ J) = \frac{(2j_p+1-2p)(2j_n+1-2n)}{(2j_p-1)(2j_n-1)}$$

$$\times \frac{(2j_p+1)(2j_n+1)}{2(2J+1)}\left(j_p \frac{1}{2} j_n \frac{-1}{2} \middle| j_p j_n J 0\right)^2$$

$$\times \left\{1 + \frac{[(2j_p+1) + (-)^{j_p+j_n+J}(2j_n+1)]^2}{4J(J+1)}\right\} F_0$$

$$+\left[pn - \frac{(2j_p+1-2p)(2j_n+1-2n)}{(2j_p-1)(2j_n-1)}\right] F_0, \tag{3·4a}$$

$$E_\sigma(j_p^p J_p = j_p,\ j_n^n J_n = j_n,\ J) = \frac{(2j_p+1)(2j_n+1)}{2(2J+1)}\left(j_p \frac{1}{2} j_n \frac{-1}{2} \middle| j_p j_n J 0\right)^2$$

$$\times \left\{\frac{[(2j_p+1) + (-)^{j_p+j_n+J}(2j_n+1)]^2}{4J(J+1)} - [1 + 2(-)^{l_p+l_n+J}]\right\} F_0, \tag{3·4b}$$

$$F_0 = \frac{1}{4\pi}\int R^2(n_p,\ l_p,\ j_p,\ \mathbf{r})\, R^2(n_n,\ l_n,\ j_n,\ \mathbf{r})\, r^2 dr, \tag{3·4c}$$

Table III. Δ_n and Δ_p calculated from Wapstra's binding energy data. The elements in the first column are the odd-odd nuclei which have one more neutron or proton than the odd A nuclei in question, and their spin are shown in the second column. The configurations in the 5th and last columns show the last neutron and proton shells of the odd-odd nuclei, respectively.

Nucleus	Spin ($\hbar$)	Δ_n (Mev)	Δ_p (Mev)	Last neutron shell	Last proton shell
B^{10}	3	2.25 ± 0.14	2.60 ± 0.11	$1p\ 3/2$	$1p\ 3/2$
N^{14}	1	2.47 ± 0.16	2.46 ± 0.16	$1p\ 1/2$	$1p\ 1/2$
N^{16}	2	1.48 ± 0.42		$1d\ 5/2$	$1p\ 1/2$
F^{18}	1	3.48 ± 0.30	3.88 ± 0.30	$1d\ 5/2$	$1d\ 5/2$
Al^{26}	5	2.07 ± 0.35	2.21 ± 0.35	$1d\ 5/2$	$1d\ 5/2$
Al^{28}	3	0.67 ± 0.40	0.68 ± 0.41	$2s\ 1/2$	$1d\ 5/2$
P^{30}	1	1.00 ± 0.40	0.90 ± 0.45	$2s\ 1/2$	$2s\ 1/2$
P^{32}	1	0.56 ± 0.42	0.56 ± 0.41	$1d\ 3/2$	$2s\ 1/2$
Cl^{34}	3	1.36 ± 0.50	0.67 ± 0.50	$1d\ 3/2$	$1d\ 3/2$
K^{38}	3	1.83 ± 0.76	1.56 ± 0.76	$1d\ 3/2$	$1d\ 3/2$

where $\left(j_p \frac{1}{2}\, j_n \frac{-1}{2} \middle| j_p j_n J 0\right)$ represents the Clebsh-Gordan coefficient. For the difference Δ_p between the pairing energies of the next proton pairs to the even-even and even-odd nucleus, we also get the same expression with (3·3) in like manner.

The values of Δ_n and Δ_p estimated from Wapstra's binding energies in connection with several even-even and odd A nuclei are listed in Table III together with the odd-odd nuclei with one more neutron or proton than these odd A nuclei. Each of the numbers in the third column is the mean value of $\Delta_n(n-1, p+1)$ and $\Delta_n(n-1, p-1)$ (n and p odd), and that in the 4th column is to be similarly understood. It is seen from this table that there are not much difference between Δ_n and Δ_p.

We want to estimate the value of V_t in the interaction (2·3) by using the values of Δ_n and Δ_p in Table III, assuming the charge independence of the nuclear force. To do this we use again the harmonic oscillator wave function for $R(n, l, j; \boldsymbol{r})$ in (3·4c). The values of $V_t/8\pi r_0^3$ was calculated thus by using $V_s/8\pi r_0^3 = -14.2$ Mev and substituting each of Δ_n and Δ_p in Table III in the left hand side of Eq. (3·3), and the obtained results are shown in Fig. 4. From this figure it seems that the values of $V_t/8\pi r_0^3$ distribute in a region from -18 Mev to -32 Mev and they slowly increase with increasing A. According to the least square method, we get for the most probable values, with the mean error, of the strength of the force for the triplet spin state

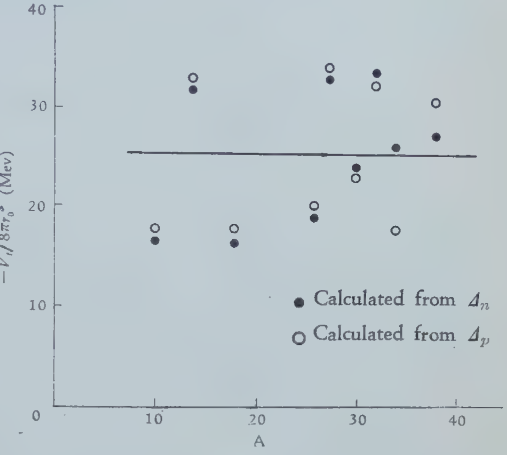

Fig. 4. $-V_t/8\pi r_0^3$ values as a function of A, calculated from the values of Δ_n and Δ_p listed in Table III by using Eqs. (3·3), (3·4) and $V_s/8\pi r_0^3 = -14.2$ Mev. The horizontal line indicates the mean of these values, 25.4 Mev.

$$V_t = -(25.4 \pm 6.7) \times 8\pi r_0^3 \text{ Mev.} \qquad (3\cdot5)$$

From (2·7) and (3·5) we get for the charge independent internucleon potential (in Mev)

$$V_{ij} = -8\pi r_0^3 \left[(25.4 \pm 6.7) \frac{1+p_\sigma}{2} + (14.2 \pm 3.1) \frac{1-p_\sigma}{2} \right] \delta(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

$$= -570 r_0^3 [1 + (0.16 \pm 0.15)\, \sigma_i \cdot \sigma_j]\, \delta(\boldsymbol{r}_i - \boldsymbol{r}_j). \qquad (3\cdot6)$$

Shwartz[6] has assumed the internucleon potential of the following type:

$$V_{ij} = -[(1-\alpha) + \alpha \sigma_i \cdot \sigma_j]\, \delta(\boldsymbol{r}_i - \boldsymbol{r}_j) \times \text{positive constant}, \qquad (3\cdot7)$$

and he has found the values of α for several odd nuclei through the consideration of their spin. His results are listed in Table IV. As seen from a comparison between

Table IV. α in Schwartz's potential: $V_{ij}=-[(1-\alpha)+\alpha\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j]\delta(\boldsymbol{r}_i-\boldsymbol{r}_j)$ $\times$positive constant.

Nucleus	Spin ($\hbar$)	α
Li^6	1	$\geqslant 0.10$
Li^8	2	$\leqslant 0.25$
B^8	2	$\leqslant 0.25$
B^{10}	3	$\leqslant 0.10$
Na^{24}	4	$\leqslant 0.42$
Cl^{36}	2	$\leqslant 0.25$
K^{40}	4	$\leqslant 0.50$
K^{42}	2	$\geqslant 0.17$
Sc^{46}	4	$\leqslant 0.30$
Co^{58}	2	$\leqslant 0.10$
Co^{60}	5	$\geqslant 0.10$; $\leqslant 0.50$
Rb^{88}	2	$\geqslant 0.17$; $\leqslant 0.50$
Sb^{122}	2	$\geqslant 0.10$

(3·6) and (3·7), referring to Table IV, the rate of combination of the spin dependent term in the potential obtained from the present analysis in general is in good agreement with that of Schwartz's potential.

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Analysis of Neutron-Electron Interaction in the Relativistic Perturbation Theory

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The recent experiments indicate that the charge distribution of the physical neutron is almost neutralized by the cancellation of the nucleon core and the meson cloud charges, which can hardly be expected from the fixed source theory. On the other hand, the relativistic *ps*-coupling theory, which indicates that both the nucleon core and the meson cloud spread out over the region of about twice the nucleon Compton wave length, seems to be more promising. In this paper, we then analyze perturbation-theoretically the effective neutron-electron potential, evaluating separately for each time-ordered graph so as to compare with the result of the fixed source theory. It is found that (1) owing to the nucleon recoil effect, the *p*-wave meson contribution is only about 1/2.3 times the one in the fixed source theory; (2) the $(-)$ energy state contribution for the nucleon core charge is as large as that of $(+)$ energy state; and (3) *s*-wave meson contributions, which are not involved in the fixed source theory, are taken into account qualitatively. Thus the main discrepancy with experiments arises from the inadequate treatment in the fixed source theory, though a part of the yet remaining discrepancy might be explained by the modification of the quantum electrodynamics as has been discussed by Yennie et al.

§ 1. Introduction

According to the experiments at Brookhaven,[1] the effective neutron-electron potential is given by $-eV_0=(-3.86\pm0.37)$ kev if the square well with a radius equal to the classical electron radius is assumed.

Since the effective potential due to the anomalous magnetic moment is about -4.1 kev,[2] these experiments show that the static potential, caused by the spread-out charge distribution, should almost vanish.

Recently, the neutron-electron interaction has been extensively investigated in the fixed source theory, calculating the static potential due to the electromagnetic interaction between the electron and the meson cloud surrounding the nucleon core. The perturbation method gave, for $g^2/4\pi=0.08$, (g is rationalized *pv*-coupling constant.) the static potential $-eV_0=-12.4$ kev,[3] which is in serious disagreement with experiments. In order to account for this discrepancy, Salzman[4] has assumed that the nucleon core itself has the charge distribution proportional to the source function, and has shown that the considerable cancellation of charge occurs between the meson cloud and the core. This idea, however, could not be included in mathematically consistent way in the frame of the fixed source theory, though it has perhaps the physical grounds. Sachs and Treiman[3] have investigated the possibility that the higher order corrections alter the meson cloud charge distribution using the Chew-Low theory. It has been found, however, that the reduction of the effective potential due to this effects is only about 20%.

On the other hand, the perturbation-theoretic treatment in the relativistic pseudoscalar coupling theory has also been done. It gives $-eV_0=-178(f^2/4\pi)$ ev[5] for the static potential, where f is rationalized *ps*-coupling constant and is related to g by $g=f\mu/m$, denoting the meson and nucleon mass by μ and m respectively. Thus, taking $g^2/4\pi=0.08$, one obtains $-eV_0=-2.58$ kev. This is about one fifth the value in the fixed source theory, and closes considerably with the experiments.

This situation is due to the fact that both the charge of the meson cloud and that of the nucleon core spread out over the region of size between meson Compton wave length and nucleon Compton wave length. In fact, the root mean square (rms) radii of the charge distributions turn out to be, for $g^2/4\pi=0.08$,

$$\langle r^2\rangle_n^{1/2}\approx 1.8\hbar/mc, \quad \langle r^2\rangle_m^{1/2}\approx -2.3\,\hbar/mc$$

for the nucleon core and the meson cloud respectively.

Apparently, one might expect that the spreads of the charges of the nucleon core and of the meson cloud would be about $\hbar/mc$ and $\hbar/\mu c$ respectively, and that cancelling out of both the charge distributions could not be expected.[6]

One of the reason that the meson cloud spreads out really over the region small compared with $\hbar/\mu c$ is that the low energy mesons, which are actually effective for the spread of charge, are emitted as *p*-waves and therefore only with small probabilities. The other reason is that the relativistic treatment predicts, owing to the nucleon recoil effect, much smaller dissociation probabilities for the high momentum mesons than in the fixed source theory.

As to the nucleon core, the spread of the charge might be of the order $\hbar/mc$, if only the process in which the nucleon recoils to the $(+)$ energy state is concerned. The actual spread, being twice as large as $\hbar/mc$, is of course due to the nucleon pair production in intermediate states.

The detailed circumstances such as were discussed in the above, however, are hardly manifested in the relativistically invariant calculation where the $(+)$ and $(-)$ energy states are treated without distinction. For this reason, we shall analyze contributing diagrams with different intermediate states separately in the similar ways as has been investigated in the analysis of magnetic moment by Goto[7] and Hasegawa et al.[8]

In § 2, we derive the rms radius of charge distribution and the effective potential for each diagram, expressing these quantities as the integral over the momenta carried by the emitted particles in intermediate states. In § 3, the discussions of these results are given from the physical point of view as far as possible, and the comparison with the fixed source theory is done. Those discussions are expected to be independent, to some extent, of the perturbation treatment.

§ 2. Derivation of the effective potential

The nucleon-meson interaction Hamiltonian is taken as

$$H_{n\pi}=if\int\bar{\psi}(x)\gamma_5\tau_i\psi(x)\phi_i(x)\,dv, \tag{2·1}$$

where f is *ps*-coupling constant, $\bar{\psi}(x)$ and $\psi(x)$ are the nucleon field operators, $\phi_i(x)$ the meson field operators, and τ_i the conventional isotopic spin variables. (Hereafter we use the natural units $\hbar=c=1$.) The interaction Hamiltonians of nucleon and meson with the external static electromagnetic field $A_\mu^{(e)}(x)$ are given by

$$H_{n\gamma}=-ie\int\bar{\psi}(x)\gamma_\mu(1+\tau_3)/2\cdot\psi(x)A_\mu^{(e)}(x)\,dv, \tag{2·2}$$

and

$$H_{\pi\gamma}=e\int\left[\frac{\partial\phi_2(x)}{\partial x_\mu}\phi_1(x)-\frac{\partial\phi_1(x)}{\partial x_\mu}\phi_2(x)\right]A_\mu^{(e)}(x)\,dv \tag{2·3}$$

respectively, where $\partial\phi_i/\partial t\equiv\pi_i$ are the canonical conjugate operators to ϕ_i. For the static electric field, which is produced by a point classical electron in our case, we have

$$A_\mu^{(e)}(x)=(0,\ 0,\ 0,\ i\varphi^{(e)}(x)), \tag{2·4}$$

$$H_{n\gamma}=e\int\psi^*(x)\,(1+\tau_3)/2\cdot\psi(x)\varphi^{(e)}(x)\,dv, \tag{2·5}$$

$$H_{\pi\gamma}=e\int[\pi_2(x)\phi_1(x)-\pi_1(x)\phi_2(x)]\varphi^{(e)}(x)\,dv. \tag{2·6}$$

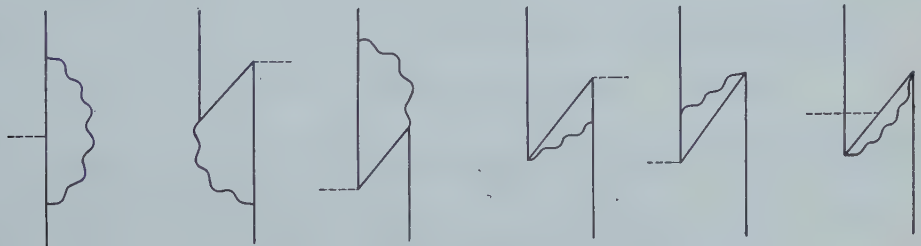
As is well known, the covariant nucleon vertex operator $\Gamma_\mu(p_1,\ p_2)$ takes the form, denoting $p_{1\nu}-p_{2\nu}=\Delta p_\nu$ and $\sigma_{\mu\nu}=\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu$,

$$\Gamma_\mu(p_1,\ p_2)=\gamma_\mu F_1((p_1-p_2)^2)+\sigma_{\mu\nu}\Delta p_\nu F_2((p_1-p_2)^2) \tag{2·7}$$

from the requirement of the relativistic invariance and gauge invariance, if p_1 and p_2 are free momenta. The first term is related to the rms radius of the charge distribution by

$$F_1((p_1-p_2)^2)=F_1(0)-\tfrac{1}{6}\langle r^2\rangle(p_1-p_2)^2+\cdots, \tag{2·8}$$

where $F_1(0)$ equals to 0 for neutron.5) The neutron-electron interaction potential is given by the relation :



(n, 1) (n, 2) (n, 3) (n, 4)

Fig. 1

$$-eV_0 = 1.36 \times \pi/3 \times \langle r^2 \rangle \text{ kev}, \tag{2·9}$$

when $\langle r^2 \rangle$ is measured in units $(\hbar/mc)^2 = 1$.

Thus if the external field is static, i. e. if there occurs no energy transfer, one needs only the quantities proportional to $(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2$ in $\Gamma_\mu(p_1, p_2)$ in order to get $\langle r^2 \rangle$ and $-eV_0$, where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are 3-dimensional momenta.

A. Nucleon contribution to electron-neutron interaction

The time-ordered graphs contributing to the interaction of the nucleon core with the static field are illustrated in Fig. 1. The contribution from $(n, 1)$ in Fig. 1, for example, is obtained as follows. According to the time-independent perturbation method, the second order scattering matrix $R^{(2)}$ is given by

$$\begin{aligned}\langle \boldsymbol{p}_1 | R^{(2)} | \boldsymbol{p}_2 \rangle &= \langle \boldsymbol{p}_1 | H_{n\pi} \frac{1}{E - H_0 + i\epsilon} H_{n\gamma} \frac{1}{E - H_0 + i\epsilon} H_{n\pi} | \boldsymbol{p}_2 \rangle \\ &= \sum_q -f^2 e u^*(\boldsymbol{p}_1) \beta \gamma_5 \tau_i u(\boldsymbol{p}_1 - \boldsymbol{q}) \cdot u^*(\boldsymbol{p}_1 - \boldsymbol{q}) (1 + \tau_3)/2 \cdot u(\boldsymbol{p}_2 - \boldsymbol{q}) \cdot u^*(\boldsymbol{p}_2 - \boldsymbol{q}) \cdot \\ &\quad \cdot \beta \gamma_5 \tau_i u(\boldsymbol{p}_2) [2\omega_q (E_{p_1} - E_{p_1 - q} - \omega_q)(E_{p_2} - E_{p_2 - q} - \omega_q)]^{-1} \varphi^{(e)}(\boldsymbol{p}_1 - \boldsymbol{p}_2),\end{aligned} \tag{2·10}$$

where $u(\boldsymbol{p}_2)$ and $u(\boldsymbol{p}_1)$ are the initial and final spinors respectively. Put

$$\langle \boldsymbol{p}_1 | R^{(2)} | \boldsymbol{p}_2 \rangle = e\varphi^{(e)}(\boldsymbol{p}_1 - \boldsymbol{p}_2) u^*(\boldsymbol{p}_1) H^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) u(\boldsymbol{p}_2), \tag{2·11}$$

then using the relations:

$$u(\boldsymbol{p}) u^*(\boldsymbol{p}) = (E_p + \alpha \boldsymbol{p} + \beta m)/2E_p, \tag{2·12}$$

$$Q \equiv \tau_i \frac{1 + \tau_3}{2} \tau_i = 1\tau_p + 2\tau_n, \tag{2·13}$$

$$(\alpha \boldsymbol{p}_2 + \beta m) u(\boldsymbol{p}_2) = E_{p_2} u(\boldsymbol{p}_2), \tag{2·14}$$

$$u^*(\boldsymbol{p}_1)(\alpha \boldsymbol{p}_1 + \beta m) = u^*(\boldsymbol{p}_1) E_{p_1}, \tag{2·15}$$

$H^{(2)}$ is written as follows:

$$H^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{Qf^2}{2 \cdot (2\pi)^3 \cdot 4} \int \frac{(E_{p_1 - q} - E_{p_1} + \alpha \boldsymbol{q})(E_{p_2 - q} - E_{p_2} + \alpha \boldsymbol{q}) d^3 q}{E_{p_1 - q} E_{p_2 - q} \omega_q (\omega_q + E_{p_1 - q} - E_{p_1})(\omega_q + E_{p_2 - q} - E_{p_2})}. \tag{2·16}$$

Now, since one needs only the term proportional to $(\boldsymbol{p}_1 \cdot \boldsymbol{p}_2)$ in order to determine the coefficient of $(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2$ in $H^{(2)}$, one proceeds to expand $H^{(2)}$ in $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ to the first power. Thus the term related to the effective potential in the low energy limit becomes

$$H^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) \rightarrow -\tfrac{1}{6} (\boldsymbol{p}_1 - \boldsymbol{p}_2)^2 (3/\pi)(f^2/4\pi) \cdot Q/2 \cdot J_{(n,1)}, \tag{2·17}$$

where $J_{(n,1)}$ is given by eq. (2·19).

Performing the same procedure for the other graphs in Fig. 1, the following results are obtained:

$$\langle r^2 \rangle_n = (3/\pi)(f^2/4\pi)(\tau_p/2 + \tau_n) \sum_i J_{(n,i)}, \tag{2·18}$$

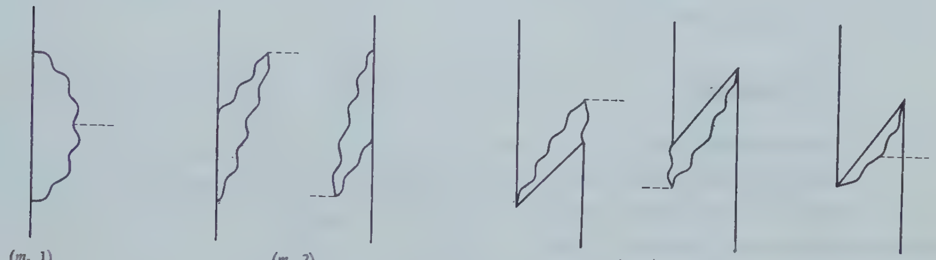
where

$$J_{(n,1)}=\frac{1}{6}\int\frac{q^4dq}{E_q^3(\omega_q+E_q-m)^2\omega_q}\left[\frac{1}{E_q}+2\frac{E_q-m}{E_q}\left(\frac{1}{\omega_q+E_q-m}+\frac{E_q}{(\omega_q+E_q-m)^2}\right)\right], \tag{2·19}$$

$$J_{(n,2)}=\frac{1}{3}\int\frac{q^4dq}{E_q^4(\omega_q+E_q-m)\,\omega_q}\left[\frac{1}{E_q}+\frac{1}{2}\,\frac{E_q-m}{E_q}\,\frac{1}{\omega_q+E_q-m}\right], \tag{2·20}$$

$$J_{(n,3)}=\frac{1}{3}\int\frac{q^4dq}{E_q^4(\omega_q+E_q+m)\,\omega_q}\left[\frac{1}{E_q}+\frac{1}{2}\,\frac{E_q-m}{E_q}\,\frac{1}{\omega_q+E_q+m}\right], \tag{2·21}$$

$$J_{(n,4)}=\frac{1}{6}\int\frac{q^4dq}{E_q^3(\omega_q+E_q+m)^2\omega_q}\left[\frac{1}{E_q}+2\frac{E_q+m}{E_q}\left(\frac{1}{\omega_q+E_q+m}+\frac{E_q}{(\omega_q+E_q+m)^2}\right)\right]. \tag{2·22}$$



(m, 1) (m, 2) (m, 3) (m, 4)

Fig. 2

B. *Meson contribution to electron-neutron interaction*

The time-ordered graphs for the meson contribution are illustrated in Fig. 2. The same procedure as for the nucleon static field interaction gives then the following results:

$$\langle r^2\rangle_m=(3/\pi)\,(f^2/4\pi)\,\tau_3\sum_i J_{(m,i)}, \tag{2·23}$$

where

$$J_{(m,1)}=\frac{1}{3}\int\frac{(E_q-m)\,q^4dq}{\omega_q^4(E_q+\omega_q-m)^2E_q}\left(\frac{1}{E_q+\omega_q-m}+\frac{\omega_q}{(E_q+\omega_q-m)^2}\right), \tag{2·24}$$

$$J_{(m,2)}=\frac{1}{6}\int\frac{(E_q-m)\,q^4dq}{\omega_q^5(E_q+\omega_q-m)^2E_q}, \tag{2·25}$$

$$J_{(m,3)}=\frac{1}{6}\int\frac{(E_q+m)\,q^4dq}{\omega_q^5(E_q+\omega_q+m)^2E_q}, \tag{2·26}$$

$$J_{(m,4)}=\frac{1}{3}\int\frac{(E_q+m)\,q^4dq}{\omega_q^4(E_q+\omega_q+m)^2E_q}\left(\frac{1}{E_q+\omega_q+m}+\frac{\omega_q}{(E_q+\omega_q+m)^2}\right). \tag{2·27}$$

The integrands of J's are illustrated in Fig. 3 as the function of the momenta carried by the virtually emitted particles. The numerical values of J, $\langle r^2\rangle^{1/2}$ and $-eV_0$ for each graph are shown in Table 1, taking $g^2/4\pi=0.08$.

Table 1. Numerical results for J, $\langle r^2\rangle^{1/2}$ and $-eV_0$ for each graph. The value of the coupling constant $g^2/4\pi=0.08$ has been chosen.

Type of graph	$J(\hbar/mc)^2$	relative percent	$\langle r^2\rangle^{1/2}$ ($\hbar$/mc)	$-eV_0$ (kev)
$(m,\ 1)$	0.186	51	−1.61	−3.66
$(m,\ 2)$	0.063	17	−0.93	−1.24
$(m,\ 3)$	0.080	22	−1.05	−1.57
$(m,\ 4)$	0.038	10	−0.73	−0.76
$(n,\ 1)$	0.097	41	1.15	1.91
$(n,\ 2)$	0.087	36	1.10	1.71
$(n,\ 3)$	0.036	15	0.71	0.70
$(n,\ 4)$	0.019	8	0.52	0.38

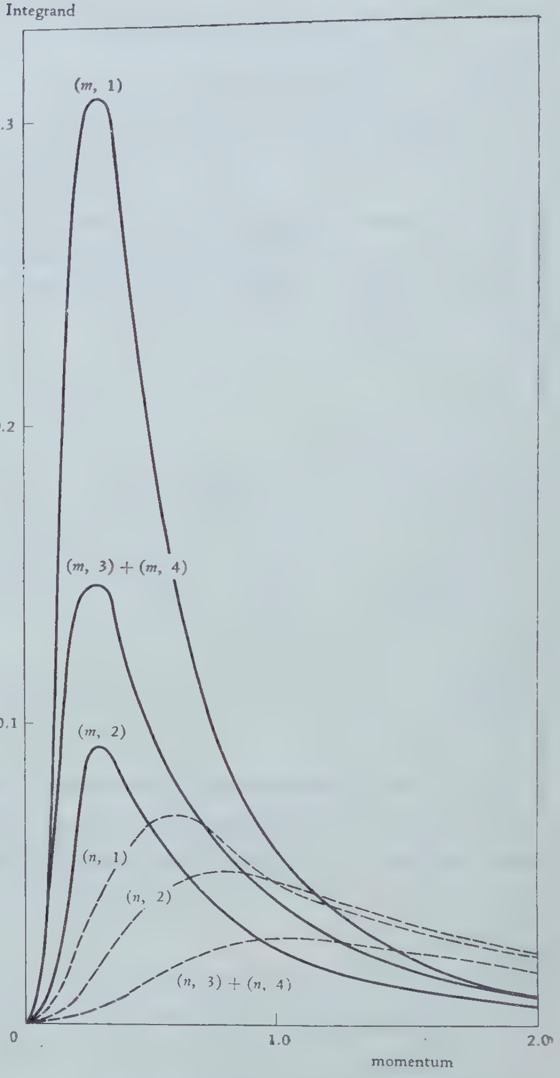


Fig. 3. The integrands of J's are illustrated as the function of the momenta carried by the virtually emitted particles. The units $\hbar=c=m=1$ are used.

§ 3. Discussion

A. The meson cloud

In the fixed source theory only $(m,\ 1)$ and $(m,\ 2)$ are considered, because the nucleon is assumed to be permanently at rest. Thus the energy difference between the initial and the intermediate nucleon state is entirely neglected, and the meson-nucleon interaction is given by $(g/\mu)\,(\sigma\mathbf{q})/\sqrt{2\omega_q}$ in momentum space. These circumstances make the meson cloud contribution much greater than the one in the relativistic theory. In fact, if one approximates $E_q-m\approx q^2/2m$ and $E_q\approx m$ in (2·24) and (2·25), $J_{(m,1)}+J_{(m,2)}$ becomes as follows:

$$J_{(m,1)}+J_{(m,2)}=(1/m^2)\,5/12\int_0^m q^6\,dq/\omega_q^7$$
$$\approx 0.46\,(1/m^2), \qquad (3\cdot1)$$

where the cut-off momentum is taken equal to mc. From (3·1) and by the use of (2·9), one gets $-eV_0\approx-9.1$ kev, which is about 2.3 times the one obtained without approximations and cutting off at mc. This situation may be interpreted roughly as follows. The physical nucleon's $\langle r^2\rangle_m$ due to the meson cloud may be proportional to the meson dissociating probability P and to the square of the spread of the meson $(\varDelta r)^2$. However, P is proportional to $1/(\varDelta E)^2$, where $\varDelta E$ is the energy difference between the initial and the intermediate state of the physical nucleon, and $\varDelta r$ may be proportionol to the meson dissociating time $t\sim\hbar/\Delta E$ and to the meson fluctuating velocity v, thus $\langle r^2\rangle_m \propto 1/(\Delta E)^4$. Therefore, if $E_q\sim m$, $\langle r^2\rangle_m$ is much overestimated.

The graphs $(m, 3)$ give the contributions from the s-wave mesons with low momenta. Now, on treating the scattering of a meson by a nucleon in the ps-coupling theory, it is convenient to eliminate the intermediate states in which the nucleon pairs are produced, thus giving the following effective s-wave "Hamiltonian" in the static limit approximately :⁹⁾

$$H=(f^2/2m)\int\psi^*(x)\phi^2(x)\psi(x)\,dv+(f/2m)^2\int\psi^*(x)\tau\cdot[\phi(x)\times\pi(x)]\psi(x)\,dv. \quad (3\cdot 2)$$

It is well known that the first so-called pair term H_c is too large to treat by means of the perturbation method, and also to explain the observed separation of s-wave scattering phase shifts δ_1 and δ_3.¹⁰⁾ This separation might be almost explained, if only the last term so-called H_π were retained, omitting H_c. Thus, if the ps-coupling theory is really correct and is properly treated, it is expected that H_c must be damped in higher approximations.

In the calculation of the charge distribution in the lowest order perturbation method H_c gives fortunately no contribution ; since H_c emits the oppositely charged mesons π^+ and π^- at the same point and at the same time, thus the meson charge distribution from H_c becomes to be neutral as a whole. And thus it is easily confirmed that $J_{(m,3)}$ except order μ/m, is equal to that expected from H_π-coupling. Thus one can expect that the s-wave contribution to the electron-neutron interaction is taken into account qualitatively.

$(m, 4)$ may be considered as responsible for the meson cloud contribution due to, so to speak, spread-out H_c over the region of order $\hbar/2mc$, since the dissociating time of the nucleon pair is of the order $\hbar/2mc^2$ and the fluctuating velocity is of the order c. It amount 10% of all the meson contributions.

B. The nucleon core

In the diagrams labelled by n, which appear in strict sense only in the relativistic theory, the broad region of virtual nucleon momenta around mc contributes to the charge distribution, because the nucleon is not so fast enough to reach far until the momentum becomes about mc.

The graph $(n, 1)$ gives the contribution due to nucleon recoil caused by the p-wave meson emission. It is thus expected that the charge spread due to this is of the order $\hbar/mc$. In the graphs $(n, 2)$ the charge spread is due partly to the meson cloud and partly to the nucleon pair. For, as the spreads of meson and nucleon pair are expected to be about $2\cdot\hbar/mc$ and $\hbar/2mc$ respectively from the results obtained in § 2 as is shown in Table 1, one gets $\hbar/mc$ as the geometrical mean. Thus these graphs, which are not included in the fixed source theory, give contributions as large as $(n, 1)$ does. The contributions from the graphs $(n, 3)$ and $(n, 4)$ are small compared with the other graphs, because there the meson exchange occurs while the nucleon pair is present in intermediate states.

C. Cut-off dependence

Recent field theoretical investigations¹¹⁾,¹²⁾ suggest that cutting off is necessary for the calculation of the radiative corrections in the present field theory. And, for the anomalous magnetic moment (a. m. m.) of nucleon, Goto[7] and Hasegawa et al.[8] have shown that cutting off the momenta at about *mc* is favorable in accounting for the discrepancy between the results of the lowest order perturbation calculation and experiments. We then investigate the cut-off dependence in our case. The results are shown in Table 2. Because the momenta effective to the nucleon core charge distribution are much larger than the momenta effective to the meson cloud one, the smaller cut-off momenta make the discrepancy between theory and experiment somewhat larger. For the same cut-off value *mc* as in the case of a. m. m., the effective potential becomes -3.8 kev. This is still about one third the one in the fixed source theory, which gives the meson contribution much larger than in our cut-off theory.

In this place, it may be worthwhile to note the charge distribution of a proton briefly. The rms radius of the proton $\langle r^2\rangle_p^{1/2}$ is obtained, by the use of eqs. (2·18) and (2·23), as follows:

$$\langle r^2\rangle_p=(3/\pi)\,(f^2/4\pi)\,(J_m+\tfrac{1}{2}J_n), \qquad (3\cdot3)$$

where $J_m=\sum\limits_i J_{(m,i)}$ and $J_n=\sum\limits_i J_{(n,i)}$. In the case of no cutting off $\langle r^2\rangle_p^{1/2}$ is about 0.56×10^{-13} cm, which is compared with the corresponding value for the neutron, -0.28×10^{-13} cm. Cutting off the momenta at *mc*, $\langle r^2\rangle_p^{1/2}$ becomes $\approx0.45\times10^{-13}$ cm, and this is somewhat small compared with the experiments, accordingly $\langle r^2\rangle_p^{1/2}$ must be at least 0.6×10^{-13} cm and is estimated to be about 0.77×10^{-13} cm.[13]

Now, cutting off the momenta at q is supposed to neglect the contribution from the inside region of the sphere the radius of which is about $\hbar/q$. However, the total charge in this region may be positive, since the π^--meson cloud spreads further than the nucleon core. To cut off the momenta simply thus violates the neutrality of the physical neutron. Then, in order to secure this neutrality it is necessary to add the + charge distribution, which may be required to spread over the outside region of that sphere in order to account for the observed rms radius of both the neutron and the proton completely. In that case, in order to preserve the success in explaining a. m. m. with the cut-off procedure, the added charge distribution should not have the contribution to a. m. m. In this piont, further investigation seems to be necessary.

Table 2. The cut-off momentum dependence of the effective potential. Numerical values are given in units kev.

Cut-off momentum	0.6 *mc*	1.0 *mc*	2.0 *mc*	3.0 *mc*
Nucleon contrib.	0.8 kev	1.8 kev	3.6 kev	4.4 kev
Meson contrib.	-4.1	-5.6	-6.7	-7.0
Total	-3.3	-3.8	-3.1	-2.6

D. K-meson effect

From the above discussion, the charge distribution of the physical neutron is concentrated in the region of the radius rather close to $\hbar/mc$, suggesting that K-meson effect will not be neglected. To estimate this effect, we assume the following interaction:

$$H=g_\Sigma \overline{\Sigma}_i K\tau_i N+g_\Lambda \overline{\Lambda}\,\overline{K}\,N+\text{conjugate complex}, \qquad (3\cdot4)$$

where Λ, Σ, N and K are field variables of hyperon, nucleon and K-meson respectively. We calculate the contribution to the charge distribution from the K-meson cloud corresponding to the graphs $(m,\ 1)$ and $(m,\ 2)$, which are most important for our effect. The results are

$$-eV_0\approx(3/\pi)\,(g_\Sigma^2/4\pi)\begin{pmatrix}0.25\\0.37\end{pmatrix}, \qquad (3\cdot5)$$

where the upper and the lower numbers correspond to the cut-off at mc and to the no-cut-off respectively. For $g_\Sigma^2/4\pi=3$, K-meson contribution is about 1 kev. In order to cancell out the neutron charge distribution $g_\Sigma^2/4\pi\approx10$ is necessary, which is too large compared with the value inferred from the approximate treatment of the K-meson nucleon collision, $g_\Sigma^2/4\pi\approx1\sim3$,[14] though this value seems to be not so reliable.

Recently the K-meson effect has been discussed by Sandri.[15] There, Σ mass is assumed to be equal to the nucleon mass, and the fixed source approximation is used. Thus, for $g_\Sigma^2/4\pi=0.4$, K-meson contribution to the effective electron-neutron potential is about 1 kev, which is overestimated about 8 times as large as ours.

§ 4. **Conclusions**

We have analyzed the neutron-electron interaction in the lowest order relativistic perturbation theory, and have investigated the possibility that the present meson theory can explain the experiments. The relativistic perturbation theory differs from the fixed source theory in the following points: (1) the p-wave meson contributions from $(m,\ 1)$ and $(m,\ 2)$, which are the only ones considered in the fixed source theory, turn out to be about 1/2.3 times smaller than in the fixed source theory, owing to the nucleon recoil effect; (2) the nucleon contributions of the graphs $(n,\ 2)$ and $(n,\ 1)$ are comparable with each other; (3) the s-wave meson contribution, which is about 30% of all meson contribution, is taken into account qualitatively, since the pair term H_c has little contribution to the charge distribution in the lowest approximation and the main contribution is due to H_π term. On account of these points, the relativistic perturbation theory can explain the experiments qualitatively. Though cutting off the momenta higher than about mc makes the discrepancy between theory and experiment somewhat larger, the neutron-electron potential is still reduced to about one third that obtained in the fixed source theory. Although, in order to explain a. m. m., it is preferable to cut off the momenta, it is not to be taken seriously, because, as has been discussed in § 3, it violates the neutrality of the physical neutron. The K-meson effect is about 1 kev for the scalar coupling constant $g_\Sigma^2/4\pi=3$. The inclusion of this effect, however, may not be sufficient to account for the experiments completely. It may be necessary to take into account the higher order corrections as has been done by Sachs and Treiman,[3] who have obtained the correction 2.4 kev using the Chew-Low theory. However, such a large correction might not be expected in the relativistic treatment owing to the recoil effect. This seems also to be supported by the evidences that the outer region of the physical nucleon is effective to the rms radius of the charge distribution and that such effect is almost included in the lowest order perturbation calculation. Thus, a part of the discrepancy between theory and experiment might be explained by the modification of the quantum electrodynamics at small distances as has been discussed by Yennie et al.,⁶⁾ although the curiously large value of the effective neutron-electron potential in the fixed source theory is largely suppressed by the nucleon recoil effect.

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Deuteron Production by Nucleons Bombarding Atomic Nuclei

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The deuteron production by the nucleon bombardment is discussed in detail. First in § 2, according to the general theory of nuclear reactions developed by Ui, the transition matrix is written as the sum of two elements, which correspond to the compound and the noncompound nuclear processes. In the processes proceeding through the compound nucleus, the indirect pickup process, in which a neutron emerging from the compound nucleus picks up any proton, gives a larger yield than the deuteron evaporation in the usual sense. The average kinetic energy of deuterons thus produced is about three times larger than that in the Maxwellian type (§ 3). In the direct pickup process of low energy $(p; d)$ reaction, the Coulomb effect is found to play an important role and the remarkable backward maximum in the angular distribution of the reaction product is predicted (§ 4). Direct pickup process at moderate and high energies is calculated by the Born approximation, assuming the Gaussian momentum distribution for the tail of the wave function of the nucleon to be picked up. The forward peak in the angular distribution for sharp energy deuteron groups corresponding to the ground state and the low-lying levels is interpreted in terms of the direct process taking place in the external region of the target nucleus (§ 5). In the limiting case of the weak coupling, which is an appropriate approach to the high energy reaction, the calculation is performed along the Chew-Goldberger's procedure on the basis of the non-degenerate Fermi gas model and the total cross section is found to obey approximately $1/v$-law (§ 6). Further the production cross section of ^1s deuterons is evaluated in the same way, and is found to give a contribution comparable to the production of deuterons in the ground state (§ 7). Main conclusions obtained in the present work are summarized in § 8.

§ 1. Introduction

This paper deals with the production of deuterons by nucleons impinging upon complex nuclei and aims at investigating the gross behaviour of the cross section, which can be directly compared with experimental observations. In the last decade the theory of nuclear reactions has been developed by means of the compound nucleus which is formed after the incident particle has entered into the target nucleus. Before the striking success of the optical model was known, it was generally assumed that particles having entered into the nucleus form the collision complex which forgets how it was formed and decays in all possible ways. The optical model, as well as the nuclear shell model, manifests that one nucleon can move freely within the nucleus without apparently changing the quantum states of other nucleons. Since the success of these models is concerned with the nuclear ground state or low excitation energy region, one may doubt whether the apparent absence of interaction between one nucleon and the rest is valid also at higher excitation energies. One way to appreciate the influence of the internucleon interaction may be achieved by computing the mean free path for a nucleon traveling in the nuclear matter. At higher energies, 10 Mev and up, the mean free path for a nucleon is considered to be comparable to nuclear radii of light nuclei. Hence, in those cases, nucleons impinging upon medium weight or heavy nuclei with small angular momenta suffer several collisions in traveling within the boundaries of the nucleus. This implies that a nucleon penetrating into the target has a small chance of leaving it without forming such a compound state that may satisfy the complete mix-up conditions. In fact the small difference in proton yields between the forward and the backward directions in the 18 Mev proton inelastic scattering[1)] predicts that the strong coupling model is still valid and realizes the main feature of the reaction phenomena in the intermediate energy region. On the contrary, the forward peak angular distributions of particle groups corresponding to low excited levels of a residual nucleus can never be interpreted on the basis of the compound nucleus procedure, but is attributed to the direct interaction between the incident particle and the tails of the wave functions for the target nucleus. The latter process, in general, is distinguished from the compound nucleus process by the term " non-compound " or " direct " process, in which no collision complex is formed. This process, chiefly arising from the surface interaction outside the nuclear radius, seems to be most significant in nuclear reactions such as the deuteron stripping and formation by pickup. Thus it is required to formulate the theory of deuteron production process so as to be capable of treating the noncompound and the compound nucleus processes on unified basis.

In the present paper the general theory of nuclear reactions developed by Ui[2)] is applied to the problem concerned here, taking account of both processes mentioned above. To define both processes formally, the surface S is introduced, which divides the configuration space into the internal and the external regions. The internal region is supposed to represent the configuration space of the compound nucleus and the external region to represent the one where the direct or noncompound processes take place. The outer boundary of the configuration space is actually located at infinity, while the surface S, i. e. the boundary of the internal region, is the place where the interactions become relatively strong and the channel description breaks down. Since the deuteron is a loosely bound system, having a spread comparable to those of light nuclei, it may be of significance to treat exactly the overlapping regions of the wave function in the neutron and the proton channels. In other words, the deuteron evaporates from the compound nucleus not only through the ordinary deuteron channel but also through the channels of its constituents. The latter is the process, in which a neutron emerging from the compound nucleus picks up any proton in the external region or vice versa. As shown in § 2, the general classification of types of deuteron production process, neglecting the contribution from the ordinary deuteron channel, will be accomplished by dividing the configuration space into four parts and the transition matrix of the process is divided into two parts, which correspond to the direct deuteron pickup process occurring in the external region and to the indirect pickup proceeding through the compound nucleus respectively. In § 3, the explicit expression for the latter process is derived into a convenient form currently employed in the statistical theory of nuclear reactions and the energy spectrum of the reaction product, together with the total reaction width, is investigated. Then the deuteron width obtained here can be brought into the ordinary evaporation theory, by replacing the deuteron width in the usual sense by it.

In the following sections, § 4 and § 5, the direct process term is examined for two cases, in which the deuteron production from the compound nucleus is subsidiary. As one may reasonably esteem, the main feature of the phenomena proceeds through the surface interaction in the deuteron pickup reactions initiated with a low energy proton beam and the Coulomb effects should play an important role, as shown in § 4.

The production of fast deuterons in nuclear events involving energies of the order of 100 Mev has been shown to take place in cosmic ray stars and in Harvard cyclotron experiments.3),4) For nucleons with such high energies, the mean free path in the nucleus becomes larger again and the overall picture can be well interpreted by the Born approximation calculation,5) neglecting all processes involving the particles other than the incident nucleon and the partner with which it is to join. However, a recent experiment6) has revealed that deuterons produced into the backward direction result in a flat angular distribution, which can never be explained by the previous theoretical calculations,5),7),8) and the formation of the compound nucleus cannot completely be neglected. For these cases, the formation of the compound state is due to nucleons which lose most of their kinetic energies after a single or multiple collisions and is considered to be very complicated as to forbid the analytical calculation ; it will be handled only with the Monté Carlo method. In § 5, the direct process, which dominates the main feature of the reaction phenomena, is evaluated by the Born approximation in accordance with the Chew-Goldberger's procedure, assuming the Gaussian momentum distribution for the tail of the nucleon to be picked up. Finally in § 6, we shall study our problem in the weak coupling limit, in which the internal region is completely disregarded and the channel description is valid for the whole configuration space. Therefore the transition matrix itself reduces to the one already treated by the previous works.5),7),8) Here we calculate the same matrix element on a little different basis, that is, by assuming the Fermi-distribution with finite temperature for the momentum distribution of the partner to be picked up. Because of the small contribution from the compound nucleus process, the conclusions for the weak coupling limit is useful in analysing the experimental observations. In § 8, main conclusions obtained in the present work are summarized and applications to the analysis of the experimental results are undertaken. To the reader who wishes to acquire some information on our process without getting involved in too much mathematics, we recommend the summaries for each section and sections 5—8.

§ 2. General formulation—Classification of processes*

In this section the general theory on (d, p) reaction developed by Ui is applied straightforwardly to derive the transition matrix into a suitable form for treating our

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* The formalism developed in this paragraph is not necessary to go further with the following sections, as the purpose of our formalism is to obtain the transition matrix in a form given in the summary for this section.

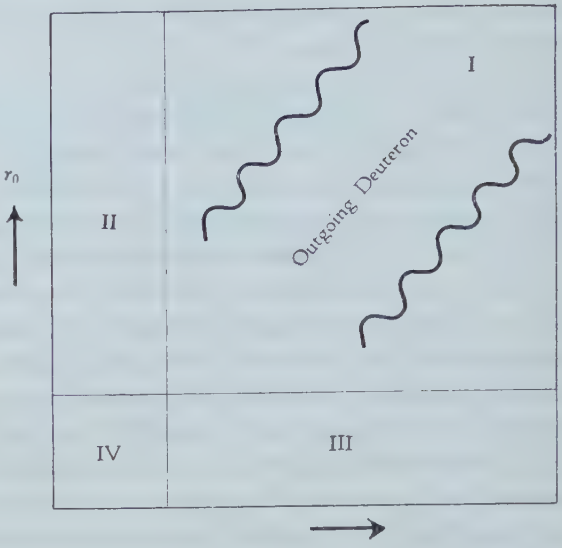

Fig. 1. Simplified one dimensional model of Butler. Region I represents the outgoing deuteron, II the nucleon 1 in the internal and the nucleon 0 in the external regions, III the nucleon 1 in the external and the nucleon 0 in the internal regions and IV compound nucleus formed by the incident nucleon 0.

problem. As mentioned in § 1, in order to take into account the loosely bound property of deuteron, we divide the configuration space into four subspaces, which is shown schematically in Fig. 1 by the simplified manner used by Butler.[28] The region I is the external region of the usual sense; and II is the region where the nucleon 1 is in the internal region and the nucleon 0 is in the external region, the region III where the roles of the nucleons 0 and 1 are exchanged, and the region IV is the internal region. The surfaces between respective pairs of them are designated by S^{12}, S^{13}, S^{24} and S^{34}.

The Hamiltonian $\mathfrak{H}$ of the total system is given by

$$\mathfrak{H}=\mathfrak{H}_c+K_0+U_0+K_1+U_1+V_{01} \quad (1)$$

where $\mathfrak{H}_c$ is the Hamiltonian of the residual core, K_0 and K_1 are the kinetic energy operators of the constituents of the deuteron, and V_{01} the interaction between them. U_0 is the interaction between the incident nucleon 0 and the residual core, and U_1 that for the nucleon picked up. Since we are concerned with the details of outgoing deuteron in the present work, it is convenient to express the transition matrix T of the process by the ingoing solution for the system. In the final state, in which we have the outgoing deuteron and the residual nucleus, the Hamiltonian of free state is $\mathfrak{H}_f=\mathfrak{H}_c+K_0+K_1+V_{01}$ and the interaction Hamiltonian is taken to be $U_f=U_0+U_1$, whereas in the initial state the corresponding quantities are $\mathfrak{H}_i=\mathfrak{H}_c+K_0+K_1+U_1$ and $U_i=V_{01}+U_0$. The ingoing wave solution of the complete system is written as

$$\Psi_f^{(-)}=\left[1+\frac{1}{E-i\varepsilon-\mathfrak{H}}(U_0+U_1)\right]\Phi_f\equiv\Omega^{(-)}\Phi_f \quad (2)$$

with Φ_f, the free state wave function in the final state, satisfying the Schrödinger equation

$$\mathfrak{H}_f\Phi_f=(\mathfrak{H}_c+K_0+K_1+V_{01})\,\Phi_f=E\Phi_f. \quad (3)$$

Making use of the quantities defined above, the T-matrix for the process concerned is given by

$$T=\langle\Psi_f^{(-)}|U_i|\Phi_i\rangle=\langle\Psi_f^{(-)}|V_{01}+U_0|\Phi_i\rangle, \quad (4)$$

where Φ_i is the free state wave function of the initial state satisfying the Schrödinger equation

$$\mathfrak{H}_i \Phi_i = (\mathfrak{H}_c + K_1 + U_1 + K_0)\, \Phi_i = E \Phi_i . \tag{5}$$

As was discussed above, it is requested to take account of the interaction between nucleon 0 and the tails of the wave function of the intermediate residual nucleus, as well as the strong interaction in the internal region. For this purpose, we divide the whole interaction acting upon the nucleon 0 into two parts: the interaction $(U_0+V_{01})^{\mathrm{II}}$ in the external region II and the $(U_0+V_{01})^{\mathrm{IV}}$ in the compound nucleus region IV. By expressing as $U_0+V_{01}=(U_0+V_{01})^{\mathrm{II}}+(U_0+V_{01})^{\mathrm{IV}}$, the T-matrix can be divided into two terms corresponding to the contributions from the compound nucleus process and the direct process as[29)]

$$\begin{aligned} T = & \langle \mathfrak{F}_f^{(-)} | (U_0+V_{01})^{\mathrm{IV}} | \Phi_i \rangle \\ & + \left\langle \mathfrak{F}_f^{(-)} \middle| \left[1 + (U_0+V_{01})^{\mathrm{II}} \frac{1}{E+i\varepsilon-\mathfrak{H}} \right] (U_0+V_{01})^{\mathrm{II}} \middle| \mathfrak{F}_i^{(+)} \right\rangle \equiv T^{(i)} + T^{(s)}, \end{aligned} \tag{6}$$

where $\mathfrak{F}$'s are the distorted waves defined by

$$\mathfrak{F}_i^{(+)} = \left[1 + \frac{1}{E+i\varepsilon-\mathfrak{H}_{\mathrm{II}}-K_0-(U_0+V_{01})^{\mathrm{IV}}} (U_0+V_{01})^{\mathrm{IV}} \right] \Phi_i \tag{7}$$

and

$$\begin{aligned} \mathfrak{F}_f^{(-)} = & \left[1 + \frac{1}{E-i\varepsilon-\mathfrak{H}_{\mathrm{II}}-K_0-(U_0+V_{01})^{\mathrm{IV}}} (U_0+V_{01})^{\mathrm{IV}} \right] \\ & \cdot \left[1 + \frac{1}{E-i\varepsilon-\mathfrak{H}_{\mathrm{II}}-K_0} (U_1-V_{01}) \right] \Phi_f \equiv \Omega_{\mathrm{IV}}^{(-)} \Phi_f , \end{aligned} \tag{8}^*$$

with

$$\mathfrak{H}_{\mathrm{II}} = \mathfrak{H}_c + K_1 + U_1 . \tag{9}$$

The second term in (6) represents the deuteron pickup process due to the interaction in the external region II, which can be regarded as a small perturbation. Further, for low and moderate incident energies, the collision partner to join is likely to be found just beneath the Fermi surface of the target nucleus and consequently has a longer tail than those of other nucleons. Hence, neglecting both U_0, in comparison with V_{01}, and the higher order terms, we obtain

$$T^{(s)} \simeq \langle \mathfrak{F}_f^{(-)} | V_{01}^{\mathrm{II}} | \mathfrak{F}_i^{(+)} \rangle . \tag{10}$$

This formula, if we take approximately only the waves in the channels i and f, is nothing

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* $\mathfrak{F}_f^{(-)}$ here defined does not identify the wave of usual form

$$\left[1 + \frac{1}{E-i\varepsilon-\mathfrak{H}_{\mathrm{II}}-K_0-V_{01}+U_1} (U_0+U_1)^{\mathrm{IV}} \right] \Phi_f .$$

This is because we are now treating the indirect pickup process, taking account of the loosely bound nature of deuteron.

else but the distorted Born approximation. This term $T^{(s)}$ describes the overall feature of the process, when the excitation energy of the compound nucleus is low enough to neglect the deuteron emission from it. The surface direct term, as well as the interference term, has been given in a precise form by Thomas,[30] when the quantum states of the initial and final states are known. The term $T^{(s)}$ is studied in § 4, § 5 and § 6.

Now let us consider the first term in (6). This term, as can be shown below, results in the deuteron production from the compound nucleus as a two steps process in which first a nucleon evaporates from the compound state IV and picks up its partner from the intermediate residual state III or II. In expressing the concept of the compound nucleus formally, the T-matrix is represented directly as a pole expansion of the energies of the levels, according to the nuclear reaction theory developed by Kapur and Peierls.[9] [10] First we expand the T-matrix in the region II where the nucleon 1 is in the internal region and the nucleon 0 in the external region of the configuration space. From the mathematical point of view, the wave matrix $\Omega_{\text{IV}}^{(-)}$ has a convenient form for expanding the T-matrix in the region II, since the wave function $[1+(E-i\varepsilon-\mathfrak{H}_{\text{II}}-K_0)^{-1}(U_1-V_{01})]\Phi_f$ is one of the wave functions in the region II. However, since $\Psi_f^{(-)}$ itself does not satisfy the boundry condition of the decaying state, it cannot be expanded as a whole. To remove this difficulty Kapur and Peierls introduced an arbitrary function χ which only satisfies the boundary condition of the entrance channel on the surface S. Then, $\Psi_f^{(-)}-\chi$ can be expanded by the complete set of the decaying states of the compound nucleus. As an alternative, Ui employed a potential L to separate the potential scattering term from the T-matrix, the rest being expanded by the eigenfunctions of the decaying states. Here, according to the latter method, we introduce the potentials L_0 and L_1, writing

$$U_1=(U_1-L_1)+L_1$$

$$U_0+V_{01}=(U_0+V_{01}-L_0)+L_0 .$$

Throughout the following formulations, the suffix IV in $(U_0+V_{01})^{\text{IV}}$ *is omitted for simplicity.* In an analogous way as in (6), we obtain

$$T^{(i)}=\langle \Omega_{\text{IV}}^{(-)}\Phi_f|U_0+V_{01}-L_0|\mathfrak{G}_i^{(+)}\rangle+\langle \mathfrak{G}_f^{(-)}|L_0|\Phi_i\rangle \equiv T^{(c)}+T^{(d)}, \tag{11}$$

with distorted waves

$$\mathfrak{G}_i^{(+)}=\left[1+\frac{1}{E+i\varepsilon-\mathfrak{H}_{\text{II}}-K_0-L_0}L_0\right]\Phi_i, \tag{12}$$

$$\mathfrak{G}_f^{(-)}=\left[1+\frac{1}{E-i\varepsilon-\mathfrak{H}_{\text{II}}-K_0-L_0}\cdot L_0\right]\cdot\left[1+\frac{1}{E-i\varepsilon-\mathfrak{H}_{\text{II}}-K_0}(U_1-V_{01})\right]\Phi_f . \tag{13}$$

On the other hand, the wave matrix $\Omega_{\text{IV}}^{(-)}$ can be decomposed into four elements:

$$\Omega_{\text{IV}}^{(-)}=\Omega_4^{(-)}\Omega_3^{(-)}\Omega_2^{(-)}\Omega_1^{(-)}, \tag{14}$$

where

$$\Omega_1^{(-)}=1+\frac{1}{E-i\varepsilon-\mathfrak{H}_{\rm II}-K_0-V_{01}-L_1+U_1}L_1, \tag{15}$$

$$\Omega_2^{(-)}=1+\frac{1}{E-i\varepsilon-\mathfrak{H}_{\rm II}-K_0}(U_1-V_{01}-L_1), \tag{16}$$

$$\Omega_3^{(-)}=1+\frac{1}{E-i\varepsilon-\mathfrak{H}_{\rm II}-K_0-L_0}L_0, \tag{17}$$

$$\Omega_4^{(-)}=1+\frac{1}{E-i\varepsilon-\mathfrak{H}}(U_0+V_{01}-L_0). \tag{18}$$

The first term in (11) represents the deuteron production through the compound nucleus process, without regard to that through the ordinary deuteron channel. The second term is the volume direct interaction term defined by Ui, being interpreted as follows; the nucleon 0 suffers only the elastic scattering due to the potential L_0, while the nucleon 1 in the target nucleus sticks to it to form a deuteron. However, this term gives no contribution to the reaction concerned here, because of negative Q-values.

Now we expand the T-matrix in the region II where the nucleon 1 is in the internal region and the nucleon 0 in the external region respectively. It is necessary to separate the coordinates of constituents of the deuteron to discuss the deuteron production through the neutron and the proton channels. For this purpose the complete set of eigenfunctions for the states of the nucleon 0,

$$K_0|\boldsymbol{k}_0\rangle=E_0|\boldsymbol{k}_0\rangle,\quad E_0=\hbar^2k_0^2/2M, \tag{19}$$

is introduced. Then expanding both sides of the Schrödinger equation

$$(E-\mathfrak{H}_{\rm II}-K_0)\,\Omega_1^{(-)}\Phi_f=(V_{01}+L_1-U_1)\,\Omega_1^{(-)}\Phi_f, \tag{20}$$

we obtain

$$\langle \boldsymbol{g}_1^{(-)}|\,(E^{\rm II}-\overline{\mathfrak{H}}_{\rm II})=\langle \Omega_1^{(-)}\Phi_f\boldsymbol{k}_0|V_{01}-\langle \boldsymbol{g}_1^{(-)}|\,(U_1-L_1) \tag{21}$$

$$(\overline{\mathfrak{H}}_{\rm II}=\text{transpose of }\mathfrak{H}_{\rm II})$$

with abbreviations,

$$\langle \boldsymbol{k}_0|\Omega_1^{(-)}\Phi_f\rangle=\boldsymbol{g}_1^{(-)}(\boldsymbol{r}_1,\ \boldsymbol{R}\,;\ \boldsymbol{k}_0) \tag{22}$$

and

$$E^{\rm II}=E-E_0. \tag{23}$$

Here $\boldsymbol{R}$ stands for all coordinates of the target nucleus except the nucleon 1. We next define the biorthogonal set, $|\mathfrak{D}_\mu^{\rm II}\rangle$ and $\langle\mathfrak{D}_\mu^{\rm II\prime}|$, of the decaying states of the intermediate residual nucleus left after the emission of nucleon 0, by

$$\mathfrak{H}_{\rm II}|\mathfrak{D}_\mu^{\rm II}\rangle=W_\mu^{\rm II}|\mathfrak{D}_\mu^{\rm II}\rangle \tag{24}$$

and

$$\mathfrak{H}_{\rm II}|\mathfrak{D}_\mu^{\rm II\prime}\rangle=W_\mu^{\rm II*}|\mathfrak{D}_\mu^{\rm II\prime}\rangle. \tag{25}$$

(* : complex conjugate)

$|\mathfrak{D}_\mu^{\rm II}\rangle$ satisfies the boundary condition of the decaying state on the surface S^{12} and $|\mathfrak{D}_\mu^{\rm II\prime}\rangle$

does the complex conjugate boundary condition to that of $|\mathfrak{D}_\mu^{\mathrm{II}}\rangle$. Then it follows that

$$\sum_\mu |\mathfrak{D}_\mu^{\mathrm{II}}\rangle\,(1/N_\mu^{\mathrm{II}})\,\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}|=1 \tag{26}$$

with the normalization conditions

$$|\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}|\mathfrak{D}_\mu^{\mathrm{II}}\rangle|^2=1,\quad \langle\mathfrak{D}_\lambda^{\mathrm{II}\prime}|\mathfrak{D}_\mu^{\mathrm{II}}\rangle=N_\mu^{\mathrm{II}}\,\delta_{\lambda\mu}. \tag{27}$$

The Green function $1/(E^{\mathrm{II}}-\mathfrak{H}_{\mathrm{II}})$, obeying the same boundary condition as those for the biorthogonal set, can be expanded by them. Hence one can accomplish the pole expansion of the T-matrix in the region II, by the aid of (21), as follows:

$$\langle\Omega_2^{(-)}\Omega_1^{(-)}\Phi_f|=\sum_{k_0,\mu}\chi_{1\mu}^{\mathrm{II}}\frac{1}{N_\mu^{\mathrm{II}}(E^{\mathrm{II}}-W_\mu^{\mathrm{II}})}\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0|, \tag{28}$$

where

$$\chi_{1\mu}^{\mathrm{II}}=\langle\boldsymbol{g}_1^{(-)}|\overline{\mathfrak{H}}_{\mathrm{II}}|\mathfrak{D}_\mu^{\mathrm{II}}\rangle-\langle\boldsymbol{g}_1^{(-)}|\mathfrak{H}_{\mathrm{II}}|\mathfrak{D}_\mu^{\mathrm{II}}\rangle \tag{29}$$

and the small quantity $i\mathcal{E}$ is omitted for brevity. Thus the pole expansion in the region II results in

$$T^{(c)}=\sum_{k_0,\mu}\chi_{1\mu}^{\mathrm{II}}\frac{1}{N_\mu^{\mathrm{II}}(E^{\mathrm{II}}-W_\mu^{\mathrm{II}})}\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|\widetilde{\Omega}_4^{(-)}(U_0+V_{01}-L_0)\,|\mathfrak{G}_i^{(+)}\rangle \tag{30}$$

($\sim$: Hermitian conjugate)

with the eigen function distorted by the potential U_0

$$|\boldsymbol{k}_0^{(-)}\rangle=|\Omega_3^{(-)}\boldsymbol{k}_0\rangle. \tag{31}$$

The quantity $\chi_{1\mu}^{\mathrm{II}}$ is proportional to the probability amplitude emitting the nucleon 1 from the intermediate residual nucleus. Our next task is to express the $T^{(c)}$-matrix (30) as the pole expansion in the compound state IV. In an analogous way to the procedure in region II, let us prepare the biorthogonal complete set $|\mathfrak{D}_\lambda^{\mathrm{IV}}\rangle$ and $\langle\mathfrak{D}_\lambda^{\mathrm{IV}\prime}|$, with eigenvalues W_λ^{IV} and $W_\lambda^{\mathrm{IV}*}$ respectively. Substitution of the explicit form for $\Omega_4^{(-)}$ in (30) leads to

$$\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|\widetilde{\Omega}_4^{(-)}=\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|+\sum_\lambda\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|U_0+V_{01}-L_0|\mathfrak{D}_\lambda^{\mathrm{IV}}\rangle\frac{1}{N_\lambda^{\mathrm{IV}}(E-W_\lambda^{\mathrm{IV}})}\langle\mathfrak{D}_\lambda^{\mathrm{IV}\prime}|. \tag{32}$$

By making use of the relation,

$$\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|V_{01}+U_0-L_0|\mathfrak{D}_\lambda^{\mathrm{IV}}\rangle=\mathfrak{Y}_{\mathrm{II}_\mu\boldsymbol{k}_0\lambda}-(E_0+W_\mu^{\mathrm{II}}-W_\lambda^{\mathrm{IV}})\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|\mathfrak{D}_\lambda^{\mathrm{IV}}\rangle \tag{33a}$$

with

$$\mathfrak{Y}_{\mathrm{II}_\mu\boldsymbol{k}_0\lambda}=\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|\overline{\mathfrak{H}}|\mathfrak{D}_\lambda^{\mathrm{IV}}\rangle-\langle\mathfrak{D}_\mu^{\mathrm{II}\prime}\boldsymbol{k}_0^{(-)}|\mathfrak{H}|\mathfrak{D}_\lambda^{\mathrm{IV}}\rangle, \tag{33b}$$

which can be easily derived from the Schrödinger equation

$$(\mathfrak{H}_{\mathrm{II}}+K_0+L_0-E)\,|\boldsymbol{k}_0^{(-)}\mathfrak{D}_\mu^{\mathrm{II}}\rangle=0, \tag{34}$$

(32) goes over into

$$\sum_{\lambda} \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda} \frac{1}{N_\lambda^{\mathrm{IV}}(E-W_\lambda^{\mathrm{IV}})} \langle \mathfrak{D}_\lambda^{\mathrm{IV}\prime}|$$

$$+(E^{\mathrm{II}}-W_\mu^{\mathrm{II}}) \sum_{\lambda} \langle \mathfrak{D}_\mu^{\mathrm{II}\prime} \boldsymbol{k}_0^{(-)} | \mathfrak{D}_\lambda^{\mathrm{IV}} \rangle \frac{1}{N_\lambda^{\mathrm{IV}}(E-W_\lambda^{\mathrm{IV}})} \langle \mathfrak{D}_\lambda^{\mathrm{IV}\prime}| . \tag{35}$$

The quantity $\mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda}$ is the amplitude of width for the emission of the nucleon 0 with momentum $\boldsymbol{k}_0$ and (33b) can be transformed into the surface integration by the use of Green's theorem. In the same manner as in (33a), we introduce the same quantity for the incident nucleon

$$\langle \mathfrak{D}_\lambda^{\mathrm{IV}\prime} | U_0+V_{01}-L_0 | \mathfrak{G}_i^{(+)} \rangle = y_{\lambda_0} - (E-W_\lambda^{\mathrm{IV}}) \langle \mathfrak{D}_\lambda^{\mathrm{IV}\prime} | \mathfrak{G}_i^{(+)} \rangle \tag{36}$$

with the definition

$$y_{\lambda_0} = \langle \mathfrak{D}_\lambda^{\mathrm{IV}\prime} | \mathfrak{H} | \mathfrak{G}_i^{(+)} \rangle - \langle \mathfrak{D}_\lambda^{\mathrm{IV}\prime} | \overline{\mathfrak{H}} | \mathfrak{G}_i^{(+)} \rangle . \tag{37}$$

On account of (36) and the relation

$$\sum_{\nu} \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \nu} \frac{1}{N_\nu^{\mathrm{IV}}} \langle \mathfrak{D}_\nu^{\mathrm{IV}\prime} | \mathfrak{G}_i^{(+)} \rangle = (W_\mu^{\mathrm{II}} - E^{\mathrm{II}}) \langle \mathfrak{D}_\mu^{\mathrm{II}\prime} \boldsymbol{k}_0^{(-)} | \mathfrak{G}_i^{(+)} \rangle , \tag{38}$$

the matrix $T^{(c)}$ can be written into

$$\sum_{\boldsymbol{k}_0 \lambda \mu} \chi_{1\mu}^{\mathrm{II}} \left[\frac{1}{N_\mu^{\mathrm{II}}(E^{\mathrm{II}}-W_\mu^{\mathrm{II}})} \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda} + \frac{1}{N_\mu^{\mathrm{II}}} \langle \mathfrak{D}_\mu^{\mathrm{II}\prime} \boldsymbol{k}_0^{(-)} | \mathfrak{D}_\lambda^{\mathrm{IV}} \rangle \right] \frac{1}{N_\lambda^{\mathrm{IV}}(E-W_\lambda^{\mathrm{IV}})} y_{\lambda_0} . \tag{39}$$

Though the potentials L_0 and L_1 are generally arbitrary, in order to make all quantities derived in this formulation identical with those in the current theory, it is necessary to take L_0 and L_1 as hard sphere potentials. Hence the second term in (39) does not give any contribution.

We obtain finally, for the T-matrix of the compound nucleus process,

$$T^{(c)} = \sum_{k_0, \mu} \chi_{1\mu}^{\mathrm{II}} \frac{1}{N_\mu^{\mathrm{II}}(E^{\mathrm{II}}-W_\mu^{\mathrm{II}})} \left\{ \sum_{\lambda} \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda} \frac{1}{N_\lambda^{\mathrm{IV}}(E-W_\lambda^{\mathrm{IV}})} y_{\lambda_0} \right\} . \tag{40}$$

Until now, the deuteron formation through the intermediate state II has been undertaken, while there is an alternative process in which the nucleon 1 emerging from the compound state picks up the nucleon 0 in the neighbourhood of nuclear surface. The T-matrix element of this process is given by

$$T^{(c)\prime} = \sum_{k_1, \mu} \chi_{0\mu}^{\mathrm{III}} \frac{1}{N_\mu^{\mathrm{III}}(E^{\mathrm{III}}-W_\mu^{\mathrm{III}})} \left\{ \sum_{\lambda} \mathfrak{Y}_{\mathrm{III}_\mu \boldsymbol{k}_1 \lambda} \frac{1}{N_\lambda^{\mathrm{IV}}(E-W_\lambda^{\mathrm{IV}})} y_{\lambda_0} \right\} . \tag{41}$$

The terms (40) and (41) will be calculated in section 3.

Summary of the classification of the processes.

In the above formulation, it has been shown that the T-matrix of the deuteron pickup reaction can be written as the sum of three terms, omitting the volume direct interaction term,

$$T = T^{(s)} + T^{(c)} + T^{(c)\prime}, \tag{42}$$

where each term is interpreted, we may summarize, as follows:

(i) $T^{(s)}$. Direct deuteron pickup reaction arising from the interaction between the incident nucleon and the tails of wave functions in the target nucleus.

(ii) $T^{(c)}$. Deuteron pickup in the evaporation process, in which the nucleon 0 emerging from the compound nucleus picks up the nucleon 1 from the intermediate residual nucleus II.

(iii) $T^{(c)\prime}$. The process similar to (ii), in which the roles of the nucleons 0 and 1 are exchanged, and the intermediate residual nucleus is III.

§ 3. Compound nucleus process—Indirect deuteron pickup in the evaporation process

In this section, we turn to calculate the cross section of the process, in which, as mentioned above, a nucleon 0 emerging from the compound nucleus picks up any nucleon 1 to form a deuteron when it passes through the nuclear boundary. Referring to this process, the T-matrix element is given by

$$T^{(c)} = \sum_{k_0,\mu} \chi^{\mathrm{II}}_{1\mu} \frac{1}{N^{\mathrm{II}}_{\mu}(E^{\mathrm{II}} - W^{\mathrm{II}}_{\mu})} \sum_{\lambda} \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda} \left\{ \frac{1}{N^{\mathrm{IV}}_{\lambda}(E - W^{\mathrm{IV}}_{\lambda})} y_{\lambda_0} \right\}. \tag{40}$$

In this formula, the factor in the bracket is concerned with the formation of a compound nucleus λ by the incident nucleon, and $\mathfrak{Y}$ stands for the emission probabily of nucleon 0. The Green function $1/(E^{\mathrm{II}} - W^{\mathrm{II}}_{\mu})$ is interpreted as the level density of the intermediate residual nucleus II and χ represents the probability that the nucleon 1 sticks to the nucleon 0 to form a deuteron.

Now that we are concerned with the process proceeding through the compound nucleus, it is convenient for our discussions to write the cross section in the dispersion formula

$$\sigma^{(c)}(0\,;d) = \frac{\pi}{k_0^2} \left| \sum_{\lambda} \frac{(\Gamma_{\lambda_0}\Gamma_d)^{1/2}}{N^{\mathrm{IV}}_{\lambda}(E - W^{\mathrm{IV}}_{\lambda})} \right|^2, \tag{43}$$

where we must take the average with respect to the initial states and perform summation over all possible final states. Therefore, let us consider the average of $\sigma^{(c)}(0\,;d)$ over the energy spread of the incident beam which contains many levels of the compound nucleus; neglecting the contribution from cross terms between different λ's, one obtains

$$\begin{aligned} \sigma^{(c)}(0\,;d) &= \frac{1}{D_C} \int \frac{\pi}{k_0^2} \cdot \frac{\Gamma_{\lambda_0}\Gamma_d}{(E - E_\lambda) + (\Gamma_\lambda^2/4)} dE \\ &= \frac{2\pi}{k_0^2} \cdot \frac{\Gamma_{\lambda_0}}{D_C} \cdot \left(\frac{\Gamma_d}{\Gamma_C} \right). \end{aligned} \tag{44}$$

Here the integral is taken over the neighbourhood of typical resonances within the interval of the energy spread. In (44), the suffix C refers to the compound nucleus and Γ_d is the deuteron width defined by

$$\Gamma_d = \sum_f \frac{MK_d}{2\pi^2\hbar^2} \int d\Omega_d \Big| \sum_{k_0,\mu} \chi_{1\mu}^{\mathrm{II}} \frac{1}{N_\mu^{\mathrm{II}}(E^{\mathrm{II}} - W_\mu^{\mathrm{II}})} \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda} \Big|^2, \tag{45}$$

where the contribution from the ordinary deuteron channel is disregarded and the summation $\sum_f$ means that over all final states. Although the derivation of above formulas stands upon several rough assumptions, yet these have convenient forms to compare with formulas derived in a number of works having investigated compound nucleus reactions.

First we get the explicit form of $\chi_{1\mu}^{\mathrm{II}}$, which is given by

$$\chi_{1\mu}^{\mathrm{II}} = \langle \mathbf{g}_1^{(-)} | \overline{\mathfrak{H}}_{\mathrm{II}} | \mathfrak{D}_\mu^{\mathrm{II}} \rangle - \langle \mathbf{g}_1^{(-)} | \mathfrak{H}_{\mathrm{II}} | \mathfrak{D}_\mu^{\mathrm{II}} \rangle. \tag{29}$$

Without regarding the effect of the potential L_1, on the wave function of the nucleon 1, one finds

$$\begin{aligned} \mathbf{g}_1^{(-)} &= \langle \boldsymbol{k}_0 | \Omega_1^{(-)} \Phi_f \rangle \simeq \langle \boldsymbol{k}_0 | \Phi_f \rangle = \mathbf{g}_1(\boldsymbol{r}_1, \boldsymbol{R}; \boldsymbol{k}_0) \\ &= (2\pi)^{-3/2} \int d\boldsymbol{r}_0 e^{-i\boldsymbol{k}_0 \cdot \boldsymbol{r}_0} \phi_d(\boldsymbol{r}_0 - \boldsymbol{r}_1)\, e^{(i/2)\cdot \boldsymbol{K}_d \cdot (\boldsymbol{r}_0 + \boldsymbol{r}_1)}\, \Psi_f(\boldsymbol{R}), \end{aligned} \tag{46}$$

where ϕ_d is the deuteron wave function. If the relative coordinate ρ is introduced, in place of $\boldsymbol{r}_0$,

$$\begin{aligned} \boldsymbol{g}_1 &= (2\pi)^{-3/2} \Psi_f(\boldsymbol{R})\, G(\boldsymbol{k}_0 - (\boldsymbol{K}_d/2))\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}_1} \\ &= (2/\pi)^{1/2} \Psi_f(\boldsymbol{R}) \sum_{l_1 m_1} i^{l_1} Y_{l_1 m_1}(\Omega_1)\, Y_{l_1 m_1}^*(\Omega_q)\, j_l(q r_1)\, G(\boldsymbol{k}_0 - (\boldsymbol{K}_d/2)), \end{aligned} \tag{47}$$

$$\boldsymbol{q} = \boldsymbol{K}_d - \boldsymbol{k}_0.$$

Here G is the Fourier transform of Φ_d and Ω_q the direction of the transfered momentum $\boldsymbol{q}$. To be more exact, taking account of the spin states of all particles concerned, let us introduce the spin-angle wave function of the system except the nucleon 0

$$\mathfrak{Y}_{Jl_1\sigma}^M = \sum (j l_1 \nu m_1 | JM)\, (\sigma_1 \sigma_c \mu_1 \mu_c | j\nu)\, \chi_{\mu_1}(\sigma_1)\, \chi_{\mu_c}(\sigma_c)\, Y_{l_1 m_1}(\Omega_1), \tag{48}$$ *

where the suffix c stands for the core. Therefore, the expression (47) is to be modified as

$$\boldsymbol{g}_1 = (2/\pi)^{1/2} \Psi_f(\boldsymbol{R})\, G(\boldsymbol{k}_0 - (\boldsymbol{K}_d/2)) \sum{}' C_J^M i^{l_1} \chi_{\mu_0}(\sigma_0)\, \mathfrak{Y}_{Jl_1\sigma}^M\, Y_{l_1 m_1}^*(\Omega_q)\, j_{l_1}(q r_1), \tag{49}$$

with the geometrical factor C_J^M

$$C_J^M = \sum_{\nu, m_1}{}' (\sigma_1 \sigma_0 \mu_1 \mu_0 | \sigma_d \mu_d)\, (\sigma_1 \sigma_c \mu_1 \mu_c | j\nu)\, (j l_1 \nu m_1 | JM). \tag{50}$$

By the aid of Green's theorem, (29) can be reduced into the surface integral

$$\begin{aligned} \chi_{1\mu}^{\mathrm{II}} = {} & (2/\pi)^{1/2} G^*(\boldsymbol{k}_0 - (\boldsymbol{K}_d/2)) \sum C_J^M (-i)^{l_1} \chi_{\mu_0}^*(\sigma_0)\, Y_{l_1 m_1}(\Omega_q)\, (-\hbar^2/2M) \\ & \cdot \int_{r_1 = R_0} \langle \Psi_f(\boldsymbol{R})\, \mathfrak{Y}_{Jl_1\sigma}^M | \mathfrak{D}_\mu^{\mathrm{II}} \Big(\frac{\partial}{\partial r_1} - f_1 \Big) j_l(q r_1)\, dS^{12}, \end{aligned} \tag{51}$$

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* The notations are the same as those used in (74). J is the total angular momentum of the system, j the channel spin. M and ν are their z-components. σ's are the spin variable with z-components μ's.

where R_0 is the channel radius, and $\mathfrak{D}_\mu^{\mathrm{II}}$, being the eigenfunction of decaying states of the intermediate residual nucleus, satisfies the following boundary condition

$$\left(\frac{\partial}{\partial r_1}-f_1\right)|\mathfrak{D}_\mu^{\mathrm{II}}\rangle=0, \quad (r_1=R_0) \tag{52a}$$

$$f_1=\left[\frac{d}{dr_1}\log h_{l_1}^{(1)}(k_1r_1)\right]_{r_1=R_0}, \tag{52b}$$

with $(\hbar^2k_1{}^2/2M)=E_1$ for the energy of emerging nucleon 1. In accordance with previous references, we define the amplitude of width for nucleon 1 by

$$\gamma_{1l_1\mu}^{\mathrm{II}}=-\left(\frac{\hbar^2k_1}{M}\right)^{1/2}\int\langle\Psi_f(\boldsymbol{R})\mathfrak{Y}_{Jl_1\sigma}^M|\mathfrak{D}_\mu^{\mathrm{II}}\left(\frac{\partial}{\partial r_1}-f_1\right)j_l(k_1r_1)\,dS^{12}. \tag{53}$$

By use of this one obtains the explicit form of $\chi_{1\mu}^{\mathrm{II}}$, in a similar form to that given by Ui,

$$\chi_{1\mu}^{\mathrm{II}}=\left(\frac{\hbar^2k_1}{2\pi M}\right)^{1/2}R_0{}^2G^*\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)\sum C_J^M(-1)^{l_1}i^{l_1+1}\chi_{\mu_0}^*(\sigma_0)Y_{l_1m_1}(\Omega_q)$$
$$\times\gamma_{1l_1\mu}^{\mathrm{II}}\left[\frac{\partial j_{l_1}(qr_1)}{\partial r_1}\cdot h_{l_1}^{(1)}(k_1r_1)-j_{l_1}(qr_1)\frac{\partial h_{l_1}^{(1)}(k_1r_1)}{\partial r_1}\right]_{r_1=R_0}. \tag{54}$$

Hereafter, aiming at the survey on the overall feature of the process, we will proceed with our calculation basing upon simplifying assumptions. To begin with, neglecting the recoil effect of the residual nucleus, it is assumed that

$$\boldsymbol{q}=\boldsymbol{K}_d-\boldsymbol{k}_0\simeq\boldsymbol{k}_1,$$

in order to eliminate the complicated term in the bracket in (51) and obtain the simple expression,

$$\chi_{1\mu}^{\mathrm{II}}\simeq\left(\frac{\hbar^2}{2\pi Mk_1}\right)^{1/2}G^*\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)\sum C_J^M(-i)^{l_1}\chi_{\mu_0}^*(\sigma_0)Y_{l_1m_1}(\Omega_q)\gamma_{1l_1\mu}^{\mathrm{II}}. \tag{55}$$

Further, regardless of the correlation between the directions of momenta of nucleon 0 and nucleon 1, we take the average of $\chi_{1\mu}^{\mathrm{II}}$ and of $\mathfrak{Y}_{\mathrm{II}_\mu\boldsymbol{k}_0\lambda}$ respectively. And the relation between two nucleons is taken into account by the overlapping between the wave functions of the deuteron and its constituents, the latters evaporating independently of one another. Such situation seems to hold, if the excitation energy of the compound state is as high as to allow the evaporation of two or more nucleons and to make the random phase assumption valid. Along this approach one finds, averaging over the directions of nucleon 1,

$$|\chi_{1\mu}^{\mathrm{II}}|_{av}^2\simeq\frac{3}{4}\cdot\frac{1}{8\pi^2}\cdot\frac{\hbar^2}{Mk_1}\left|G\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)\right|^2\Gamma_{1\mu}(k_1), \tag{56}$$

in which the width $\Gamma_{1\mu}(=\gamma_{1\mu}^{\mathrm{II}\,2})$ of the usual sense is introduced. Next, let us perform summations over $\boldsymbol{k}_0$ and μ. Because of random signs of $\mathfrak{Y}_{\mathrm{II}_\mu\boldsymbol{k}_0\lambda}$, only the immediate-neighbour resonances near E^{II}, the excitation energy of the intermediate residual nucleus, contribute appreciably to the cross section; consequently we can omit the sum over μ and replace the energy denominator by the average level spacing $D(E^{\mathrm{II}})$ near E^{II}. We then obtain the following estimate,

$$\sum_{k_0,\mu} \chi_{1\mu}^{\mathrm{II}} \frac{1}{N_\mu^{\mathrm{II}}(E^{\mathrm{II}}-W_\mu^{\mathrm{II}})} \cdot \mathfrak{Y}_{\mathrm{II}_\mu \boldsymbol{k}_0 \lambda} \simeq \left(\frac{3}{32\pi}\right)^{1/2} \frac{\hbar^2}{M} \int \frac{d\boldsymbol{k}_0}{D(E^{\mathrm{II}})} \left(\frac{\Gamma_{0C}\Gamma_{1\mu}}{k_1 k_0}\right)^{1/2} G\left(\boldsymbol{k}_0 - \frac{\boldsymbol{K}_d}{2}\right). \tag{57}$$

For simplicity, if we take the exponential deuteron wave function

$$\phi_d(\rho) = (\alpha/2\pi)^{1/2}(e^{-\alpha\rho}/\rho), \quad \alpha = (MB_d/\hbar^2)^{1/2} \tag{58}$$

we find the Fourier transform

$$G\left(\boldsymbol{k}_0 - \frac{\boldsymbol{K}_d}{2}\right) = \frac{2(2\pi\alpha)^{1/2}}{\alpha^2 + (\boldsymbol{k}_0 - \boldsymbol{K}_d/2)^2}. \tag{59}$$

The integration over the direction of $\boldsymbol{k}_0$ leads to

$$\left(\frac{3}{4}\pi^2\alpha \frac{\hbar^2}{ME_d}\right)^{1/2} \int \frac{dE_0}{D(E^{\mathrm{II}})} \left(\frac{\Gamma_{0C}\Gamma_{1\mu}}{k_1 k_0}\right)^{1/2} \log \frac{B_d + (\sqrt{2E_0} + \sqrt{E_d})^2}{B_d + (\sqrt{2E_0} - \sqrt{E_d})^2}, \tag{60}$$

where E_0 and E_d are the kinetic energies of the nucleon 0 and the deuteron produced, and B_d the binding energy of the deuteron.

Since we are treating the process involving an enormous number of excited levels, the following relation should be employed to obtain the appropriate formula of nucleon width

$$\Gamma_{0C} = \frac{1}{w_C(E_C)} \cdot \frac{2ME_0}{\pi^2\hbar^2} \sigma_C^{(0)}(E_0). \tag{61}$$

Here $w_C(E_C)$ is the level density in the compound nucleus with the excitation energy E_C; $\sigma_C^{(0)}(E_0)$ is the cross section for the formation of compound nucleus C by incident particle 0 with kinetic energy E_0. The relation (61) for the two opposite processes can be derived from simple statistical considerations.[13)] Substituting (61) in (60) and on the grounds of certain approximations, one gets

$$\left(\frac{3}{2\pi^2} \cdot \frac{\alpha\sigma_1\sigma_0}{E_d}\right)^{1/2} \int (E_0 E_1)^{1/4} \log \frac{B_d + (\sqrt{2E_0} + \sqrt{E_d})^2}{B_d + (\sqrt{2E_0} - \sqrt{E_d})^2} \cdot \exp\left(-\frac{E_0 + S_0}{2\Theta_C}\right) \cdot dE_0 \tag{62}$$

where σ_0 and σ_1 are the compound nucleus formation cross sections for nucleons 0 and 1 respectively, which are assumed to be slowly-varying within the change of E_0 and E_1. S_0 is the separation energy of nucleon 0 from the compound nucleus and Θ_C is the temperature of the compound state. Intuitively speaking, it is probable that both constituents of the deuteron have approximately equal kinetic energies; in fact, the integrand of (62) has its maximum around $E_0 \simeq E_d/2$. Hence we estimate the order of magnitude of (62) as

$$\left(\frac{\alpha\sigma_1\sigma_0}{3\pi^2}\right)^{1/2} E_d \log\frac{B_d+4E_d}{B_d}\cdot\exp\left[-\frac{E_d+2S_0}{4\theta_C}\right], \tag{63}$$

which gives rise to the deuteron width expressed by

$$\Gamma_d=\sum_f\frac{4M^2}{3\pi^3\hbar^4}\sigma_1\sigma_0\cdot B_d^{1/2}E_d^{3/2}\left[\log\frac{B_d+4E_d}{B_d}\right]^2\cdot\exp\left[-\frac{E_d+2S_0}{2\theta_C}\right]. \tag{64}$$

The sum over final states can be replaced by the integration ;

$$\sum_f\rightarrow dE_f/D(E_f)$$

with $E_f=E_C-E_d-S_d$ for the excitation energy of the residual nucleus. Making use of approximate expansion,

$$w_f(E_f)\simeq w_C(E_C)\exp[-E({}_d+S_d)/\theta_C],$$

one obtains

$$\Gamma_d=\frac{4M^2}{3\pi^3\hbar^4}B_d^{1/2}\sigma_1\sigma_0\cdot w_C(E_C)\int_0^{W_d}E_d^{3/2}\left[\log\frac{B_d+4E_d}{B_d}\right]^2\exp\left[-\frac{3E_d+2S_d+2S_0}{2\theta_C}\right]\cdot dE_d, \tag{65}$$

where W_d is the energy available for the emission of a deuteron from the compound nucleus and equal to the excitation energy E_C minus the separation energy of a deuteron S_d. The logarithmic term in the integrand, being slowly varying compared with the rest, can removed from the integral and taken at the maximum of the integrand. Further, expanding the integrand around its maximum into a Gaussian form, the integration from $-\infty$ to $+\infty$ gives the final result

$$\Gamma_d=\frac{1000}{81}\cdot\frac{M^2}{\pi^{5/2}\hbar^4}B_d^{1/2}\theta_C^{7/2}\left[\log\frac{3B_d+20\theta_C}{3B_d}\right]^2\sigma_1\sigma_0 w_C(E_C)\exp\left[-\frac{5}{2}-\frac{S_d+S_0}{\theta_C}\right]. \tag{66}$$

Now, if we introduce the nucleon width

$$\Gamma_0=\frac{2M}{\pi^2\hbar^2}\sigma_0\cdot\theta_C^2\cdot\exp\left[-\frac{S_0}{\theta_C}\right] \tag{67a}$$

and the deuteron width of the usual sense, due to the evaporation through ordinary deuteron channels

$$\Gamma_d^{(0)}=\frac{4M}{\pi^2\hbar^2}\sigma_d\cdot\theta_C^2\exp\left[-\frac{S_d}{\theta_C}\right], \tag{67b}$$

the formula (66) can be rewritten into a more convenient form

$$\Gamma_d=\frac{\Gamma_0\Gamma_d^{(0)}}{D_C(E_C)}\cdot\frac{125\pi^2}{81}\cdot\frac{\sigma_1}{\sigma_d}\cdot\left(\frac{B_d}{\pi\theta_C}\right)^{1/2}\cdot e^{-5/2}\left[\log\frac{3B_d+20\theta_C}{3B_d}\right]^2 \tag{68 *}$$

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* Due to the approximation (46), this formula for the deuteron width may yield a larger magnitude than the actual one. Using a frivolous model this error is found to be not so large as to alter our conclusion.

$$\approx \frac{\Gamma_0 \Gamma_d^{(0)}}{D_C(E_C)} \cdot \frac{\sigma_1}{\sigma_d} \cdot \frac{1}{\sqrt{\theta_C}} \left[\log \frac{3B_d + 20\theta_C}{3B_d} \right]^2 \quad (\theta_C \text{ in Mev}).$$

Though the result obtained here is not quantitatively reliable because of crude approximations, yet $\Gamma_d/\Gamma_d^{(0)}$ can become much larger than unity, even if we assume the barrier penetration factors for both cases are the same. As can be understood obviously, the Coulomb penetrability in Γ_d is only for one nucleon of the constituents of deuteron, contrary to that in $\Gamma_d^{(0)}$ due to the transmission of the deuteron as the whole through the nuclear potential barrier.

It is of interest to see the energy spectrum of deuterons emitted through the process proposed here and how it differs from the ordinary Maxwellian type. From the integrand of (65), the average kinetic energy of deuterons is estimated, with the same uncertainty as (66), as

$$\langle E_d \rangle_{av} \simeq 6.4\theta_C \tag{69}$$

which is much larger than that obtained from the ordinary evaporation theory. Based upon this prediction it follows the deuterons due to the pickup evaporation scarcely suffer the effect of the Coulomb barrier, which is usually lower than the average kinetic energy.

Summary for the indirect pickup process

(i) The deuteron width due to the indirect pickup process may swell 10～100 times larger than that due to the ordinary deuteron evaporation.

(ii) The energy spectrum of deuterons produced through the indirect pickup is of complicated form dependent upon the temperature of the compound nucleus and differs from the Maxwellian type of the usual one. The average kinetic energy of deuterons is about 3 times larger than that of the ordinary case.

§ 4. Direct pickup process in the low energy $(p; d)$ reaction —Effect of the Coulomb potential—

According to the scanty knowledge from the experimental observation, the characteristic feature of the deuteron pickup process, involving discrete levels for the residual nuclei, seems to be a forward-peak angular distribution. Although the Born approximation treatment on the matrix element $T^{(s)}$ may give an adequate explanation of this phenomenon, yet it is worth while to examine the effect of Coulomb forces acting upon the incident proton and the emergent deuteron, which is commonly neglected. Since a great compexity is introduced in the theoretical calculation to take into account the Coulomb effect, we think it an appropriate way to study it on a frivolous example, namely, the low energy limit where the Coulomb effect dominates. In this section, $T^{(s)}$, the first term of (42), is investigated in the $(p; d)$ reaction where the incident energy, as well as the emergent energy, is so small as to violate the validity of the Born approximation. Therefore, the results obtained in this section facilitate us to discuss the angular distribution of deuterons from $(p; d)$ reaction on heavy nuclei, subject to certain approximations. In cases concerned here, the deuteron emission from the compound nucleus is reasonably neglected. Because of the large Coulomb repulsion, nuclear effects on the incident and emergent particles, may be also negligible and it allows to take a pure Coulomb wave function, instead of distorted waves. Hence we obtain approximately

$$T^{(s)} \simeq \langle \Psi_f | V_{np} | \Psi_i \rangle, \tag{70}$$

with

$$\Psi_i = \Phi_i(J_i M_i\,;\ \boldsymbol{R},\ \boldsymbol{r}_n)\,\chi_{\mu_p}(\sigma_p)\,\varphi_p(\boldsymbol{r}_p) \tag{71a}$$

and

$$\Psi_f = \Phi_f(J_f M_f\,;\ E_f\,;\ \boldsymbol{R})\,\chi_{\mu_d}(\sigma_d)\,\Psi_d(\boldsymbol{r}_p,\ \boldsymbol{r}_n), \tag{71b}$$

where φ_p and Ψ_d are the Coulomb wave functions describing the motions of proton and deuteron and satisfy the Schrödinger equation respectively,

$$\left[-\frac{\hbar^2}{2M}(\nabla_p^2+\nabla_n^2)+V_{np}+\frac{Ze^2}{r_p}-E_d+B_d\right]\Psi_d=0, \tag{72}$$

$$\left[-\frac{\hbar^2}{2M}\nabla_p^2+\frac{Ze^2}{r_p}-E_p\right]\varphi_p=0. \tag{73}$$

Φ_i is the wave function of the target nucleus, with total angular momentum J_i and its z-component M_i, and Φ_f is that of the residual nucleus with corresponding quantities, J_f and M_f. χ's are spin wave functions and σ spin variable with z-component μ for respective cases. E_p and E_d are the kinetic energies of proton and deuteron respectively. B_d is the binding energy of deuteron. In the external region, the wave function Ψ_i is decomposed, in a usual manner, as

$$\begin{aligned}&\Psi_i(J_i M_i\,;\ \boldsymbol{R},\ \boldsymbol{r}_n)\\ &\quad=\sum_{j_n\sigma_n}(J_f j_n M_f \nu_n|J_i M_i)\,(l_n\sigma_n m_n\mu_n|j_n\nu_n)\,\Psi_f(J_f M_f\,;\ \boldsymbol{R})\,\chi_{\mu_n}(\sigma_n)\,\varphi_n(l_n m_n\,;\ \boldsymbol{r}_n),\end{aligned} \tag{74}$$

where j_n is the neutron spin and ν_n its z-component and $\chi(\sigma_n)\varphi_n(\boldsymbol{r}_n)$ is the spin-space wave function of the neutron to be picked up. The substitution of (74) and (71) in (70) leads to

$$T^{(s)}=\sum(I_f j_n M_f \nu_n|J_i M_i)\,(l_n\sigma_n m_n\mu_n|j_n\nu_n)\,(\sigma_n\sigma_p\mu_n\mu_p|\sigma_d\mu_d)\cdot I(l_n m_n) \tag{75}$$

with

$$I(l_n m_n)=\iint\limits_{\text{external region}}\Psi_d^*(\boldsymbol{r}_p,\ \boldsymbol{r}_n)\,\varphi_p(\boldsymbol{r}_p)\,\varphi_n(l_n m_n\,;\ \boldsymbol{r}_n)\,d\boldsymbol{r}_p d\boldsymbol{r}_n. \tag{76}$$

By taking the average with respect to the initial states and performing a summation over the final states, the cross section of this process is obtained through

$$\frac{d\sigma(p\,;d)}{d\Omega_d}=\frac{M^2}{2\pi^2\hbar^4}\cdot\frac{K_d}{k_p}\cdot\frac{1}{(2i_p+1)\,(2J_i+1)}\cdot\sum_{i,f}|T^{(s)}|^2 \tag{77}$$

where i_p is the intrinsic spin of the proton, $\boldsymbol{k}_p$ and $\boldsymbol{K}_d$ are the incident and the emergent momenta respectively. The summations in respect to the initial and the final states yield a factor 3/4, namely, *a priori* probability for the neutron and proton to have parallel spins, and (77) reduces to

$$\frac{d\sigma(p\,;d)}{d\Omega_d}=\frac{3}{4}\cdot\frac{M^2}{2\pi^2\hbar^4}\cdot\frac{K_d}{k_p}\sum_{j_n m_n}\cdot\frac{1}{2l_n+1}\,|I(l_n,\ m_n)\,|^2. \qquad (78)$$

Now we evaluate integral $I(l_n,\ m_n)$ given by (76). According to the usual method conventionally employed, the calculation of such an integral necessiates a partial wave expansyon and rather extensive calculations tailored to each particular reaction under consideration, but, in our work, a similiar alternative as used by Biedenharn[11] and Ter-Martirosyan[12] is utilized to compensate for the complexity arising from a partial wave expansion. If the coordinates $\boldsymbol{r}$ and ρ, by $\boldsymbol{r}=(\boldsymbol{r}_p+\boldsymbol{r}_n)/2$ and $\rho=\boldsymbol{r}_n-\boldsymbol{r}_p$, are introduced, the Schrödinger equations for Ψ_d and φ_p are rewritten as

$$\left[-\frac{\hbar^2}{4M}\nabla_r{}^2+\frac{Ze^2}{|\boldsymbol{r}-(\rho/2)\,|}-\frac{\hbar^2}{M}\nabla_\rho{}^2+V_{np}(\rho)-E_d+B_d\right]\Psi_d(\boldsymbol{r},\ \rho)=0, \qquad (79)$$

$$\left[-\frac{\hbar^2}{2M}\nabla_p{}^2+\frac{Ze^2}{|\boldsymbol{r}-(\rho/2)\,|}-E_p\right]\varphi_p(\boldsymbol{r},\ \rho)=0. \qquad (80)$$

Since the effective range ρ_0 of the interaction V_{np} is considered to be much smaller than the channel radius R_0, the polarization of the deuteron due to the Coulomb repulsion can be neglected. Therefore the solutions of (79) and (80) are approximately given by

$$\Psi_d(\boldsymbol{r},\ \rho)\simeq\varphi_d(\boldsymbol{r})\,\phi_d(\rho) \qquad (81a)$$

$$\varphi_p(\boldsymbol{r},\ \rho)\simeq\varphi_p(\boldsymbol{r}). \qquad (81b)$$

In fact, the linear term in the expansion of $\varphi(\boldsymbol{r}\pm\rho/2)$ in powers of ρ drops out when we integrate over the direction of ρ. The quadratic term is a small quantity of magnitude of about E(Mev)/40. In (81a), ϕ_d is the internal wave function for deuteron. Allowing the same approximation for φ_n, the integral $I(l_n m_n)$ is factored, by the aid of (81)

$$I(l_n m_n)\simeq W\cdot\int\Psi_d{}^*(\boldsymbol{r})\,\varphi_p(\boldsymbol{r})\,\varphi_n(l_n m_n\,;\ \boldsymbol{r})\,d\boldsymbol{r} \qquad (82)$$

with

$$W=\int V_{np}(\rho)\,\phi_d{}^*(\rho)\,d\rho. \qquad (83)$$

It is convenient to express the neutron wave function as

$$\varphi_n(r)=u_{l_n}(r)\,Y_{l_n m_n}(\Omega_n), \qquad (84)$$

with

$$u_{l_n}(r)=(2MR_0/\hbar^2)^{1/2}\gamma_{j_n f}[u_{l_n}(R_0)\,]^{-1}v_{l_n}(r)\cdot e^{-\kappa_n r}/r \qquad (85)$$

and

$$v_{l_n}(r)=\sum_{\nu=0}^{\infty}\frac{(l+\nu)\,!}{\nu!\;(l-\nu)\,!\;(2\kappa_n r)^{\nu}}. \tag{86}$$

Where $\gamma_{j_n f}$ is the reduced-width amplitude for the separation in the external region of the ground state of the target nucleus into a neutron and a state of the final nucleus ; κ_n is the wave number associated with the binding energy E_n of the neutron in the ground state of the target. In the case which we are treating, when Z is large, E_d is small and the inequalities

$$\alpha_d=\frac{Ze^2}{\hbar}\left(\frac{M}{E_d}\right)^{1/2}>1, \tag{87a}$$

and

$$\alpha_p=\frac{Ze^2}{\hbar}\left(\frac{M}{2E_p}\right)^{1/2}>1 \tag{87b}$$

are fulfilled, the angular distribution and the variation of the cross section with energy depend weakly on the angular momentum l_n. Thus both the spherical harmonics and the function v_{l_n} can be removed from the integral and calculated at the point ($r=r_1$ and $\Omega=\Omega_1$) where the value of other rapidly varying function in the integrand has its maximum. For an estimate we can use the value of $\kappa_n r_1$ obtained by the saddle point method in calculating the integral of the type of (82), which gives a magnitude of about 5∼10 for $\kappa_n r_1$ in our cases concerned. Therefore, it is more convenient to calculate the integration over the whole space, because of the small contribution from the internal region if it is included in the integration. Now substituting the exact Coulomb wave functions

$$\varphi_p(\boldsymbol{r})=(2\pi\alpha_p)^{1/2}\exp(-\pi\alpha_p+i\boldsymbol{k}_p\cdot\boldsymbol{r})\,F(-i\alpha_p,\;1,\;i(k_p r-\boldsymbol{k}_p\cdot\boldsymbol{r})), \tag{88}$$

$$\varphi_d{}^*(\boldsymbol{r})=(2\pi\alpha_d)^{1/2}\exp(-\pi\alpha_d-i\boldsymbol{K}_d\cdot\boldsymbol{r})\,F(-i\alpha_d,\;1,\;i(K_d r+\boldsymbol{K}_d\cdot\boldsymbol{r})) \tag{89}$$

in (82), the integration goes over into

$$\begin{aligned}I(l_n m_n)=&W\cdot v_{l_n}(r_1)\,Y_{l_n m_n}(\Omega_1)\cdot\frac{F(-i\alpha_p,\;-i\alpha_d,\;1,\;-\xi)}{1+\xi}\cdot\frac{2\pi^2(\alpha_p\alpha_d)^{1/2}}{\kappa_n{}^2+(k_p-K_d)^2}\cdot\\&\cdot\left[\frac{(\kappa_n-ik_p)^2+K_d{}^2}{\kappa_n{}^2+(k_p-K_d)^2}\right]^{i\alpha_p}\cdot\left[\frac{(\kappa_n-iK_d)^2+k_p{}^2}{\kappa_n{}^2+(k_p-K_d)^2}\right]^{i\alpha_d}\cdot\exp[-\pi(\alpha_p+\alpha_d)+(\alpha_d\zeta_d+\alpha_p\zeta_p)],\end{aligned} \tag{90}$$

where

$$\zeta_p=\arctan\frac{2\kappa_n k_p}{K_d{}^2+\kappa_n{}^2-k_p{}^2},\quad 0\leq\zeta_p<\pi \tag{91a}$$

$$\zeta_d=\arctan\frac{2\kappa_n K_d}{k_p{}^2+\kappa_n{}^2-K_d{}^2},\quad 0\leq\zeta_d<\pi/2 \tag{91b}$$

and

$$\xi=\xi_0\sin^2(\theta_d/2)=\frac{4k_p K_d}{\kappa_n{}^2+(k_p-K_d)^2}\sin^2(\theta_d/2). \tag{92}$$

θ_d is the angle between $\boldsymbol{k}_p$ and $\boldsymbol{K}_d$. The derivation of (90) is given in reference 12, in which the integral representation for the confluent hypergeometric function

$$F(-i\alpha,\ 1,\ i(kr-\boldsymbol{k}\cdot\boldsymbol{r}))=e^{i(kr-\boldsymbol{k}\cdot\boldsymbol{r})}\frac{\Gamma(1)}{\Gamma(1+i\alpha)}\int_0^\infty u^{i\alpha}J_0(2\sqrt{i(kr-\boldsymbol{k}\cdot\boldsymbol{r})u})\,e^{-u}du \tag{93}$$

is used. Making use of (90) and on account of

$$\frac{4\pi}{2l+1}\sum_{m=-l}^{l}|Y_{lm}(\Omega_1)|^2=1, \tag{94}$$

we obtain the cross section formula

$$\frac{d\sigma(p\,;d)}{d\Omega_d}=\frac{3}{4}\cdot\frac{\pi M^2}{2\hbar^4}\cdot\frac{K_d}{k_p}\cdot\frac{\alpha_p\alpha_d}{[\kappa_n^2+(k_p-K_d)^2]^2}\cdot\left|\frac{F(-i\alpha_p,\ -i\alpha_d,\ 1,\ -\xi)}{1+\xi}\right|^2\cdot|W|^2$$
$$\cdot\sum_{j_n}|\gamma_{j_nf}\cdot v_{l_n}(r_1)|^2\cdot\exp[-2\pi(\alpha_p+\alpha_d)+2(\alpha_d\zeta_d+\alpha_p\zeta_p)]. \tag{95}$$

The value of the hypergeometric function has been evaluated by Sommerfeld[14] using the saddle point method, from which we have

$$|F(i\alpha_p,\ i\alpha_d,\ 1,\ -\xi)|^2=\begin{cases}\dfrac{1+\xi}{2\pi\alpha_d\xi}\cdot\dfrac{\exp 2(\alpha_d\zeta_d'+\alpha_p\zeta_p')}{[(4\lambda/\xi)-(1-\lambda)^2]^{1/2}}, & \text{for } \xi>\xi' \quad (96a)\\ |1+\xi|^2, & \text{for } \xi\to 0 \quad (96b)\end{cases}$$

where

$$\zeta_d'=\arccos\frac{2+(1-\lambda)\xi}{2\sqrt{1+\lambda}},\quad 0\leq\zeta_d'<\pi/2 \tag{97a}$$

$$\zeta_p'=\arccos\frac{(1-\lambda)\xi-2\lambda}{2\lambda\sqrt{1+\xi}},\quad 0\leq\zeta_p'<\pi/2 \tag{97b}$$

and $\qquad \lambda=\alpha_p/\alpha_d.$

The critical value ξ' is determined by the applicability condition of the saddle point method, which affords

$$\xi'^2(1+\xi')^{-1}\sqrt{(\alpha_p\alpha_d/\xi')-(\alpha_p-\alpha_d)^2}=\min\{\lambda,\ \lambda^{-1}\}. \tag{98}$$

From the above condition, this approximation is of good validity if the residual nucleus is in the ground state or near ground states. However the calculations obtained here are useful for the study on general cases. Thus one can write the cross section formula as

$$\frac{d\sigma(p\,;d)}{d\Omega_d}=\frac{3}{4}\cdot\frac{M^2}{4\pi\hbar^4}\cdot\frac{K_d}{k_p}\cdot\frac{\alpha_p}{[\kappa_n^2+(k_p-K_d)^2]^2}\cdot|W|^2\cdot K(\xi)$$
$$\cdot\sum_{j_n}|\gamma_{j_nf}\cdot v_{l_n}(r_1)|^2\cdot\exp[-2\pi(\alpha_p+\alpha_d)+2(\alpha_p\zeta_p+\alpha_d\zeta_d)], \tag{99}$$

where the function $K(\xi)$ is expressed by

$$K(\xi)=\begin{cases}\dfrac{1}{\xi(1+\xi)[(4\lambda/\xi)-(1-\lambda)^2]^{1/2}}\exp 2(\alpha_p\zeta_p'+\alpha_d\zeta_d'), & \text{for } \xi>\xi' \quad (100a)\\[2ex] 2\pi\alpha_d, & \text{for } \xi\to 0. \quad (100b)\end{cases}$$

The angular distribution of the reaction product is determined by the dependence of function $K(\xi)$ on θ_d, which has a maximum for $\theta_d=\pi$ i. e., $\xi=\xi_0$. For small $(\pi-\theta_d)$, $K(\xi)$ is expanded in powers of $(\xi_0-\xi)$, neglecting the small change of the denominator in $K(\xi)$; thus we obtain

$$K(\xi)\simeq K(\xi_0)\exp[-(\pi-\theta_d)^2/\delta^2]. \tag{101}$$

Here δ^2 is the width of the angular distribution, which is given by

$$\delta^2=\frac{(\sqrt{E_p}+\sqrt{2E_d})^2+|E_n|}{\omega\sqrt{B_d(B_d-E_f)}} \tag{102}$$

with

$$\omega=\frac{Ze^2}{\hbar}\left(\frac{2M}{B_d}\right)^{1/2}=0.21\,Z. \tag{103}$$

The width is smaller, for the larger Z and the smaller E_d, as it should be. Although the result obtained above is not quite reliable, yet it may be concluded that in general the angular distribution of the particles produced in the $(p\,;d)$ reaction is shifted to backward directions if the Coulomb repulsion is taken into account. In order to calculate the total cross section, we integrate (99) over Ω_d; noting $d\Omega_d=(4\pi/\xi_0)\,d\xi$, one obtains the integral

$$\int K(\xi)\,d\Omega_d=\frac{4\pi}{\xi_0}\int_0^{\xi_0}K(\xi)\,d\xi\simeq\frac{4\pi}{\xi_0}\,K(\xi_0)\int_0^{\infty}\exp\left[-\frac{(\pi-\theta_d)^2}{\delta^2}\right]\,d\xi$$
$$=\pi\delta^2K(\xi_0) \tag{104}$$

and consequently

$$\sigma(p\,;d)=\frac{3}{4}\cdot\frac{\pi M^2}{32\hbar^4}\cdot\frac{1}{k^2\kappa_{df}^2}\cdot|W|^2\cdot\sum_{j_n}|\gamma_{j_nf}\cdot v_{l_n}(r_1)|^2\cdot\exp[-\Phi(E_p,E_d)], \tag{105}$$

where

$$\Phi(E_p,\ E_d)=\sqrt{2}\,\omega(\eta_p\sqrt{B_d/2E_p}+\eta_d\sqrt{B_d/E_d}), \tag{106}$$

$$\eta_p=\arctan\frac{2\sqrt{E_p}(\sqrt{|E_n|}-\sqrt{2(B_d-E_f)})}{E_p-(\sqrt{|E_n|}-\sqrt{2(B_d-E_f)})},\quad 0\leq\eta_p<\pi \tag{107a}$$

$$\eta_d=\arctan\frac{2\sqrt{E_d}(\sqrt{|E_n|}-\sqrt{2(B_d-E_f)})}{E_d-(\sqrt{|E_n|}-\sqrt{2(B_d-E_f)})},\quad 0\leq\eta_d<\pi \tag{107b}$$

$$\kappa_{df}=[M(B_d-E_f)/\hbar^2]^{1/2}. \tag{108}$$

This cross section formula is valid for $B_d\gg E_f$; at heavy nuclei there exist several levels satisfying this condition. The formula (105) gives the cross section associated with each deuteron group corresponding to the low lying levels of the residual nucleus. Therefore the energy spectrum of the deuteron is obtained by taking account of contributions from all these final states, which can be done multiplying $dE_d/D_f(E_f)$ on the cross section formula (105). $D_f(E_f)$ is the level distance of the residual nucleus at the excitation energy E_f. Thus the energy spectrum is proportional to

$$\frac{\gamma_{j_n f}^2}{D_f(E_f)} \cdot \frac{1}{\kappa_{df}^2} \exp[-\Phi(E_p,\ E_d)], \tag{109}$$

which depends upon the behaviour of $\gamma_{j_n f}^2/D_f(E_f)$, namely, the strength function associated with the excited states of the residual nucleus. In particular, one can obtain the cross section formula near the threshold

$$\sigma(p\,;\,d)_{\text{threshold}} \propto \exp[-\sqrt{2}\,\pi\omega(B_d/\epsilon_c)^{1/2}], \tag{110}$$

where ϵ_c is the excess energy over the threshold. This result may be expected intuitively by assuming the penetrability to play the most important role.

In conclusion, the pronounced backward maximum that occurs for large values of α should be distinguished in the experimental observations, if slow deuterons can be discriminated from protons, when the yield of the latter is much larger than that of the former.

§ 5. Direct pickup process—Born approximation

In this section we shall calculate the surface direct term $T^{(s)}$, which arises from the interaction between the incident nucleon and the tail of the wave function of the nucleon to be picked up. Although this has been already investigated by a number of authors, almost of these are concerned with the angular distribution of definite energy particle groups corresponding to the low-lying levels of the residual nucleus. In order to obtain the total cross section of the process, one should take into account all contributions from an enormous number of excited states of the residual nucleus. To do this we evaluate the matrix element $T^{(s)}$ by applying the Chew-Goldberger's procedure and assuming the Gaussian momentum distribution for the tail of the collision partner. In the recent analysis made by Elton and Gomes[15)] this momentum distribution for nucleons in the diffuse rim has been revealed to be successful in explaining the angular and the energy distributions of protons inelastically scattered by heavy nuclei.

Now the required T-matrix element is then, in the Born approximation,

$$T^{(s)} = \langle e^{(i/2)\boldsymbol{K}_d\cdot(\boldsymbol{r}_1+\boldsymbol{r}_0)}\,\phi_d(\boldsymbol{r}_0-\boldsymbol{r}_1)\,\Psi_f(\boldsymbol{R})\,|V_{01}|\,e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}_0}\,\Psi_i(\boldsymbol{R}\,;\,\boldsymbol{r}_1)\rangle, \tag{111}$$

where the integration over the coordinate $\boldsymbol{r}_1$ should be performed only in the external region. If it is taken over all space, the T-matrix reduces to the same as treated by previous authors[5),7)] and goes over into the weak coupling limit, for which an attempt will be made in the next paragraph. If the relative coordinate $\rho=\boldsymbol{r}_1-\boldsymbol{r}_0$ is introduced in place of $\boldsymbol{r}_0$, the matrix element may be factored:

$$T^{(s)}=\langle\phi_d(\boldsymbol{\rho})|V_{01}(\boldsymbol{\rho})|e^{i(\boldsymbol{k}_0-\boldsymbol{K}_d/2)\cdot\boldsymbol{\rho}}\rangle\langle|e^{i(\boldsymbol{K}_d-\boldsymbol{k}_0)\cdot\boldsymbol{r}_1}\Psi_f(\boldsymbol{R})|\Psi_i(\boldsymbol{R};\boldsymbol{r}_1)\rangle. \tag{112}$$

If we remember that V_{01} and ϕ_d are connected through the Schrödinger equation for the ground state of the deuteron, then the first factor may be reduced to

$$-\left[B_d+\frac{\hbar^2}{M}\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)^2\right]G\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right), \tag{113}$$

where G is again the Fourier transform of ϕ_d. Making use of the independent particle model wave function, the target wave function Ψ_i in the external region is decomposed in the product of the core wave function and that of the nucleon 1, when the expansion coefficients are represented by the reduced-width amplitude.[16)~19)] The signs of these coefficients referring to different states of the nucleon 1 are considered as likely to be positive as negative so that the cross terms can be disregarded. Above simplifications lead to the approximate relation

$$\langle e^{i(\boldsymbol{K}_d-\boldsymbol{k}_0)\cdot\boldsymbol{r}_1}\Psi_f(\boldsymbol{R})|\Psi_i(\boldsymbol{R};\boldsymbol{r}_1)\rangle^2\simeq\sum_\alpha\frac{\gamma_{i,f\alpha}^2}{\theta_\alpha^2}|\phi_\alpha(\boldsymbol{k}_1)|^2, \tag{114}$$

with

$$\phi_\alpha(\boldsymbol{k}_1)=\int_{|r_1|>R_0}u_\alpha(\boldsymbol{r}_1)e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1}d\boldsymbol{r}_1 \tag{115}$$

and $\boldsymbol{k}_1=\boldsymbol{K}_d-\boldsymbol{k}_0$.

In the above formula, $\gamma_{i,f\alpha}$ is the reduced width amplitude for the separation of the ground state of the target nucleus into a nucleon state u_α and a state Ψ_f of the final nucleus. θ_α^2 is the single-particle reduced width and is expected to be the order of magnitude of $\hbar^2/MR^2$. It may be assumed that the reduced width $\gamma_{i,f\alpha}^2$ is a slowly varying function of α, and can be replaced by the most probable value $\gamma_{i,f}^2$. Now the cross section for emission of a deuteron into the solid angle $d\Omega_d$, leaving the residual nucleus in the state Ψ_f, is given by

$$d\sigma^{(s)}=\frac{3}{4}\cdot\frac{M^2}{2\pi^2\hbar^4}\cdot\frac{K_d}{k_0}|T^{(s)}|^2\cdot d\Omega_d. \tag{116}$$

The factor 3/4 represents *a priori* probability for the neutron and proton to have parallel spins. Multiplying the final state density of the residual nucleus and substituting (113) and (114) in (116), one obtains the energy spectrum of deuterons produced as

$$\frac{\partial^2\sigma^{(s)}}{\partial E_d\partial\Omega_d}=\frac{3}{4}\cdot\frac{M^3R^2}{2\pi^2\hbar^6}\cdot\frac{K_d}{k_0}\cdot\frac{\gamma_{i,f}^2}{D_f(E_f)}\left[B_d+\frac{\hbar^2}{M}\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)^2\right]^2 \cdot G^2\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)\sum_\alpha|\phi_\alpha(\boldsymbol{k}_1)|^2. \tag{117}$$

The essential point in our method is to replace the sum of $|\phi_\alpha|^2$ over all α by an appropriate momentum distribution for the nucleon 1. Actually this distribution has such a complicated form as to make the analytical calculation impossible. Then to avoid the algebraic complexity of $\phi_\alpha(\boldsymbol{k}_1)$ and to make a rough estimation, we assume a Gaussian distribution and may put as

$$\sum_\alpha |\phi_\alpha(\boldsymbol{k}_1)|^2 = (2\pi\hbar^2/M\Theta_g)^{3/2}\cdot P_e\cdot \exp[-E_1/\Theta_g], \tag{118}$$

where P_e is the probability of finding the nucleon 1 in the external region. The parameter Θ_g represents the non-degeneracy effect and reduces to the temperature of the ground state of the target nucleus in the weak coupling model, in which the internal region vanishes. The validity of the approximation (118) is quite hypothetical, but is supported by the success in the phenomenological analyse of high energy nucleon scattering.[15)]

Taking the exponential deuteron wave function (58) for brevity, the substitution of (118) in (117) leads to the cross section formula

$$\frac{\partial^2\sigma^{(s)}}{\partial E_d\partial\Omega_d} = 12\sqrt{2}\,\pi R^2\cdot s_{i,f}\cdot P_e\cdot(B_d E_d/E_0\Theta_g^3)^{1/2}\exp(-E_1/\Theta_g), \tag{119}$$

with the abbreviation $s_{i,f}=\gamma_{i,f}^2/D_f(E_f)$ and the incident energy $E_0=\hbar^2 k_0^2/2M$.

From the formula (119), the deuteron energy $E_d^{\max}$ corresponding to the peak of the energy spectrum is approximately

$$E_d^{\max} = [\sqrt{2E_0}\cos\theta_d + \sqrt{2E_0\cos^2\theta_d + 4\Theta_g}]^2/16, \tag{120}$$

neglecting the energy dependence of $s_{i,f}$, and the typical values

$$E_d^{\max} = \begin{cases} [E_0+\Theta_g+\sqrt{E_0(E_0+2\Theta_g)}]/4 & \text{for } \theta_d=0^\circ \\ \Theta_g/4 & \text{for } \theta_d=90^\circ \\ [E_0+\Theta_g-\sqrt{E_0(E_0+2\Theta_g)}]/4 & \text{for } \theta_d=180^\circ \end{cases} \tag{121}$$

are in rough agreement with the experimental observations, in view of that the sharp energy distribution of deuterons corresponding to the ground state or low lying levels of the residual nucleus may be due to the energy dependence of the strength function $s_{i,f}$. In a strong coupling theory, the function $s_{i,f}$ may be approximately represented by the monotonously varying function $s=(\pi KR)^{-1}$. If this expression is tentatively employed, the energy $E_d^{\max}$ should have larger values than those given by (121). However, the function does not appear to be in accord with recent experimental results on the total neutron cross section. The actual strength function should fluctuate according to the details of the level structure and may approach to the strong-coupling value when many levels are concerned intimately. Further it should be worthy to mention that the energy spectrum depends strongly upon both the initial state of the target nucleus and the state of emerging deuteron as well, since $\gamma_{i,f}^2$ is the reduced width for the separation of the ground state of the target nucleus into a nucleon state of a specific energy corresponding to the residual state of the excitation energy E_f. The fluctuation in the energy spectrum may be expected in view of the so-called giant resonance nature of the strength function, according to the intermediate coupling model proposed by Lane, Thomas and Wigner.[18)] In fact the above trend has been found in a recent report[19)] on deuteron pickup reaction initiated by 20 Mev protons bombarding heavy nuclei.

On the other hand, the strongly forward peak angular distribution can be seen, also from (119), as

$$\frac{\partial^2 \sigma^{(s)}}{\partial E_d \partial \Omega_d} \simeq \left(\frac{\partial^2 \sigma^{(s)}}{\partial E_d \partial \Omega_d} \right)_{\theta_d = 0^\circ} \cdot \exp[-(\theta_d^2/\delta_0^2)], \tag{122}$$

with the width defined by

$$\delta_0^2 = \Theta_g / \sqrt{2E_0 E_d}, \tag{123}$$

which characterizes the angular distribution. The peak of the angular distribution becomes flat for small deuteron energies, as it should be expected.

The total cross section of the direct pickup process is obtained by integrating (122) over θ_d and E_d, assuming $s_{i,f}$ to be slowly varying,

$$\sigma_t^{(s)} = \frac{12\pi R^2}{\sqrt{2}} \cdot \left(\frac{B_d}{E_0} \right)^{1/2} \langle s_{i,f} \rangle_{av} \cdot P_e \cdot [1 + E(y)], \quad \text{for } E_0 \gg \Theta_g \tag{124}$$

where $E(y)$ is the error integral defined by

$$E(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt,$$

$y = (\sqrt{2(E_0 - Q)} - \sqrt{E_0}) / \sqrt{\Theta_g}$ and Q is the threshold energy. The above expression is approximated for $y \gg 1$, which is the case at a high incident energy, as

$$\sigma_t^{(s)} = \frac{12\pi R^2}{\sqrt{2}} \cdot \left(\frac{B_d}{E_0} \right)^{1/2} \cdot \langle s_{i,f} \rangle_{av} \cdot P_e = 12.7\,\pi R^2 \cdot \langle s_{i,f} \rangle_{av} \cdot P_e \cdot E_0^{-1/2}, \tag{125}$$

(E_0 in Mev)

which is proportional to the inverse root of the incident energy, in agreement with Heidmann's numerical result. Further it should be noted that the formula (125) does not involve the quantity Θ_g explicitly, though Θ_g is closely related with the probability P_e. The dependence of the cross section on Θ_g will be discussed in the following section.

Summary for the direct pickup process.

(i) The cross section of the process is proportional to the inverse root of the incident energy ($1/v$—*law*).

(ii) The cross section depends weakly upon the shape of the momentum distribution of a nucleon to be picked up and is proportional to the probability of finding the nucleon in the external region.

(iii) The angular distribution of high energy deuterons produced shows a sharp forward peak, which becomes sharper with increasing incident energy and depends upon the shape of the momentum distribution. On the contrary, the low energy deuterons show a flat angular distribution.

(iv) The energy spectrum formula contains the strength function closely related to the nuclear structure of the initial and the final nuclei.

§ 6. High energy pickup reaction — Weak coupling model

From the standpoint of the formalism developed in § 2, the treatment in this section is regarded to be limiting case, in which the contribution from the internal region is perfectly disregarded and the interaction is relatively weak all over the configuration space. Hence we may call this the case of the weak coupling limit. Present method is not quite valid, but is supported by certain grounds that the target nucleus looks like a rather transparent object for high energy nucleons and, moreover, the incident nucleon travels so rapidly within the nucleus that it has a small chance of exciting the core of the target nucleus. With these assumptions one may obtain the cross section formula, replacing $\sum_\alpha |\phi_\alpha|^2$ in (117) by the momentum distridution of nucleon 1 in the whole space. Now if we put

$$g(E_1) = \sum_\alpha \Big| \int_{\text{all space}} u_\alpha(\boldsymbol{r}_1)\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1} d\boldsymbol{r}_1 \Big|^2, \tag{126}$$

the energy spectrum of deuterons is written as

$$\frac{\partial^2\sigma}{\partial E_d \partial \Omega_d} = \frac{3}{4} \cdot \frac{M^3 R^2}{2\pi^2 \hbar^6} \cdot \frac{K_d}{k_0} \cdot \frac{\gamma^2_{i,f}}{D_f(E_f)} \cdot \left[B_d + \frac{\hbar^2}{M}\left(\boldsymbol{k}_0 - \frac{\boldsymbol{K}_d}{2}\right)^2 \right]^2 \cdot G^2\left(\boldsymbol{k}_0 - \frac{\boldsymbol{K}_d}{2}\right) g(E_1), \tag{127}$$

where R is the radius of the nucleus and G is the Fourier transform of the deuteron wave function.

So far as the momentum distribution of nucleons is concerned, the Fermi model is often utilized in calculating binding energies of heavy nuclei or in studying the nuclear optical potential. However, the completely degenerate Fermi distribution inevitably predicts a peculiar angular distribution of reaction products, due to its sharp upper limit, in contradiction to the experimental evidences not only in our process but also in the

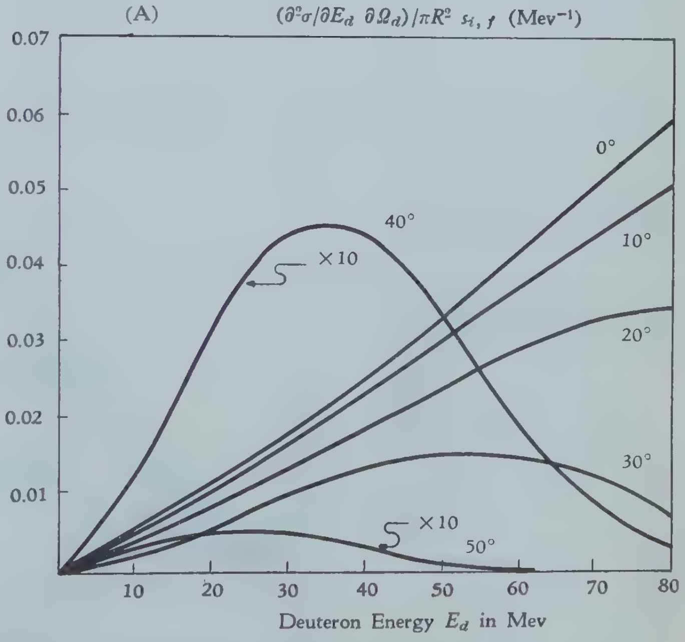

Fig. 2. Energy spectrum, divided by $\pi R^2 s_{i,f}$, of deuterons produced. The energy of the incident nucleon is 90 Mev and the temperature Θ_0 is 8 Mev and the Fermi energy E_F is 35 Mev. Figures attached to respective curves indicate scattered angles.

photo- and nucleon meson production[20)~22)] or in the high energy photoejection of nucleons.[23)] A semi-empirical calculation of the pickup process has been made in the work of Chew and Goldberger in which the wave function of the nucleon to be picked up is taken from the experiments themselves. This method has been found to be very useful in analysing the experiments and obtaining the information on the momentum distribution of nucleons in a complex nucleus. Thus it has been revealed that experimental results can never be interpreted by assuming the nucleus to consist of the degenerate Fermi gas of neutrons and protons, non-interacting but confined to the nuclear volume. The correction to the Fermi model has been made by Heidmann by taking account of the interaction between the nucleon to be picked up and another left in the residual nucleus after the collision.[7)] This procedure has resulted in agreement with experimental observations in a rough way and enabled one to draw some conclusions concerning the energy dependence of the process. Actually the momentum distribution predicted from the experimental results is approximately of the Gaussian type, if we note the leading term of the empirical formula given by Selove.[4)] If we take this distribution tentatively, the cross section is given by putting $P_e=1$ in the formula (119). Accordingly, all the conclusions summarized for the surface pickup process can be fitted to the total cross section of high energy pickup reactions, whence the quantity Θ_g is interpreted as the temperature of the nuclear ground state. In view of the summary (i) in the last section, we may reasonably assume that the total cross section is written as

$$\sigma_t=\pi R^2\langle s_{i,f}\rangle_{av}\mathfrak{A}/\sqrt{E_0}, \tag{128}$$

which is supported by the

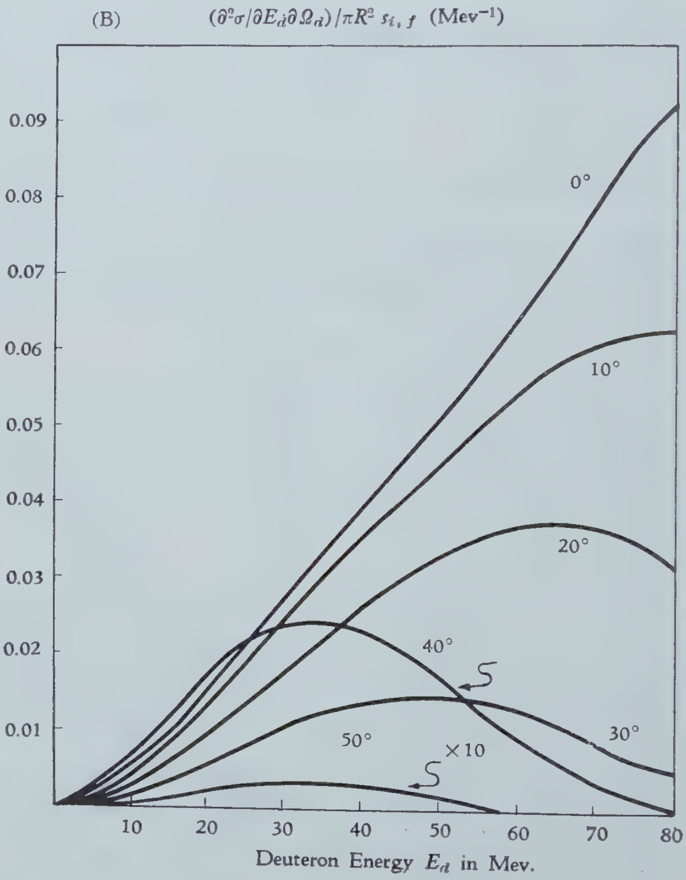

Fig. 3. Energy spectrum, divided by $\pi R^2 s_{i,f}$, of deuterons produced. The energy of the incident nucleon is 90 Mev and the temperature Θ_g is 8 Mev and the Fermi energy E_F is 25 Mev. Figures attached to respective curves indicate scattered angles.

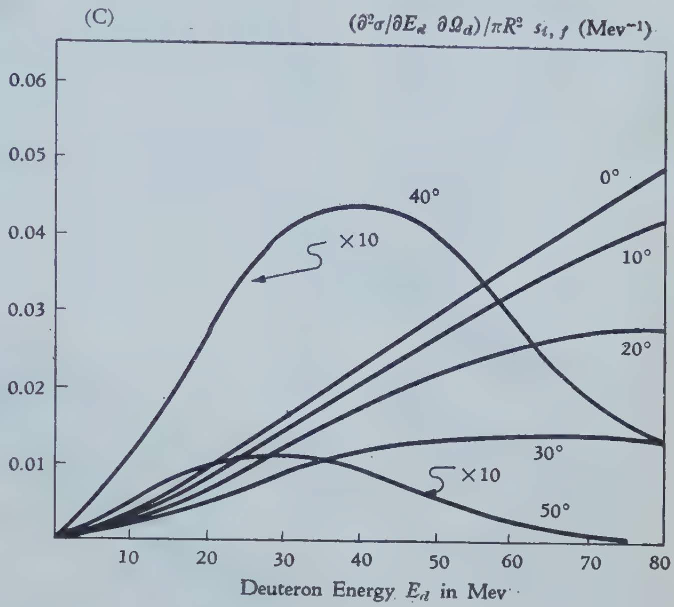

Fig. 4. Energy spectrum, divided by $\pi R^2 s_{i,f}$, of deuterons produced. The energy of the incident nucleon is 90 Mev and the temperature θ_g is 12 Mev and the Fermi energy E_F is 35 Mev. Figures attached to respective curves indicate scattered angles.

numerical result obtained by Heidmann. By the summary (ii), $\mathfrak{A}$ is nearly independent of the magnitude of the temperature θ_g and is approximately equal to 12.7 Mev$^{1/2}$ (E_0 in Mev) for the Gaussian distribution. The latter conclusion does not seem to be entirely unquestionable, because of the crude assumption of the Gaussian distribution and approximate calculations. One may rather consider that the cross section is closely related to the shape of the momentum distribution of the partner to join to form a deuteron. Therefore it seems worth while to examine this point taking another example for the momentum distribution, namely, the Fermi distribution with a finite temperature θ_g. Thus we may put

$$g(E_1)=\frac{6\pi^2}{k_F^3 f_F(\theta_g)[1+\exp(E_1-\mu)/\theta_g]}\equiv\frac{6\pi^2}{k_F^3 f_F(\theta_g)}h_F(E_1), \tag{129}$$

where k_F is the Fermi momentum at zero temperature. The function $f_F(\theta_g)$ is determined by[24]

$$f_F(\theta_g)=1+\frac{1}{8}\left(\frac{\pi\theta_g}{E_F}\right)^2-\frac{19}{1920}\left(\frac{\pi\theta_g}{E_F}\right)^4+\cdots, \tag{130}$$

where E_F is the Fermi energy and μ, θ_g times the chemical potential, is given by

$$\mu=E_F\left[1-\frac{1}{12}\left(\frac{\pi\theta_g}{E_F}\right)^2-\frac{1}{81}\left(\frac{\pi\theta_g}{E_F}\right)^4+\cdots\right] \tag{131}$$

The substitution of (129) into (127) leads to

$$\frac{\partial^2\sigma}{\partial E_d\partial\Omega_d}=\frac{9M^{3/2}R^2}{8\hbar^3E_F^{3/2}f_F(\theta_g)}\cdot\left(\frac{E_d}{E_0}\right)^{1/2}\cdot\frac{\gamma_{i,f}^2}{D_f(E_f)}G^2\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)$$
$$\cdot\left[B_d+\frac{\hbar^2}{M}\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)^2\right]^2\cdot h_F(E_1). \tag{132}$$

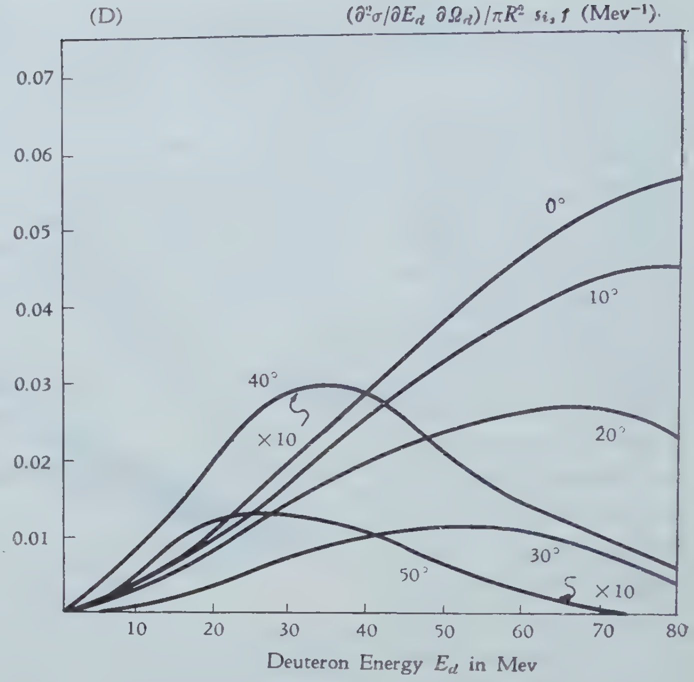

Fig. 5. Energy spectrum, divided by $\pi R^2 s_{i\cdot f}$, of deuterons produced. The energy of the incident nucleon is 90 Mev and the temperature Θ_g is 12 Mev and the Fermi energy E_F is 25 Mev. Figures attached to respective curves indicate scattered angles.

The level structure of the nucleus appears in the above formula through the factor $\pi R^2(\gamma_{i,f}^2/D_f)$, the rest being determined by the values of E_F and Θ_g. These two quantities E_F and Θ_g are considered to be always constant except for small fluctuations according to the structure of the target nucleus. The energy spectra, divided $\pi R^2(\gamma_{i,f}^2/D_f)$, of deuterons produced by 90 Mev incident nucleons, calculated by the formula (132) for different values of E_F and Θ_g, are shown in Fig. 2∼5. The numerical values for respective cases are given in Table 1. Though our choice of the temperature for the ground state of the target nucleus is quite hypothetical, yet these are, within our uncertainities, in agreement with other values obtained by Watanabe[25)] and Heidmann. The calculated curves themselves are not the same as those observed in the experiments, since the function $s_{i,f}(=\gamma_{i,f}^2/D_f)$ involved in the formula depends upon the deuteron energy. In the strong coupling theory, the function $s_{i,f}$ may be approximately represented by a monotonously varying function $(\pi KR)^{-1}$. If this expression is tentatively applied, the energy spectrum increases more rapidly with increasing deuteron energy than those shown in Fig. 2∼5. The total cross section of the process is obtained by the graphical integration, which is also given in table 1. It is known from the table that the simple

Table 1. Total cross section calculated by (132)

case	Θ_g	E_F	μ	$\sigma_t/\pi R^2\langle s_{i,f}\rangle_{av}$	$\mathfrak{A}$
(A)	8 Mev	35 Mev	33.5 Mev	1.57	14.9 Mev$^{1/2}$
(B)	8	25	22.6	1.65	14.7
(C)	12	35	31.0	1.38	13.1
(D)	12	25	18.5	1.29	12.3
formula (128)	Gaussian distribution			1.34	12.7

formula (128) can yield the right order of the total cross section and further the total cross section is weakly dependent upon the magnitudes of the temperature and the Fermi energy. For increasing the value of Θ_g, the total cross section decreases slowly as can be expected by (124). On the contrary the energy spectrum and the angular distribution, as shown in Fig. 6, are strongly dependent upon the choice of the temperature and the Fermi energy. The dependence of the angular distribution on the choice of these quantities will be discussed in § 8. Further in § 8, we shall find that the formula (128) is of much use in explaining the experiment on the pickup reaction by high energy protons bombarding Carbon nuclei.

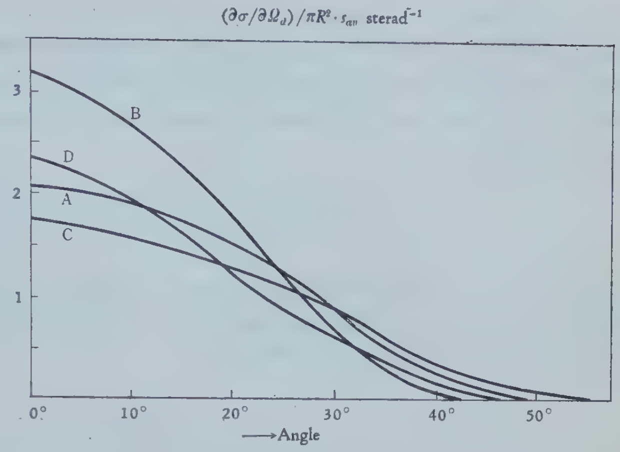

Fig. 6. Angular distribution of deuterons, divided by $\pi R^2 \langle s_{i,f}\rangle_{av}$, integrated over the deuteron energy. Figures attached to respective curves indicate the cases referred in the text (Table 1).

Summary for high energy pickup reactions

Through above considerations, we have found some knowledge on the overall behaviour of the deuteron pickup reaction, which can be summarized as follows :

(i) The total cross section is approximately proportional to the inverse root of the incident energy.

(ii) The total cross section is proportional to $R^2\langle s_{i,f}\rangle$ and then roughly proportional to the radius of the target nucleus, since the quantity $\langle s_{i,f}\rangle_{av}$ behaves like R^{-1}.

(iii) The total cross section is weakly dependent upon the Fermi energy and the temperature of ground state of the target nucleus.

§ 7. The production of deuterons into ^1s state

In this section, we shall give a treatment for the production of deuteron into ^1s state, which should be actually observed as a $(p;\ p,\ n)$ type reaction in the experimental detection. The present method of evaluation is quite analogous to that in the last section, based upon the weak coupling assumption. Hence one obtains the cross section for this process

$$\frac{\partial^2\sigma(^1s)}{\partial E_d\partial\Omega_d}=\frac{1}{4}\cdot\frac{M^2}{2\pi^2\hbar^4}\cdot\frac{K_d}{k_0}\cdot\frac{1}{D_f(E_f)}\left|T(^1s)\right|^2$$
$$=\frac{3M^3R^2}{4\hbar^6}\cdot\frac{K_d}{k_0k_F{}^3f_F(\Theta_g)}\cdot s_{i.f}\left|G^{(s)}\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)\right|^2\cdot\left[-\varepsilon_d+\frac{\hbar^2}{M}\left(\boldsymbol{k}_0-\frac{\boldsymbol{K}_d}{2}\right)^2\right]^2h_F(E_1). \tag{133}$$

Here the factor 1/4 represents *a priori* probability for the neutron and proton to have antiparallel spins and $G^{(s)}$ is the Fourier transform of the deuteron wave function for the singlet s-state of small positive energy $\mathcal{E}_d=\hbar^2P^2/M$. The spatial part of the wave function for our problem may be written approximately, following Hulthen[26] but omitting the second term, as

$$\phi_d^{(s)}(\rho)=\frac{1}{\sqrt{\mathfrak{V}_d(a^2+P^2)}}\cdot\frac{\sin(P\rho+\delta)}{\rho\sin\delta}, \tag{134}$$

where $\mathfrak{V}_d^{-1/2}$ is the normalization factor, in which $\mathfrak{V}_d$ is equal to the volume of the deuteron, a^{-1} is the scattering length and δ the phase shift. Through elementary calculations one gets the Fourier transform

$$G^{(s)}(q)=\frac{4\pi}{\sqrt{\mathfrak{V}_d(a^2+P^2)}}\cdot\frac{1}{P^2-q^2}, \tag{135}$$

which fortunately does not contain the phase shift δ. The substitution of (135) in (134) leads to the differential cross section

$$\frac{\partial^2\sigma(^1s)}{\partial E_d\partial\Omega_d}=\frac{6\pi^2R^2}{\mathfrak{V}_d}\cdot\left(\frac{\hbar^2}{ME_F}\right)^{3/2}\cdot\frac{s_{i,f}}{f_F(\theta_g)}\cdot\left(\frac{E_d}{E_0}\right)^{1/2}\cdot\frac{1}{\mathcal{E}_a+\mathcal{E}_p}\cdot{}^F h(E_1) \tag{136}$$

with

$$\mathcal{E}_a=\hbar^2a^2/M=1.43 \text{ Mev}.$$

Since $\mathcal{E}_d$ is considered to be small positive energy, it should be reasonable that the neutron proton pair is mainly produced into the forward direction, because the formula (136) shows a sharp forward peak angular distribution. Again employing the Gaussian approximation, integrations of (136) over the angle and the energy give rise to the total cross section

$$\sigma_t(^1s)=\frac{4\pi^2R^2}{\mathfrak{V}_d}\cdot\frac{\langle s_{i,f}\rangle_{av}}{k_0}\cdot\frac{1}{a^2+P^2}. \tag{137}$$

As can been seen easily from (137), the total cross section obeys again $1/v-$ law and is approximately proportional to the nuclear radius R; both are the same as in the deuteron pickup reaction treated in the preceding section. It is of interest to compare the cross sections in two cases: one is for the production of deuteron and the other for that of the neutron-proton pair in ^1s state. Using the cross section formula (125) with $P_e=1$, one finds the ratio

$$\frac{\sigma_t(^1s)}{\sigma_t(G)}=\frac{\sigma(0\,;p,n)}{\sigma_t(0\,;d)}=\frac{1}{4\alpha R_d^3}\cdot\frac{1}{a^2+P^2}$$

$$\simeq\frac{100}{R_d^3}\cdot\frac{1}{3.18+2.23\,\mathcal{E}_d},\ \begin{pmatrix}\mathcal{E}_d \text{ in Mev}\\ R_d \text{ in } 10^{-13}\text{ cm}\end{pmatrix} \tag{138}$$

where the radius of deuteron, R_d, is introduced, in replacing $\mathfrak{V}_d$ by $(4\pi/3)R_d^3$. The above ratio for various values of R_d and $\mathcal{E}_d$ is given in Fig. 7. From this figure, the $(p:pn)$ type cross section due to the production of deuterons into virtual state is unexpectedly large and this process cannot be disregarded in analysing the nuclear reaction data.

Summary for the production of deuteron into ¹s-state.

The production of the neutron-proton pair through the excited state of deuteron plays an important role in high energy nuclear reactions and gives a cross section comparable to that of the ordinary pickup reaction.

§ 8. Summary and applications to the analysis of experimental results

Quantitative cross section measurements for deuteron pickup reactions have been made up to the present time with only a limited number of elements, mostly for light nuclei, for which the statistical model employed here may not be of good validity. In this section we summarize main conclusions obtained in this work and compare our theory with these measurements. Firstly let us begin with the overall behaviour of the total cross section.

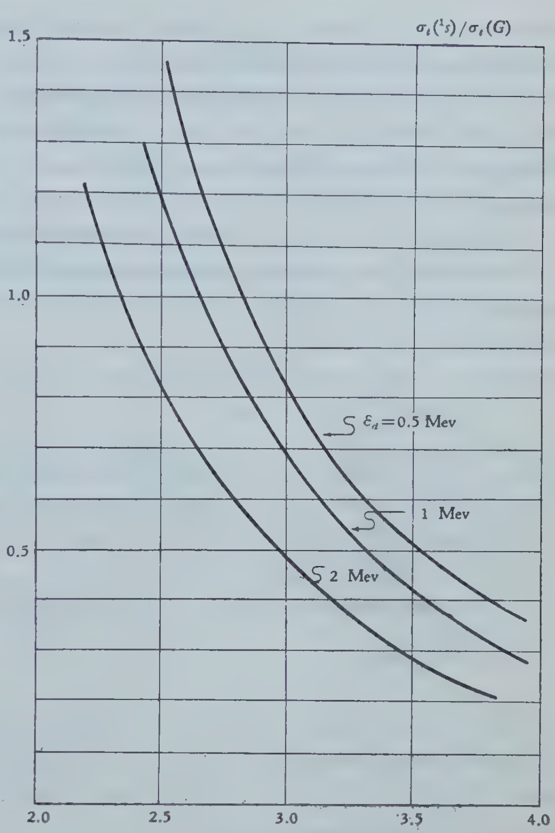

Fig. 7. Ratio of the cross sections for the production of deuteron and for that of neutron-proton pair through ${}^1s$ state of deuteron, *vs.* the radius of deuteron R_d. Figures attached to respective curves indicate the value of the deuteron energy ε_d in the singlet *s*-state.

8.1 Total cross section

The cross section of the process can be written, neglecting the interference term, in the formula conventionally employed as

$$\sigma_t(0\,;\,d)=\sigma^{(s)}(0\,;\,d)+\sigma^{(c)}(0:d), \tag{139}$$

where 0 stands for the incident particle. $\sigma^{(s)}$ is the cross section due to the surface direct process and $\sigma^{(c)}$ due to the compound nucleus process. From the standpoint of our formalism, these two processes can be discriminated in the presence of the surface S, which is located at the channel radius R_0 and divides the configuration space into two subspaces. In the outer region from this surface S, the interaction between the incident particle and the target nucleus is relatively weak and the direct process takes place. On the contrary, the inner region, where the interaction becomes strong, represents the configuration space of the compound nucleus.

For low incident energies, since the production of deuterons through the conventional compound nucleus process may be of negligible magnitude, the direct process dominates the main feature of the reaction phenomena. Therefore, for this case, the Coulomb penetrability determines the overall behaviour of the excitation curve, which is expressed as a function of the excess energy over the threshold as shown by the formula (110).

For moderate and high incident energies, the cross section $\sigma^{(s)}$ is proportional to the probability, P_e, of finding the collision partner 1 in the external region and given by the approximate formula (125) as

$$\sigma^{(s)}(0\,;\,d)=12.7\pi R^2\langle s_{i,f}\rangle_{av}\cdot P_e/\sqrt{E_0} \qquad (125)$$

$$(E_0 \text{ in Mev}).$$

Here R is the radius of the target nucleus and E_0 is the incident energy. $\langle s_{i,f}\rangle_{av}$ is the average value of the strength function accociated with the initial and the final states and roughly equal to $(\pi KR)^{-1}$. Then this quantity $\langle s_{i,f}\rangle_{av}$ is considered to vary from 0.07 to 0.03 for medium weight and heavy nuclei.

The cross section $\sigma^{(c)}$ due to the compound nucleus process is given by (44) in the following form

$$\sigma^{(c)}(0\,;\,d)=\sigma_C^{(0)}\cdot\Gamma_d/\Gamma_C, \qquad (140)$$

where $\sigma_C^{(0)}$ is the cross section for the formation of a compound nucleus by the incident nucleon 0. Γ_C is the total width of the compound state and Γ_d is the deuteron width, which is generally a sum of the deuteron width due to the indirect pickup process and that of the usual evaporation theory. By the calculation in § 3, it has been concluded that the deuteron width due to the indirect pickup process is much larger than that due to the deuteron evaporation of usual sense. Hence, neglecting the contribution from the latter the deuteron width Γ_d is estimated by the formula (66)

$$\Gamma_d\simeq\frac{\Gamma_0\Gamma_d^{(0)}}{D_C}\cdot\frac{1}{\sqrt{\theta_C}}\cdot\left[\log\frac{3B_d+20\,\theta_C}{3B_d}\right]^2 \quad (\theta_C \text{ in Mev}). \qquad (66)$$

In this formula a factor which approximately equals to unity is omitted. D_C and θ_C is the level distance and the temperature of the compound nucleus respectively ; Γ_0 is the neutron width and $\Gamma_d^{(0)}$ is the deuteron width given by the usual evaporation theory. From the above formula one can easily see that the value of $\Gamma_d/\Gamma_d^{(0)}$ rises much larger than unity. As can be seen from the derivation of the formula (66) in § 3, it is not adequate to take distance of all energetically possible levels as D_C. It is because the number of levels concerned in the process is about 10^{-3} times smaller than that of all energetically possible levels, owing to the selection rule due to the conservation of spin and isotopic spin.[32] Taking account of this correction and using the level density parameter given in the text book of Blatt and Weisskopf, the ratio $\Gamma_d/\Gamma_C\approx\Gamma_d/\Gamma_0$ is estimated as varying from 0.1 to 0.01 for medium weight and heavy nuclei with 20 Mev excitation energies.

In order to calculate the cross section of the process by the formula (133), it is of primary importance to determine the magnitudes of $\sigma_C^{(0)}$ and P_e. Now if we regard both quantities, $\sigma_C^{(0)}$ and P_e, as functions of the channel radius R_0, one can choose the most probable value of R_0 in order to reproduce the experimental cross section with the above formula. While the nuclear radius R is fixed in the course of changing the magnitude of R_0. Therefore P_e diminishes for a larger value of R_0 than R, whereas $\sigma_C^{(0)}$ increases; and P_e approaches to unity for small R_0, whence $\sigma_C^{(0)}$ becomes smaller. Since the mean free path for an incident nucleon has a minimum near $E_0 \sim 15$ Mev and the interaction appears relatively strong, the probability P_e may have a minimum and the compound formation cross section $\sigma_C^{(0)}$ becomes maximum.

To obtain estimates of these quantities we make an assumption that an incident nucleon, after suffering n collisions in the target nucleus, loses most of its kinetic energy and forms a compound nucleus. With this assumption, the total cross section of the direct process is estimated as $(n^2/4)\pi\bar{\lambda}^2$ in terms of the mean free path $\bar{\lambda}$. Further the total reaction cross section may be assumed to be approximately πR^2 for a wide range of incident energy, beside small fluctuations. Hence we get the following estimates, for $R > (n\bar{\lambda}/2)$ and $R > R_0$,

$$\sigma_C^{(0)} \equiv \pi R_0^2 = \pi\{R^2 - (n^2/4)\bar{\lambda}^2\} \tag{141}$$

and

$$P_e = 1 - (R_0/R)\{1 + (n^2\bar{\lambda}^2/8R^2)\}. \tag{142}$$

As the interaction becomes weaker and the mean free path becomes longer, the probability P_e approaches to unity and $\sigma_C^{(0)}$ vanishes, as they should be. The mean free path equals

Table 2. Yield of deuterons

Incident energy		Medium Weight nuclei	Heavy nuclei
	Total reaction cross section	1400 mb	2400 mb
	$\langle s_i, f\rangle_{av}$	0.07	0.03
	Γ_d/Γ_C	10^{-1}	10^{-2}
$E_0 = 20$ Mev	$(n^2/4)\pi\bar{\lambda}^2$	400 mb	400 mb
$n = 2$	$\sigma_C^{(0)}$	1000 mb	2000 mb
	P_e	0.1	0.03
	$\sigma^{(c)}(0; d)$	100 mb	20 mb
	$\sigma^{(s)}(0; d)$	30 mb	10 mb
	$\sigma_t(0; d)$	130 mb	30 mb
$E_0 = 100$ Mev	$(n^2/4)\pi\bar{\lambda}^2$	—	1800 mb
$n = 3$	$\sigma_C^{(0)}$	negligible	600 mb
	P_e	1	0.4
	$\sigma^{(c)}(0; d)$	negligible	10 mb
	$\sigma^{(s)}(0; d)$	120 mb	40 mb
	$\sigma_t(0; d)$	120 mb	50 mb

to nearly 3.5×10^{-13} cm and then increases gradually up to 5×10^{-13} cm as the incident energy swells from 15 Mev to 100 Mev.[33] For moderate incident energies, assuming $n=2$, the total cross section of the direct process is approximately 400 mb and P_e decreases from 0.1 to 0.03 for medium weight and heavy nuclei. In nuclear reactions involving the incident energy of 100 Mev and up, we should put $n=3$, which predicts $P_e\simeq1$ and $R_0\simeq0$. This is the case of the weak coupling limit investigated in § 6.

Now one can obtain a crude estimate for the yield of deuterons by nucleons bombarding complex nuclei. The results obtained and the numerical values of parameters are shown in Table 2. However it should be remembered that *this table is intended only for illustrative purpose and quantitatively unreliable because of inaccurate choices for the numerical constants contained in the formula.*

One can see from the table that the total cross section $\sigma_t(0\,;d)$ does not vary so rapidly but shows a plateau in the above energy region. This plateau is partly due to the second term in the bracket of the formula (124), which causes a deviation from $1/v$-law for incident energies not much larger than the reaction threshold Q. Nevertheless the cross section shows a rapid decrease obeying $1/v$-law for incident energies above 100 Mev. In the reactions at light nuclei the $1/v$-decrease may appear for incident energies below 100 Mev, since the nuclear radius is comparable with the mean free path. The gross behaviour of the cross section is shown in Fig. 10 for the illustrative purpose.

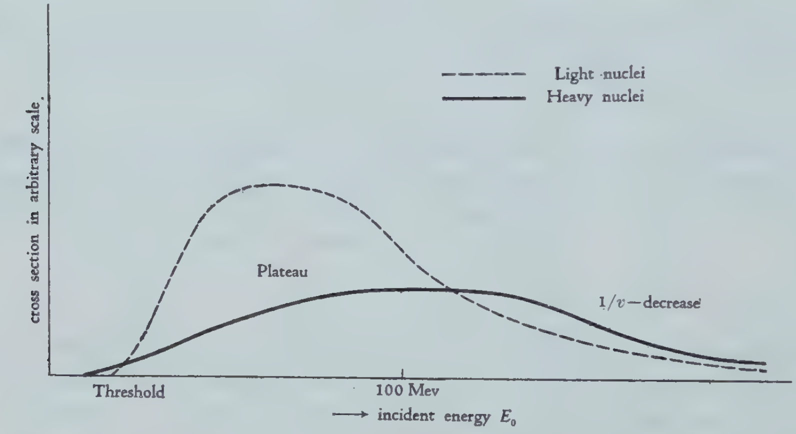

Fig. 10. Schematic representation for the behaviour of the cross section.

8·II Energy spectrum of deuterons due to the indirect pickup process.

The energy spectrum of deuterons due to the direct process is already discussed in § 6. Then here we concentrate our efforts to study the spectrum of deuterons produced through the evaporation process. The energy spectrum of deuteron indirectly picked up is expressed by the integrand of (65) and is written, more generally, as

$$\partial\sigma^{(e)}/\partial E_d \propto E_d^a f(E_d)\exp(-bE_d/\Theta_D), \qquad (143)$$

with the slowly-varying function $f(E_d)$ of the deuteron energy E_d and two parameters a and b. If we put $a=b=f(E_d)=1$, the formula (143) yields the energy spectrum predicted from the conventional evaporation theory. However, as shown in § 3, we obtain the crude estimates

$$\begin{aligned} &a=5/2, \\ &b=3/2, \\ &f(E_d)=\log[(B_d+4E_d)/B_d]^2. \end{aligned} \tag{144}$$

Therefore the energy spectrum of deuterons emitted through the indirect pickup process does never identify with that due to the ordinary deuteron channels, the latter showing approximately the Maxwellian distribution multiplied by the barrier penetrability. As expected from the E_d^a dependence of the energy spectrum, the average kinetic energy of deuterons indirectly picked up is three times larger than that expected by the usual evaporation theory.

Since the choice (144) is not quantitatively reliable, an alternative proposed is to determine these three quantities semi-empirically from the experimental data themselves. Though the Berkeley cyclotron experiment, in which reaction products are measured at large angles, is available for this purpose, it is necessary to calculate the total number of deuterons produced in several steps of evaporation, because of its high excitation energy. This situation would make the determination of these parameters very difficult. Therefore it is desirable to make an experiment of deuteron production in the moderate energy region, by which the above predictions will be tested.

8· III The angular distribution of the sharp energy group in the $^{12}C(p; d)$ reaction.

The calculation on the angular distribution is done by (132) for the sharp energy group in the $^{12}C(p; d)$ reaction, which is shown in Fig. 8. The nuclear radius of Carbon nucleus is taken as 3.3×10^{-13} cm and the quantity $s_{i,f}$ for the ground state of the residual nucleus is taken as 0.3, in accordance with the analysis of the reduced width from the $(d; p)$ reaction.[31] The completely degenerate Fermi distribution predicts a peculiar angular distribution with its sharp upper limit, which contradicts with the experimental angular distribution. In our treatment this difficulty is partly removed by introducing the non-degenerate Fermi model. However, since the momentum distribution function h_F defined by (129) is about equal to unity for smaller values of E_1 than the critical energy μ given by (131), the angular distribution calculated decreases more slowly than the experimental one with increasing angle in the extreme forward direction. Moreover, the calculated curve indicates a more rapid descent than the observed one, when the energy E_1 excesses over μ. This trend in the angular distribution becomes less remarkable, as the effect of non-degeneracy in the target nucleus becomes more appreciable, that is, the magnitude of the temperature becomes larger. On the other hand, the value of μ is closely related with the choice of the nuclear radius, since the Fermi energy, being

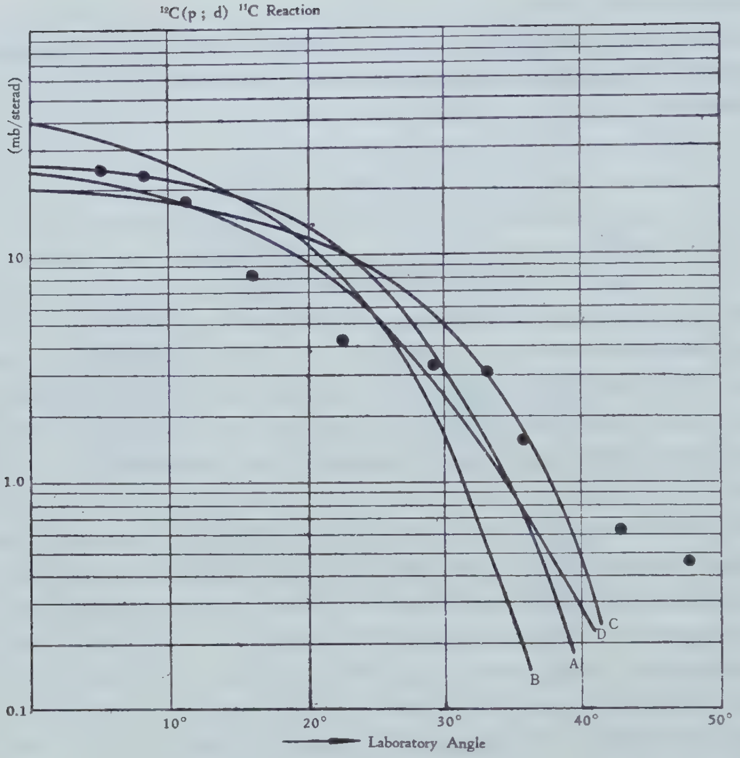

Fig. 8. Angular distribution of the sharp energy group in the $^{12}C(p;\, d)^{11}C$ reaction, calculated for four cases of parameters. Circles indicate experimental points.

given by $E_F = (\hbar^2/2M)\,(3\pi^2 N/V)^{2/3} = (54/r_0^2)$ Mev (r_0 in 10^{-13} cm), is proportional to the inverse square of the nuclear radius. The choice of the small nuclear radii $r_0 A^{1/3}$, which corresponds to the large value of μ, results in broadening the forward peak. A recent measurement[6] on the $(p;\, d)$ reaction initiated by 300 Mev protons has shown a flat angular distribution of deuterons in the backward directions, which may be due to complicated processes such as the indirect pickup processes treated in § 3.

8·IV ^{11}C activation cross sections by protons bombarding Carbon nuclei.

The reaction $^{12}C(p;\, d \text{ or } np)^{11}C$ has been studied by several investigators because of its use as a monitor for proton beam at energies above the 19.5 Mev threshold. Fig. 9 gives a compilation of experimental data on ^{11}C activation cross section with high energy protons. It should be noted that the cross section approximately obeys the $1/v$-law as expected from the theoretical prediction in the preceding paragraphs. Using the following numerical values

$$R = 3.3 \times 10^{-13}\ \text{cm}, \quad \mathcal{E}_d = 1\ \text{Mev}$$

$$\langle s_{i,f} \rangle_{av} = 0.1, \qquad R_d = 3.3 \times 10^{-13}\ \text{cm}$$

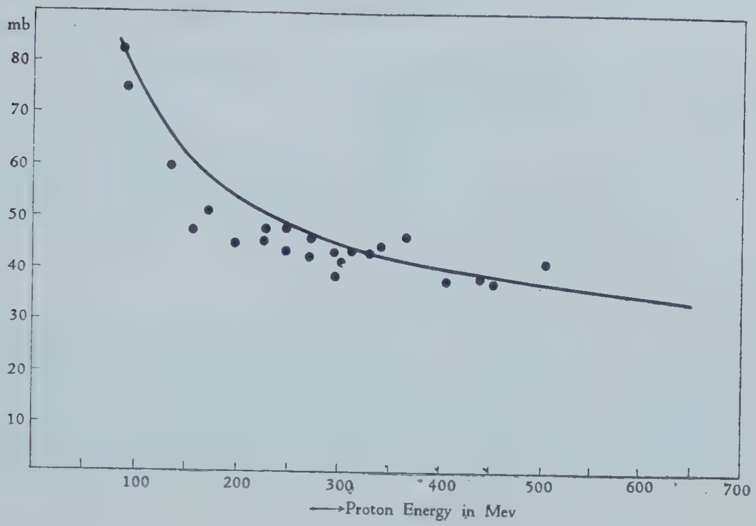

Fig. 9. ^{11}C activation cross section by protons bombarding carbon nuclei. Circles indicate the experimental data and the solid curve indicates the calculated one. The compilation of experimental data is taken from reference 27).

we obtain the ^{11}C activation cross section σ_{ac}, by use of (125) and (138), as

$$\sigma_{ac} \simeq \sigma_t(G) + \sigma_t(^1s)$$

$$= \frac{431}{\sqrt{E_0}} + \frac{234}{\sqrt{E_0}} = \frac{665}{\sqrt{E_0}} \text{ mb} \quad (E_0 \text{ in Mev}).$$

The calculated cross section is shown in Fig. 9, in fair agreement with the experimental observations, though our choice of parameters is rather hypothetical. However the agreement at incident energies above 300 Mev is rather accidental, since the effect of meson production becomes important in this energy region.

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Band Theory of Superexchange Interaction

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A spin dependent energy of NiO crystal is given. One-electron orbits of the Bloch type were used as linear combinations of 2p- and 3d-orbits of O^{--} and Ni^{++} ions. For the antiferromagnetic state the ordered antiparallel spin arrangement was assumed and then two alternant bands corresponding to α and β spin Ni^{++} ions are mixed. Energy difference between parallel and antiparallel spin state is formulated. Rough numerical estimation, using s-orbits instead of 3d-orbits, shows that the mixing of two alternant bands plays an important role in the superexchange interaction.

§ 1. Introduction

The theory of superexchange interaction was initiated by P. W. Anderson[1] using a method of spin operator. Since then many authors have investigated the properties of antiferromagnets using the spin operator method with considerable successes. It may be said that the method of spin operator gives results which agree well with experiments at least phenomenologically. However, when we attempt to extend Anderson's theory to the real crystal containing enormous number of ions, the non-orthogonality of one-electron orbits of an atomic type makes the method divergent and unsuitable for numerical calculation, as J. C. Slater[2] pointed out.

The problem of superexchange interaction may be to find spin dependent energy of actual crystals. We take a NiO crystal as an example of antiferromagnets. The energy of the totally ionic state of NiO in which each ion is doubly ionized is independent of spin orientations of Ni^{++} ions, if the overlaps between adjacent Ni^{++} ions are neglected. P. W. Anderson has given a spin dependent energy by mixing with the ground state a covalent state in which 2p electrons of O^{--} ions transfer into adjacent metal ions. J. Yamashita and the author have given[3] a spin dependent energy of MnO crystal assuming that the non-orthogonality of atomic orbitals is small so that we can use the same procedure as P. O. Loewdin[4] used in treating the cohesion of alkaline halide crystals. P. O. Loewdin's treatment is applicable only for the system which is described by a single Slater determinant. Our problem was to find one-electron orbits which minimize the energy of the state which is represented by the determinant constructed from the one-electron orbits. It can be shown that the Slater determinant which is constructed from one-electron orbits which satisfy Hartree-Fock's one-electron equation minimizes the total energy of the system. When the system is a crystal and has a translational symmetry, there exists a solution of the one-electron equation which is of the Bloch type and is spread over the entire crystal.

One-electron orbits obtained as solutions of the Hartree-Fock equation are mutually orthogonal and the difficulty mentioned above does not occur. It may be said, however, that the simple band theory, when applied to NiO, gives a very high energy because there appear nickel ions of various degree of ionization, thus losing the Coulomb energy, whereas the bonding effect is small because the width of 3d-bands of nickel may be small. This is not our solution. P. O. Loewdin[5)] has shown that if the alternant bands are used, the formation of excessive ions of the high degree of ionization is prevented. We approach from this side and describe the state as follows: three d-states of Ni^{++} ions (of $d\varepsilon$-type) as well as 2p-orbits of O^{--} ions are all filled by α and β spin electrons and as to $d\gamma$-type orbits, one alternant band corresponding to Ni^{++} ions with α spin is filled up by α spin electrons and β spin electrons fill up the other alternant band corresponding to Ni^{++} ions with β spin. Then these two alternant bands are mixed together so that α spin electrons have a probability on $d\gamma$-orbits of Ni^{++} ions with β spin and β spin electrons on those of Ni^{++} ions with α spin, these probabilities being determined to make the total energy minimal.

However if all the $3d\gamma$-electrons of the NiO crystal have one spin direction, the formation of excessive ions is also prevented and the parallel spin state may be as good as the ordered antiparallel spin state described above. The purpose of this paper is to find the difference between the energies of these two states and to give the superexchange interaction in the band picture.

§ 2. Energy of the antiparallel spin state

In the following calculation we neglect all overlaps except those between the d-orbits and the 2p-orbit of an adjacent O^{--} ion which has axial symmetry about the axis connecting the Ni^{++} and O^{--} ions. The unit cell of NiO is shown in Fig. 1. One cell contains two Ni^{++} ions each having α and β spin and two O^{--} ions. We arbitrarily specify one of these two O^{--} ions as corresponding to the Ni^{++} ion with α spin ("A" in Fig. 1) and the other to the ion with β spin ("B" in Fig. 1). We denote with $v_i(\boldsymbol{r}-\boldsymbol{R})$ $(i=1, 2, \cdots, 5)$ five d-orbits of Ni^{++} ion with position specified by radius vector $\boldsymbol{R}$. (v_1, v_2) and (v_3, v_4, v_5) are $d\gamma$- and $d\varepsilon$-type orbits respectively. We denote with $u_t(\boldsymbol{r}-\boldsymbol{R})$ $(t=1, 2, 3)$ 2p-orbits of O^{--} ion with a position vector $\boldsymbol{R}$. Then we construct Bloch sums of these atomic orbitals,

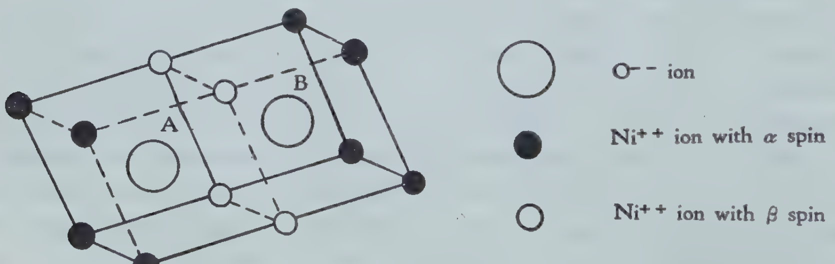

A
B
O^{--} ion
Ni^{++} ion with α spin
Ni^{++} ion with β spin

Fig. 1 Unit cell of NiO

$$V_{ik}(\boldsymbol{r})=1/\sqrt{N}\cdot\sum_{\boldsymbol{R}}{}^{-}\exp(i\boldsymbol{kR})\,v_{i'}(\boldsymbol{r}-\boldsymbol{R}),\quad (i=1,\cdots,5\,;\,i'=1,\cdots,5) \tag{1}$$

$$V_{ik}(\boldsymbol{r})=1/\sqrt{N}\cdot\sum_{\boldsymbol{R}}{}^{+}\exp(i\boldsymbol{kR})\,v_{i'}(\boldsymbol{r}-\boldsymbol{R}),\quad (i=6,\cdots,10\,;\,i'=1,\cdots,5) \tag{2}$$

$$U_{tk}(\boldsymbol{r})=1/\sqrt{N}\cdot\sum_{\boldsymbol{R}}{}^{\mathrm{A}}\exp(i\boldsymbol{kR})\,u_{t'}(\boldsymbol{r}-\boldsymbol{R}),\quad (t=1,2,3\,;\,t'=1,2,3) \tag{3}$$

$$U_{tk}(\boldsymbol{r})=1/\sqrt{N}\cdot\sum_{\boldsymbol{R}}{}^{\mathrm{B}}\exp(i\boldsymbol{kR})\,u_{t'}(\boldsymbol{r}-\boldsymbol{R}),\quad (t=4,5,6\,;\,t'=1,2,3) \tag{4}$$

where $\sum_{R}^{\mp}$ in eq. (1) or (2) means a summation only over the Ni^{++} ions with β or α spin and $\sum_{R}^{\mathrm{A,B}}$ in eq. (3) or (4) a summation over only half of the whole O^{--} ions which are specified by A or B. N is the number of unit cells in the crystal.

If $V_{ik}(i=3, 4, 5, 8, 9, 10)$ and $U_{tk}(t=1, 2, 3, 4, 5, 6)$ are filled by α and β electrons and moreover α electrons fill up $V_{ik}(i=6, 7)$ and β electrons $V_{ik}(i=1, 2)$, then the state is totally ionic and spins are alternately ordered. There are $14N$ electrons with α spin whereas $16N$ orbitals are available. Our problem is to require $14N$ orbitals to associate with $14N$ α spin electrons and other $14N$ orbits to associate with $14N$ β spin electrons as linear combinations of $16N$ orbitals V_{ik} and U_{tk} so as to make the energy of the system minimal. However, if in addition there are $2N$ electrons with α spin, then available orbitals are all filled with α spin electrons. Similarly more $2N$ electrons with β spin will fill up all the orbits of 2p- and 3d-type of O^{--} and Ni^{++} ions, the wave function of this system being uniquely described by a single Slater determinant. We may as well require the $2N$ missing orbits for α electrons, which we denote with ψ_{1k} and ψ_{2k}, as linear combinations of V_{ik} and U_{tk}. We primarily consider the case of α spin electrons. The following argument applies also to β spin electrons. The total energy is written in terms of the coefficients of V_{ik}, U_{tk} in the expressions of ψ_{1k}, ψ_{2k}. In this case the number of coefficients is less than when we require $14N$ orbits and the calculation will be much easier. We assume ψ_{1k} and ψ_{2k} can be expressed in terms of V_{ik} and U_{tk} as follows:

$$\psi_{1k}=(1+a_k)V_{1k}+\sum_{t=1}^{6}b_{tk}U_{tk}+\sum_{i=6}^{10}c_{ik}V_{ik}, \tag{5}$$

$$\psi_{2k}=(1+A_k)V_{2k}+\sum_{t=1}^{6}B_{tk}U_{tk}+\sum_{i=6}^{10}C_{ik}V_{ik}. \tag{6}$$

The mixing of $V_{2k}, \cdots, V_{5k}$ into ψ_{1k} and of V_{1k}, V_{3k}, V_{4k}, V_{5k} into ψ_{2k} may be small as we neglected the direct overlaps between Ni^{++} ions.

Our total wave function is such as follows: we fill up $16N$ orbitals with α electrons assuming as if there were $2N$ more electrons and set up a Slater determinant with $16N$ electrons. We can rearrange without changing the wave function the $16N$ orbits so as to make $2N$ orbitals equal to ψ_{1k} and ψ_{2k}, then we drop these $2N$ orbits out of the determinant and set up a new Slater determinant involving only $14N$ electrons. It is hard to write down this Slater determinant. However, the density matrix corresponding to this determinant is that of the state of $16N$ electrons minus that of ψ_{1k} and ψ_{2k}. It is easier to handle with this density matrix than with the wave function.

As we shall see later (eqs. (28), (29)), b_{tk} and B_{tk} are expressed in terms of integrals whose integrands are products of U_{tk} and V_{ik}. We assume, like Loewdin, that the overlap between adjacent Ni^{++} and O^{--} ions is small. So we regard b_{tk} and B_{tk} as well as overlap integral between V_{ik} and U_{tk}

$$S_{it}=\int \bar{V}_{ik}(1)\,U_{tk}(1)\,dv_1 \tag{7}$$

as small quantities of first order. Similarly c_{ik} and C_{ik} are regarded as small quantities of the second order, as we shall see later in eqs. (30), (31). a_k is expressed in terms of b_{tk} and c_{ik} from normalization condition of ψ_{1k}:

$$\begin{aligned} a_k+\bar{a}_k+|a_k|^2=&-\sum_t(b_{tk}S_{1t}+\bar{b}_{tk}S_{t1})-\sum_t|b_{tk}|^2-\sum_{i=6}^{10}|c_{ik}|^2\\ &-\sum_{t=1}^{6}\sum_{i=6}^{10}(\bar{b}_{tk}c_{ik}S_{ti}+b_{tk}\bar{c}_{ik}S_{it})-\sum_{t=1}^{6}(\bar{a}_kb_{tk}S_{1t}+a_k\bar{b}_{tk}S_{t1}). \end{aligned} \tag{8}$$

A similar equation holds for A_k. We develop the density matrix of ψ_{1k} and ψ_{2k} in powers of a, b, c, A, B and C and retain up to the fourth order terms. For instance from eq. (5), using eq. (8),

$$\begin{aligned} \bar{\psi}_{1k}(2)\,\psi_{1k}(1)=&\bar{V}_1V_1+\sum_t\{b_t(\bar{V}_1U_t-S_{1t}\bar{V}_1V_1)+\text{comp.conj.}\}+\sum_t|b_t|^2(\bar{U}_tU_t-\bar{V}_1V_1)\\ &+\sum_{i=6}^{10}[c_i\{\bar{V}_1V_i+\sum_{t=1}^{6}\bar{b}_t(\bar{U}_tV_i-S_{ti}\bar{V}_1V_1)\}+\text{comp.conj.}]+\sum_{i=6}^{10}|c_i|^2(\bar{V}_iV_i-\bar{V}_1V_1)\\ &+\sum_{t=1}^{6}\{\bar{a}b_t(\bar{V}_1U_t-S_{1t}\bar{V}_1V_1)+\text{comp.conj.}\}+\sum_{t\neq t'}\bar{b}_tb_{t'}\bar{U}_tU_{t'}. \end{aligned} \tag{9}$$

In the above equation, $\bar{V}_1U_t$, for instance, is the abbreviation of $\bar{V}_{1k}(2)\,U_{tk}(1)$ and the dependence of b_{tk}, V_{tk}, etc., on $\boldsymbol{k}$ is not explicitly written. $\bar{V}_iU_t$ is regarded to be of the first order, while $\bar{V}_iV_{i'}$ $(i=1,\cdots,5\,;\,i'=6,\cdots,10)$ and $\bar{U}_tU_{t'}\,(t\neq t')$ are regarded to be of the second order. Then the first term of the right-hand side of eq. (9) is of the zeroth order, the second and the third term are of the second order and the rest is of the fourth order. As mentioned above the α spin part of the density matrix of our total wave function is the density matrix of $16N$ electrons, which we denote with $M_T(2,1)$, minus those of ψ_{1k} and ψ_{2k}. The latter is $\bar{\varphi}_{1k}\varphi_{1k}+\bar{\varphi}_{2k}\varphi_{2k}$, where φ_{1k}, φ_{2k} are the orthogonalized orbits constructed from ψ_{1k}, ψ_{2k}, and can be expressed in terms of ψ_{1k}, ψ_{2k} and we retain up to the fourth order terms. Then the α spin part of the density matrix of our wave function is

$$\begin{aligned} M_\alpha(2,1)=&M_T(2,1)-\sum_k(1+|S_{12}|^2)\,(\bar{\psi}_{1k}(2)\,\psi_{1k}(1)+\bar{\psi}_{2k}(2)\,\psi_{2k}(1))\\ &+\sum_k(S_{21}\bar{\psi}_{1k}(2)\,\psi_{2k}(1)+\text{comp. conj.}), \end{aligned} \tag{10}$$

where

$$S_{12}=\int\bar{\psi}_{1k}(1)\,\psi_{2k}(1)\,dv_1=\sum_t(B_tS_{1t}+\bar{b}_tS_{t2}+\bar{b}_tB_t). \tag{11}$$

$M_T(2,1)$ can be expressed[5)] in terms of various overlap integrals and products like $\bar{u}_t(\boldsymbol{r}_2-\boldsymbol{R}')v_i(\boldsymbol{r}_1-\boldsymbol{R})$. Any integral whose integrand is a product $\bar{v}_i(\boldsymbol{r}_1-\boldsymbol{R})u_t(\boldsymbol{r}_1-\boldsymbol{R}')$, $\boldsymbol{R}'$ being a nearest neighbouring lattice site of $\boldsymbol{R}$, is regarded as a small quantity of the first order. Any product $v_i(\boldsymbol{r}_2-\boldsymbol{R})u_t(\boldsymbol{r}_1-\boldsymbol{R}')$ is also regarded as a small quantity of the first order if $\boldsymbol{R}'$ is a nearest neighbouring lattice site of $\boldsymbol{R}$ and is neglected if $\boldsymbol{R}'$ is more apart from $\boldsymbol{R}$. We develop M_T in powers of the small quantities and retain up to the second order terms. Then eq. (10) is

$$M_\alpha=M_\alpha^{(0)}+M_\alpha^{(2)}+M_\alpha^{(4)}, \tag{12}$$

where

$$M_\alpha^{(0)}=M_A+M_B^+, \tag{13}$$

$$\begin{aligned} M_\alpha^{(2)}=2S^2M_A+3S^2(M_B^++M_B^-)-S(M_C^++M_C^-) \\ -\sum_{tk}\{B_t(\bar{V}_2U_t-S_{2t}\bar{V}_2V_2)+\text{comp.conj.}\}-\sum_{tk}|B_t|^2(\bar{U}_tU_t-\bar{V}_2V_2) \\ -\sum_{tk}\{b_t(\bar{V}_1U_t-S_{1t}\bar{V}_1V_1)+\text{comp.conj.}\}-\sum_{tk}|b_t|^2(\bar{U}_tU_t-\bar{V}_1V_1), \end{aligned} \tag{14}$$

$$\begin{aligned} M_\alpha^{(4)}=-\sum_k|S_{12}|^2(\bar{V}_1V_1+\bar{V}_2V_2)+\sum_k\{S_{21}\sum_t(B_t\bar{V}_1U_t+\bar{b}_t\bar{U}_tV_2+\bar{b}_tB_t\bar{U}_tU_t)+\text{comp.conj.}\} \\ -\sum_k\sum_{i=6}^{10}|C_i|^2(\bar{V}_iV_i-\bar{V}_2V_2) \\ -\sum_k\sum_{i=6}^{10}[C_i\{\bar{V}_2V_i+\sum_t\bar{B}_t(\bar{U}_tV_i-S_{ti}\bar{V}_2V_2)\}+\text{comp.conj.}] \\ -\sum_{tk}\{\bar{A}B_t(\bar{V}_2U_t-S_{2t}\bar{V}_2V_2)+\text{comp.conj.}\}-\sum_{t\neq t'}\sum_k\bar{B}_tB_{t'}\bar{U}_tU_{t'} \\ -\sum_k\sum_{i=6}^{10}|c_i|^2(\bar{V}_iV_i-\bar{V}_1V_1) \\ -\sum_k\sum_{i=6}^{10}[c_i\{\bar{V}_1V_i+\sum_t\bar{b}_t(\bar{U}_tV_i-S_{ti}\bar{V}_1V_1)\}+\text{comp.conj.}] \\ -\sum_{tk}\{\bar{a}b_t(\bar{V}_1U_t-S_{1t}\bar{V}_1V_1)+\text{comp.conj.}\}-\sum_{t\neq t'}\sum_k\bar{b}_tb_{t'}\bar{U}_tU_{t'}, \end{aligned} \tag{15}$$

$$M_A=\sum_{t\boldsymbol{R}}\bar{u}_t(\boldsymbol{r}_2-\boldsymbol{R})u_t(\boldsymbol{r}_1-\boldsymbol{R}), \tag{16}$$

$$M_B^+=\sum_{\boldsymbol{R}}{}^+\sum_{i=1}^{5}\bar{v}_i(\boldsymbol{r}_2-\boldsymbol{R})v_i(\boldsymbol{r}_1-\boldsymbol{R}), \tag{17}$$

$$M_B^-=\sum_{\boldsymbol{R}}{}^-\sum_{i=1}^{5}\bar{v}_i(\boldsymbol{r}_2-\boldsymbol{R})v_i(\boldsymbol{r}_1-\boldsymbol{R}), \tag{18}$$

$$M_C^+=\sum_{\boldsymbol{R}}{}^+\sum_{\boldsymbol{R}'}\bar{v}_i(\boldsymbol{r}_2-\boldsymbol{R})u_t(\boldsymbol{r}_1-\boldsymbol{R}'), \tag{19}$$

$$M_C^-=\sum_{\boldsymbol{R}}{}^-\sum_{\boldsymbol{R}'}\bar{v}_i(\boldsymbol{r}_2-\boldsymbol{R})u_t(\boldsymbol{r}_1-\boldsymbol{R}'), \tag{20}$$

where in (19) and (20) $\boldsymbol{R}$ and $\boldsymbol{R}'$ are nearest neighbouring lattice sites and $v_i(\boldsymbol{r}-\boldsymbol{R})$, $u_t(\boldsymbol{r}-\boldsymbol{R}')$ are those 3d-, 2p-orbits which are axially symmetric about the axis connecting the $\boldsymbol{R}$ and $\boldsymbol{R}'$ lattice points. S is the overlap integral between these two orbits. We have neglected the contribution to the fourth order density matrix from M_T. We have seen that the contribution of this term to the energy difference of the parallel and the antiparallel state is rather small in the case of the four electron model[3].

For β electrons we develop the density matrix in the same way,

$$M_\beta = M_\beta^{(0)} + M_\beta^{(2)} + M_\beta^{(4)}, \tag{21}$$

then the total density is

$$\begin{aligned} M(2,1) &= \bar{\alpha}(2)\alpha(1)M_\alpha(2,1) + \bar{\beta}(2)\beta(1)M_\beta(2,1) \\ &= M^{(0)} + M^{(2)} + M^{(4)}, \end{aligned} \tag{22}$$

where
$$\begin{aligned} M^{(0)} &= \bar{\alpha}\alpha M_\alpha^{(0)} + \bar{\beta}\beta M_\beta^{(0)}, \\ M^{(2)} &= \bar{\alpha}\alpha M_\alpha^{(2)} + \bar{\beta}\beta M_\beta^{(2)}, \\ M^{(4)} &= \bar{\alpha}\alpha M_\alpha^{(4)} + \bar{\beta}\beta M_\beta^{(4)}. \end{aligned} \tag{23}$$

The total energy of the system is given[5] by

$$E = \int H_1 M(1,1)\,dv_1 + \tfrac{1}{2}\int \begin{vmatrix} M(1,1) & M(1,2) \\ M(2,1) & M(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2\,, \tag{24}$$

where H_1 is the sum of the kinetic and potential energies arising from all the atomic cores. H_1 operates only on the second argument 1 of $M(1,1)$ in the integrand of the first term of eq. (24). On substituting (22) in (24),

$$\begin{aligned} E = &\int H_1 \{M^{(0)}(1,1) + M^{(2)}(1,1) + M^{(4)}(1,1)\}\,dv_1 \\ &+ \tfrac{1}{2}\int \begin{vmatrix} M^{(0)}(1,1) + M^{(2)}(1,1) & M^{(0)}(1,2) + M^{(2)}(1,2) \\ M^{(0)}(2,1) + M^{(2)}(2,1) & M^{(0)}(2,2) + M^{(2)}(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2 \\ &+ \int \begin{vmatrix} M^{(0)}(1,1) & M^{(4)}(1,2) \\ M^{(0)}(2,1) & M^{(4)}(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2 \\ &+ \int \begin{vmatrix} M^{(2)}(1,1) & M^{(4)}(1,2) \\ M^{(2)}(2,1) & M^{(4)}(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2 \\ &+ \tfrac{1}{2}\int \begin{vmatrix} M^{(4)}(1,1) & M^{(4)}(1,2) \\ M^{(4)}(2,1) & M^{(4)}(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2 \\ = &\int H_1 M^{(0)}(1,1)\,dv_1 + \tfrac{1}{2}\int \begin{vmatrix} M^{(0)}(1,1) & M^{(0)}(1,2) \\ M^{(0)}(2,1) & M^{(0)}(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2 \\ &+ \int H(1)\,\{M^{(2)}(1,1) + M^{4)}(1,1)\}\,dv_1 \\ &+ \tfrac{1}{2}\int \begin{vmatrix} M^{(2)}(1,1) & M^{(2)}(1,2) \\ M^{(2)}(2,1) & M^{(2)}(2,2) \end{vmatrix} \frac{e^2}{r_{12}}\,dv_1 dv_2 \end{aligned}$$

$$+\int\begin{vmatrix} M^{(2)}(1,1) & M^{(4)}(1,2) \\ M^{(2)}(2,1) & M^{(4)}(2,2) \end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2$$

$$+\tfrac{1}{2}\int\begin{vmatrix} M^{(4)}(1,1) & M^{(4)}(1,2) \\ M^{(4)}(2,1) & M^{(4)}(2,2) \end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2, \tag{25}$$

where $$H(1)=H_1+e^2\int M^{(0)}(2,2)/r_{12}dv_2-e^2\int M^{(0)}(2,1)\mathrm{P}_{12}/r_{12}\,dv_2. \tag{26}$$

P_{12} is a permutation operator which permutes the argument 1 and 2 and operates only on the second argument 1 of $M^{(2)}(1,1)+M^{(4)}(1,1)$.[5)]

We now determine b_t, B_t, c_i, C_i so as to make (25) minimal. b_t and B_t are involved in the second order part as well as in the fourth order part when (13), (14) and (15) are substituted in (25). We determine B_t, b_t so as to make only the second order terms minimal and insert the values thus obtained into the fourth order part, the error being higher than the fourth order. The part of E which involves b_t in the second order is

$$-\sum_{tk}|b_t|^2\{(U_t|H|U_t)-(V_1|H|V_1)\}-\sum_{tk}[b_t\{(V_1|H|U_t)-S_{1t}(V_1|H|V_1)\}+\text{comp.conj.}]$$

$$-\tfrac{1}{2}\sum_{tt'}\sum_{kk'}\{b_{tk}\bar{b}_{t'k'}\int\bar{V}_{1k}(1)\,\bar{U}_{t'k'}(2)\,(e^2/r_{12})\,V_{1k'}(1)\,U_{tk}(2)\,dv_1dv_2+\text{comp.conj.}\}$$

where $$(U_t|H|U_t)=\int\bar{U}_t(1)H(1)U_t(1)dv_1,\ \text{etc.} \tag{27}$$

The first and second terms come from $\int H(1)M^{(2)}(1,1)dv_1$ and the last term from $\frac{1}{2}\int\begin{vmatrix} M^{(2)} & M^{(2)} \\ M^{(2)} & M^{(2)} \end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2$. The last term of (27) is rewritten as $-Q_1\sum_{tk}|b_{tk}|^2$, if b_{tk} is assumed to be of the form $(U_{tk}|h|V_{1k})$ where h is some one-electron operator with the translational symmetry of the lattice and further if the Coulomb integral between $v_1(\boldsymbol{r}-\boldsymbol{R})$ and $u_t(\boldsymbol{r}-\boldsymbol{R}')$ is assumed to be independent of the position of $\boldsymbol{R}'$ when $\boldsymbol{R}'$ is a nearest neighbouring lattice site to $\boldsymbol{R}$. We write this integral as Q_1. Then the minimum of (27) is given by

$$b_{tk}=(U_t|H-(V_1|H|V_1)|V_1)/\{(V_1|H|V_1)-(U_t|H|U_t)-Q_1\}$$

$$=(U_t|H-(V_1|H|V_1)|V_1)/\varDelta_1. \tag{28}$$

The denominator of (28) is independent of $\boldsymbol{k}$, according to our initial assumption concerning overlaps between atomic orbitals, so we have written it as $\varDelta_1$. Similarly

$$B_{tk}=(U_t|H-(V_2|H|V_2)|V_2)/\varDelta_1. \tag{29}$$

$(V_2|H|V_2)$ is equal to $(V_1|H|V_1)$ because $H(1)$ has cubic symmetry about a lattice point and V_1 and V_2 are of dγ-type. We write it as E_d. Similarly we write $(U_t|H|U_t)$ as E_p.

Inserting these values, eq. (25) involes as unknown quantities only c_i and C_i which we determine according to the minimum principle. Then

$$c_i=\{1/(E_d-(V_i|H|V_i)-Q_2)\}\{(V_i|H|V_1)+\sum_t(V_i|H-E_d|U_t)(U_t|H-E_d|V_1)/\varDelta_1\} \tag{30}$$

$$C_i=\{1/(E_d-(V_i|H|V_i)-Q_2)\}\{(V_i|H|V_2)+\sum_t(V_i|H-E_d|U_t)(U_t|H-E_d|V_2)/\varDelta_1\} \tag{31}$$

which, we can see, vanishes for $i=8,\ 9,\ 10$ (dε-type) from symmetry consideration. Q_2 represents Coulomb integrals between two dγ-orbits which are $2R_0$ apart from each other, R_0 being the nearest $\mathrm{Ni}^{++}-\mathrm{O}^{--}$ distance. We neglected small differences between these integrals which arise from the combination of dγ_1 and dγ_2. Then the denominators of (30) and (31) are identical for $i=6,\ 7$, which we write as $\varDelta$. $E_d-(V_i|H|V_i)$ does not vanish even for $i=6,\ 7$. It is the energy difference of an electron when it is on α spin Ni^{++} ions and when on β spin Ni^{++} ions.

Now that we have determined the values of the unknown parameters, we can express eq. (25) in terms of various integrals. However, we want only the difference between eq. (25) and the corresponding expression for the parallel spin state. We give here only the expression of the part of eq. (25) which are needed for the calculation of the energy difference. Using values for b_{tk}, etc., expressing Bloch sums in terms of atomic orbitals and summing over $\boldsymbol{k}$,

$$\begin{aligned}\int HM^{(4)}dv_1+\tfrac{1}{2}\int\begin{vmatrix}M^{(4)} & M^{(4)}\\ M^{(4)} & M^{(4)}\end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2\\ =24N(p'^2-\varDelta_1^2S^2)(p^2+2q^2)/\varDelta_1^3-12N\{p^2(u|H|u')+4pq(u|H|u'')\}/\varDelta_1^2\\ -12N[|(v_1u|uv_1')-p\{p-(uv_1|v_1v_1)-(uv_2|v_2v_2)\}/\varDelta_1|^2+|(v_2u|uv_2')|^2]/\varDelta,\end{aligned} \tag{32}$$

where $p=(u|H-E_d|v_1)$, $p'=(u|H-E_p-Q_1|v_1)$, $q=(u''|H-E_d|v_1)$,

$$(uv|st)=\int\bar{u}(1)\bar{v}(2)\frac{e^2}{r_{12}}s(1)t(2)\,dv_1dv_2.$$

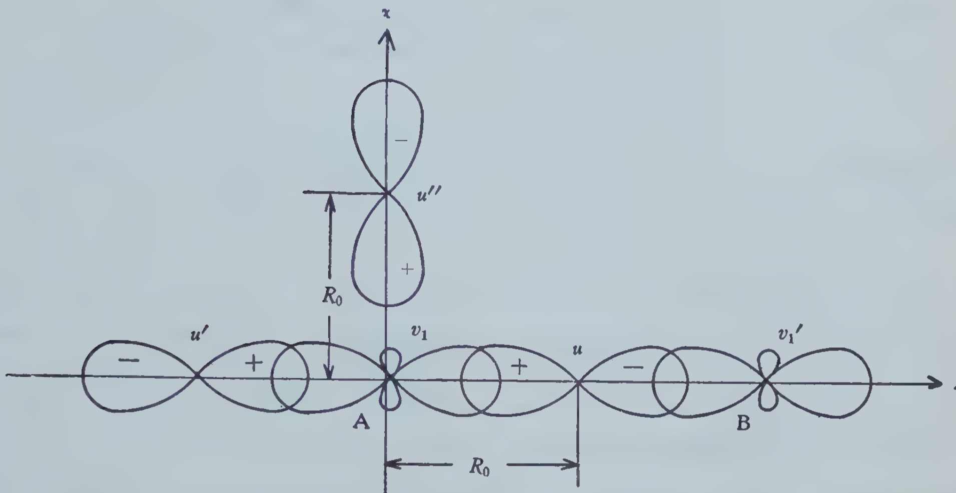
The configuration of the various orbitals appearing in the above integrals is shown in Fig. 2. v_1 and v_2 have an angular dependency of the form $(3z^2-r^2)/r^2$ and $(x^2-y^2)/r^2$ respectively. For the later calculation we give also the expression for M to the second order. Inserting b_{tk}, etc., given by (28), (29) and summing over $\boldsymbol{k}$,

$$M^{(0)}+M^{(2)}=\bar{\alpha}(2)\alpha(1)(M_1+M_4)+\bar{\beta}(2)\beta(1)(M_2+M_3), \tag{33}$$

where

$$\begin{aligned}M_1&=(\tfrac{1}{2}+S^2)M_A+(1+3S^2)M_B^+-SM_C^-,\\ M_2&=(\tfrac{1}{2}+S^2)M_A+(1+3S^2)M_B^--SM_C^-,\\ M_3&=(\tfrac{1}{2}+S^2-r^2)M_A+3r'^2M_B^+-r'M_C^+,\\ M_4&=(\tfrac{1}{2}+S^2-r^2)M_A+3r'^2M_B^--r'M_C^+,\end{aligned} \tag{34}$$

$$r=p/\varDelta_1,\qquad r'=p'/\varDelta_1. \tag{35}$$



v_2 and v_2' are of the form $(x^2-y^2)/r^2$ and located

on the lattice point A and B respectively

Fig. 2 Configuration of various orbits

§ 3. Energy of the parallel spin state

In the ferromagnetic state all available orbitals are filled by α electrons and there are two holes of β electrons on each Ni^{++} ion. These holes may appear with a certain probability on O^{--} ions, and the total energy can be minimized with respect to this probability. The procedure is quite analogous to the antiparallel case and here we give only the result needed for the calculation of the energy difference.

$$M^{(0)}+M^{(2)}=\bar{\alpha}(2)\,\alpha(1)\,(M_1+M_2)+\bar{\beta}(2)\,\beta(1)\,(M_3+M_4)\,. \tag{36}$$

$$\int HM^{(4)}(1,\,1)\,dv_1+\tfrac{1}{2}\int\begin{vmatrix} M^{(4)} & M^{(4)} \\ M^{(4)} & M^{(4)} \end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2$$

$$=24N(p'^2-\varDelta_1^2S^2)\,(p^2+2q^2)/\varDelta_1^3+12Np^2\{(p'/\varDelta_1)^2-S^2\}/\varDelta_1$$

$$-12N\{p^2(u|H|u')+4pq(u|H|u'')\}/\varDelta_1^2. \tag{37}$$

§ 4. Energy difference between the parallel and the antiparallel spin state

The energy difference between the parallel and the antiparallel spin state comes from the fourth order parts of (25), since the zeroth and second order terms in the parallel state are identical with the respective terms for the antiparallel state. The difference of the term $\int HM^{(4)}dv_1+\frac{1}{2}\int\begin{vmatrix} M^{(4)} & M^{(4)} \\ M^{(4)} & M^{(4)} \end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2$ of eq. (25) in the fourth order is the difference between (32) and (37), which is

$$\begin{aligned}\varDelta E_1 &= (37)-(32)\\ &= 12N/\varDelta\{|(v_1u|uv_1')-p\{p-(uv_1|v_1v_1)-(uv_2|v_2v_2)\}/\varDelta_1|^2+|(v_2u|uv_2')|^2\}\\ &\quad +12Np^2\{(p'/\varDelta_1)^2-S^2\}/\varDelta_1.\end{aligned} \tag{38}$$

The difference of the term $\frac{1}{2}\int\begin{vmatrix} M^{(0)}+M^{(2)} & M^{(0)}+M^{(2)} \\ M^{(0)}+M^{(2)} & M^{(0)}+M^{(2)}\end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2$ in (25) is from (33) and (36)

$$\begin{aligned}\varDelta E_2 &= -\int\begin{vmatrix} M_1-M_3 & M_2-M_4 \\ M_1-M_3 & M_2-M_4\end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2\\ &= -2N(p/\varDelta_1)^4\{2(u_xu_y|u_yu_x)+(u_xu_x|u_xu_x)\}\\ &\quad -12N(p/\varDelta_1)^2\{(uv_1|v_1u)+(uv_2|v_2u)\}\\ &\quad +12N(p/\varDelta_1)^2(uu|v_1v_1')-24N(p/\varDelta_1)^3(uu|uv_1).\end{aligned} \tag{39}$$

Here u_x and u_y are those 2p-orbits of an O^{--} ion which have angular dependency of the form x/r and y/r respectively. The term of $\int\begin{vmatrix} M^{(2)} & M^{(4)} \\ M^{(2)} & M^{(4)}\end{vmatrix}\frac{e^2}{r_{12}}dv_1dv_2$ is identical for the two cases up to the fourth order part. Thus we have given a spin dependent energy of NiO crystal in the band formalism. The first term in the right-hand side of (38) favours the antiparallel state and comes from the fact that in the antiparallel state the alternant bands mix, thus lowering the energy. The first term of (39) may originate from the fact that when two 2p electrons transfer simultaneously to two Ni^{++} ions, the energy of the resulting oxygen atom is lower by Hund's rule when the two electrons have the same spin and this is so in the ferromagnetic case, thus favouring the ferromagnetic state. The second term of (39) may correspond to the original mechanism of Anderson for superexchange interaction in Heitler-London scheme when $S=0$. However, exact parallelism between the energy difference given by the second term of (38) and (39) together and that by the Heitler-London scheme was not found.

We have estimated values of (38) and (39) using s-orbits instead of 3d-orbits of Ni^{++} ion. $\varDelta+Q_2$ is approximately the third ionization energy minus the second ionization energy of nickel. We estimated this to be about 15 e.v. Q_2 is approximately $e^2/2R_0=$ 3.45 e.v. $\varDelta_1$ is estimated by the following expression

$$(4\alpha-1)e^2/R_0+E_a-I_2,$$

where α is the Madelung constant of NaCl structure, E_a is the electron affinity of O^- ion and I_2 is the second ionization energy of nickel. Then $\varDelta_1$ is about 11.6 e.v. As the radial part of u_t and s-orbit mentioned above we used the nodeless Slater type function $R(r)=\sqrt{4\delta^5/3}\,r\exp(-\delta r)$, with $\delta=2.16$ in atomic units. When u_t and the s-orbit are apart by the actual lattice parameter of NiO crystal, then $S=0.0625$. Absolute values of b_{tk} and c_{jk} are of the order of $|p/\varDelta_1|$ and $|(v_1u|uv_1')-p\{p-(uv_1|v_1v_1)\}/\varDelta_1|/\varDelta$ respectively. They are 0.045 and 0.007 which are small compared with unity, thus confirming our initial assumption. Our estimate is as follows:

first term of (38) = 0.00690N e.v.
second term of (38) = −0.00125 „
first term of (39) = −0.00004 „
second term of (39) = −0.00167 „
third term of (39) = −0.00031 „
fourth term of (39) = 0.00171 „ .

The total energy difference is 0.00534N e.v. The first term of (38) is the largest of all terms. $-p\{p-(uv_1|v_1v_1)\}/\Delta_1$ is about five times larger than $(v_1u|uv_1')$ and gives the main contribution to the first term of (38) and also to the total energy difference. Though our rough numerical estimation does not allow us to speak in quantitative terms, it may be said that mixing of alternant bands plays an important role in the superexchange interaction.

The author wishes to express his cordial thanks to Prof. J. Yamashita for his kind advice in the course of study.

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Letters to the Editor

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On the Spin-Lattice Relaxation Process in Ferromagnets

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August 20, 1957

Recently many kinds of mechanisms have been proposed for ferromagnetic relaxation process. However, there is not yet any reasonable explanation for the experimental result. In this paper we calculate the spin-lattice relaxation time in ferromagnetic matter.

We consider the exchange integral to be modulated by lattice vibration where we assume the adiabatic approximation to be useful, and that atomic distance dependence of exchange integral has an exponential type i.e. $J'\exp(-\alpha R)$, and expand it in the deviation $\boldsymbol{u}$ from lattice point. Accordingly, the Hamiltonian is :

$$H=-J\sum_{j,\delta}\boldsymbol{s}_j\cdot\boldsymbol{s}_{j-\delta}+\frac{\alpha J}{a}\sum_{j,\delta}(\delta\cdot\boldsymbol{u}_j-\boldsymbol{u}_{j-\delta})(\boldsymbol{s}_j\cdot\boldsymbol{s}_{j-\delta}). \qquad (1)$$

The first term is the common exchange coupling and the second the spin-lattice interaction Hamiltonian which we required. a is atomic distance, $\boldsymbol{\delta}$ the vector to the nearest neighbour lattice point and $\boldsymbol{j}$ the lattice site label. In the wave number space[1),2)] the above interaction Hamiltonian is written as

$$H_{\text{int}}=-\frac{2JS\alpha}{a}\sqrt{\frac{\hbar}{2MN}}\sum_{\lambda,\sigma,\mu,i}w_{\mu i}^{-1/2}[a_\lambda^* a_\sigma b_{\mu i}\times\delta_{\lambda,\sigma+\mu}F(\sigma,\mu)+a_\lambda^* a_\sigma b_{\mu i}^*\delta_{\sigma,\lambda+\mu}G(\sigma,\mu)], \qquad (2)$$

where

$$\begin{aligned}F(\sigma,\mu)&=\sum_\delta(\boldsymbol{e}_{\mu i}\cdot\delta)(1-e^{i\sigma\delta})\times(1-e^{i\mu\delta}),\\G(\sigma,\mu)&=\sum_\delta(\boldsymbol{e}_{\mu i}\cdot\delta)(1-e^{i\sigma\delta})\times(1-e^{-i\mu\delta}).\end{aligned} \qquad (3)$$

However, our interaction Hamiltonian is different from that used by Kittel-Abrahams[3)] in their discussion of spin-lattice relaxation process. They used the phenomenological magnetoelastic interaction[3),4)] which is derived from the anisotropy energy the origin of which is believed to be the interaction between spin and lattice systems through the partially quenched orbital angular momentum.

In the present calculation the spin and the phonon systems are assumed to be in thermal equilibrium at temperature $T+\Delta T$ and T, respectively. Then the change of phonon number is

$$dm_{\mu i}/dt=P_A-P_B \qquad (4)$$

where P_A and P_B are transition probabilities which are given by

$$P_A=\frac{4\pi J^2S^2\alpha^2}{MNa^2w_{\mu i}}\sum_\lambda(m_{\mu i}+1)(n_\lambda+1)\times n_{\lambda+\mu}|G(\lambda+\mu,\mu)|^2\delta(E_i-E_f), \qquad (5)$$

$$P_B = \frac{4\pi J^2 S^2 \alpha^2}{MNa^2 w_{\mu i}} \sum_\lambda (n_{\lambda+\mu}+1) n_\lambda m_{\mu i} \times |F(\lambda, \mu)|^2 \delta(E_i - E_f).$$

Adopting the long wave approximations for F and G, we calculate the energy transfered to the lattice per unit time which is defined by

$$dQ/dt = \sum_{\mu i} \mathcal{E}_{\mu i} dm_{\mu i}/dt. \tag{6}$$

Now we take account of only the case of longitudinal wave of phonons and finally take three times of those results to include the two transversal waves. We develop dQ/dt in term of $\Delta T/T$ and obtain

$$\frac{dQ}{dt} = \frac{\Delta T}{T} \cdot \frac{3\alpha^2 \hbar N a^8 (kT)^9}{512\pi^3 M (\hbar c)^8} \int_0^{x_0} \frac{e^x dx}{e^x - 1} \int_{y_1}^{y_2} y^4 \left(y - \frac{\hbar^2 c^2}{2JSa^2 kT}\right)^4 \frac{e^y dy}{(e^{x+y}-1)(e^y-1)}, \tag{7}$$

where

$$y_1 = \max\left[0, \frac{\hbar^2 c^2}{2JSa^2 kT} - \frac{2\hbar c x^{1/2}}{(2JSa^2 kT)^{1/2}}\right],$$

$$y_2 = \min\left[\frac{2\pi}{a}\left(\frac{3}{4\pi}\right)^{1/3} \frac{\hbar c}{kT}, \frac{\hbar^2 c^2}{2JSa^2 kT} + \frac{2\hbar c x^{1/2}}{(2JSa^2 kT)^{1/2}}\right], \tag{8}$$

$$x_0 = \frac{8\pi^2 JS}{kT}\left(\frac{3}{4\pi}\right)^{2/3}.$$

The spin-lattice relaxation time T_L which is the time constant in the exponential decay of ΔT is given by[3)]

$$T_L = C_S C_L / C_S + C_L \cdot \Delta T / \dot{Q} \tag{9}$$

where C_S and C_L are specific heats of spin and lattice systems, respectively. C_S is estimated by the use of spin wave and C_L by the Debye approximation. In the case of pure Ni, the numerical values are given by

$$J = 1.8 \times 10^{-14}\ \text{erg}, \quad C = 5 \times 10^5\ \text{cm/sec},$$

$$M = 10^{-22}\ \text{gr}, \quad a = 2.5 \times 10^{-8}\ \text{cm},$$

$$T_c = 631\,^\circ\text{K}, \quad \kappa = 5.29 \times 10^{-7}, \tag{10}$$

where κ is the compressibility and T_c the Curie point. The parameter α is estimated from the experiment[5)] which shows the variations of the Curie point with applied pressure i.e.

$$a\alpha = \Delta J/\Delta R \cdot a/J = 1/T_c \cdot 3/\kappa \cdot \Delta T_c/\Delta p = 3{,}15. \tag{11}$$

Eqs. (7) ~ (11) tell us that

$$T_L = 2{,}2 \times 10^{-11}\ \text{sec (at } 300\,^\circ\text{K)},$$

$$= 5{,}0 \times 10^{-11}\ \text{sec (at } 160\,^\circ\text{K)}.$$

The value at 300°K is the same in order with but slightly shorter than Bloembergen's estimation 5×10^{-10} sec at room temperature.[6)] Because the long wave approximation is not really valid in such a high temperature range, so that an underestimation of T_L is expected actually. However, it is clear that we cannot neglect the above mechanism playing at least an important part in the spin-lattice relaxation process. In ferromagnetic resonance absorption, the heat capacity of spin system is so large that spin-lattice relaxation may not be estimated by Bloembergen's experiment, and in addition also the use of the Bloch type equation in this case is not fully justified. There is, however, a method to single out the above process by measuring the absorption of ultrasound in ferromagnetic substance.[7)]

The authors wish to thank Professor K. Tomita, Mr. K. Sugihara and Mr. S. Takeno for their valuable suggestions and helpful discussions on this work.

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The Artificial Satellite and the Relativistic Red Shift

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September 25, 1957

Recently a suggestion for the experimental verification of general relativity has been put forward by Singer.[1] He proposes to equip an artificial satellite and to measure directly the time dilatation by electronic device.

The formula that Singer derives for this purpose has been extended by Hoffmann[2] who has taken into account two not less important factors, viz., the rotation and the eccentricity of the earth. But the way he treats the rotation is rather artificial and does not follow directly from the solutions of the field equation. We shall do that here.

The metric tensor in the field of the rotating earth, as given by Clark,[3] Das,[4] is the following set*

$$g_{11}=-1-\frac{2M}{R_E+h}+\frac{AR_E^2\omega^2}{7(R_E+h)^3}\times\left(1-\frac{5}{3}\cos^2\theta\right),$$

$$g_{22}=-(R_E+h)^2\times\left[1-\frac{AR_E^2\omega^2}{21(R_E+h)^3}\cdot\sin^2\theta\right],$$

$$g_{33}=-(R_E+h)^2\sin^2\theta\times\left[1+\frac{AR_E^2\omega^2}{7(R_E+h)^3}\cdot\sin^2\theta\right],$$

$$g_{44}=1-\frac{2M}{R_E+h}+\frac{AR_E^2\omega^2}{7(R_E+h)^3}\times\left(\frac{2}{3}-\sin^2\theta\right),$$

$$g_{12}=\frac{AR_E^2\omega^2}{21(R_E+h)^2}\cdot\sin 2\theta,$$

$$g_{34}=\frac{2A\omega}{R_E+h}\cdot\sin^2\theta. \tag{1}$$

where M, A, ω, R_E are the mass, angular momentum, angular velocity, radius of the earth respectively and h denotes the distance of the test-satellite from the earth's surface.

For a satellite moving circularly in the fixed plane $\theta=\pi/2$, the ratio of its proper-time to the coordinate time (as calculated from (1)) is given by

$$\left(\frac{ds}{dt_s}\right)^2\simeq 1-\frac{3M}{R_E+h}+\frac{6\sqrt{M}A\omega}{(R_E+h)^{5/2}}-\frac{5AR_E^2\omega^2}{42(R_E+h)^3}-\frac{3MAR_E^2\omega^2}{14(R_E+h)^4}. \tag{2}$$

In the coordinate system freely rotating with the earth the metric tensor in the set (1) transforms into

$$g'_{44}=g_{44}+2\omega g_{34}+\omega^2 g_{33} \quad \& \text{ etc.}$$

Therefore, for a clock fixed ω, r, t, the rotating earth (i. e. $h=0$) the ratio of the proper-time to the coordinate time is

---

* The solutions are approximate (upto the terms containing ω^2) and in the spherical polar coordinates.

$$\left(\frac{ds}{dt_E}\right)^2 \simeq 1-\frac{2M}{R_E}-R_E^2\,\omega^2\sin^2\theta$$
$$+\frac{2A\omega^2}{21R_E}+\frac{27A\omega^2\sin^2\theta}{7R_E}. \quad (3)$$

From (2) & (3) Δ, the relative shift comes out to be

$$\Delta=\frac{dt_s-dt_E}{dt_E}\simeq(M/R_E)$$
$$\times[1.5\,(1+h/R_E)^{-1}-1]$$
$$-\frac{1}{2}R_E^2\,\omega^2\sin^2\theta-(3\sqrt{M}A\omega)/(R_E+h)^{5/2}$$
$$+\frac{A\omega^2}{7R_E}\cdot\left(\frac{1}{3}+\frac{27}{2}\sin^2\theta\right)+\frac{5AR_E^2\,\omega^2}{84\,(R_E+h)^3}. \quad (4)$$

The numerical magnitudes of the last three correction terms* for the colatitude $\theta=\pi/2$ are respectively 1.7×10^{-20}, 3.4×10^{-22}, 1.05×10^{-23} and falls outside the experimental range as given by Singer.

The first order eccentricity term as introduced by Hoffmann is the same as that following from Einstein's linearised field equation.

In order of relative importance next to these corrections come the influences of the moon and the other planets. But for the practical purposes these can be safely ignored.

For the perigee motion of the satellite the effect of the earth's spin is not negligible. In fact, the ratio of the first order spin correction to the Einstein term is $4\omega T/15\pi(1+h/R_E)^2\sim0.03$.

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* Our corrections, as given by (4), are not exactly applicable to the Russian satellite! This is because it will move from the north to the south and so cannot have any fixed orbital plane.

Efficiency of Acceleration of Relativistic Particles in the Crab Nebula

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October 2, 1957

The energy density of cosmic rays comparable to that of star light suggests the surprisingly high efficiency of converting the thermal energy into the kinetic energy of relativistic particles. Although the storage of cosmic rays in the galactic magnetic field lowers the overall efficiency, there must be some objects, in which the acceleration is very efficient, since most of stars, like the sun, are known to be rather poor producers of cosmic rays. In this note we would like to show on the semi-quantitative basis that the high efficiency is really seen in the Crab nebula, the remnant of a supernova in 1054.

Our arguments are based on the hypotheses, which seem to be plausible, that (a) the electromagnetic radiations are caused by the synchrotron radiation of relativistic electrons[1)] and (b) the so-called rapid process of forming transuranic elements took place in the explosion of the supernova in 1054.[2)]

The hypothesis (a) was examined by Oort and Walraven[1)] who were able to deduce the rate of energy consumption in the form of relativistic electrons. According to our revised analysis of the spatial and energy distributions of the electrons, their mean lifetime due to the diffusion through nebulous magnetic fields is estimated at about 100 years; consequently, the rate of energy consumption is obtained as about 10^{38} erg $\sec^{-1}$. If the escape of the electrons through the diffusion were insignificant, the rate of energy consumption would decrease to 10^{37} erg $\sec^{-1}$, this being the lower limit.

The hypothesis (b) is successful to explain the light curve of supernovae as well as the origin of heavy elements. According to Burbidge et al.[2] the yields of various nuclides are determined by referring to the light curve and the abundances of heavy elements. Within the arbitrariness of the determination the present rate of energy generation is estimated to lie between $(3\sim 10)\times 10^{36}$ erg $\sec^{-1}$. To this $^{250}C_m$ is found to give the largest contribution. The balance sheet with regard to the outcome and income of energy, applicable to the present Crab nebula, is shown in Table 1.

Table 1. Rates of energy supply and consumption (erg $\sec^{-1}$)

Consumption		Supply	
Electromagnetic radiation	6×10^{36}	Radioactivity	3×10^{36} $\sim 10^{37}$
Electrons	$10^{37}\sim$ 10^{38}		

Table 1 shows that the energy supply is just enough or may be insufficient to make up the consumption due to high energy particles. The situation is more serious, if one compares the energies per particle in these two cases; of the order of MeV for the radioactive decay products and on the average tens GeV for the high energy particles. The above comparison suggests us that the energy sources other than the radioactivity are mainly responsible for the acceleration of the high energy particles, electrons and nuclear particles; the latter ones turn into cosmic rays and their rate of energy consumption may be as large as that of electrons.

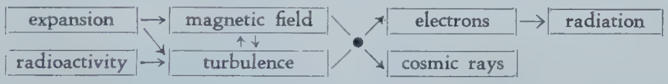
The largest amount of energy is contained in the expanding motion with velocity of about 1.1×10^{8} cm $\sec^{-1}$.[3] The turbulent velocity is about one third of the above one, and the turbulent energy is regarded as large as the magnetic energy; the corresponding magnetic field strength is 10^{-3} gauss. These forms of energy are supposed to be generated chiefly in the initial stage of explosion, perhaps within a year after explosion. Excluding the first fifty days, the rates of energy consumption and supply are integrated over the last nine hundred years, assuming that the rates vary slowly in time as the radioactive energy generation rate. Thus we obtain Table 2, in which various modes of energy are compared.

From these sequences and the energy contents in Table 2 we can draw the following conclusions. (i) The efficiency of acceleration of relativistic particles is surprisingly good; the turbulent and magnetic energies mav be converted almost completely to relativistic particles for a rather short period. (ii) The Fermi acceleration is not a major cause of acceleration, because this would result in the cosmic ray energy some thousand times larger (proportional to the mass of particles accelerated) than the electron energy, much larger than any form of energy. This conclusion is also supported by the analysis of the energy spectrum of electrons alone.[1]

Table 2. Energy contents (erg) in various modes

Expansion	3×10^{49}	Electromagnetic radiation	$(0.5\sim0.9)\times10^{48}$
Turbulence	3×10^{48}	Electrons	$(0.3\sim3)\times10^{48}$
Magnetic field	3×10^{48}	Radioactivity	$(0.6\sim1.5)\times10^{48}$

Table 2 suggests the following sequences of energy conversion:



(iii) The moving light ripple, which was suggested by Oort and Walraven$^{1)}$ as responsible for the injection of relativistic particles, should not be regarded as the motion of a gas, because the kinetic energy of the gas would correspond to the rate of energy consumption as large as 10^{39} erg $\sec^{-1}$, far larger than any other rate of energy supply.

Detailed discussions on the bearing of our analysis will be given in a forthcoming paper. Our thanks are due to Professor C. Hayashi, with whom we enjoyed stimulating discussions.

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A New Approach to a Bound State

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The extended wave matrix which generates not only the eigenvectors for scattering states but also those for bound states from those of the free Hamiltonian is introduced. By using this operator, the problems in the non-perturbation approach are discussed in a general fashion in the case of the system involving bound states.

Introduction

In the recent approach to the quantum field theory,[1)-3)] the eigenfunctions of the total Hamiltonian play a great part. In this field of physics, there still remain many complicated problems even though a number of assumptions such as the existence of the true vacuum were accepted. For example, one often assumes the unitarity of the wave-matrix, under which many definite conclusions can be derived by means of the 'representation of the incoming fields.' If the unitarity condition of the wave-matrix is violated, as in the case of the system including some bound states, all the eigenvectors of the total Hamiltonian cannot be covered either by a set of functions generated by the wave matrix from those of the free Hamiltonian or by the set of eigenfunctions in the 'incoming field representations.'[4)-6)] On account of this situation, we cannot prove the positive definiteness of the weight function, $\pi(-a)$, in Källen's work,[1)7)] and also cannot remain in the framework of Lehmann, Symanzik, Zimmermann's formulation.[3)] Therefore, it is worth while to study how we can obtain some information on the eigenfunctions which enable us to formulate or to discuss the problems in a general fashion in the case of the system involving bound states. For this purpose, we shall construct an operator tentatively called an extended wave matrix, which plays a role of the wave matrix,[8)9)] but is wider than the latter, namely, it generates not only eigenvectors for scattering states but also those for bound states from those of the free Hamiltonian.

We shall give a general procedure to construct the extended wave matrix under several assumptions which will be mentioned in detail (§ 1). In the first step we introduce the 'intermediate Hamiltonian' which has a similar spectrum to that of the total Hamiltonian and same eigenvectors with those of the free Hamiltonian. One of the equations, which define that operator, supplies the condition which determines the energy of the bound state. In § 2, the above procedure is illustrated with the example of the Lee model.[10)] The possibility of restoration of problems stated in the beginning will be discussed in § 3 by using this extended wave matrix.

§ 1. General procedure

In spite of the general conjecture[11)–14)] that the series of perturbation expansion may diverge, it may be expected that the similar character to that of the free Hamiltonian will be very useful in the future theory because its particle-like description seems to be suitable for the description of the experimental results. We shall firstly assume that

A) any eigenvector $\Psi_\mu(g)$, of the total Hamiltonian H, converges weakly to an eigenvector Φ_μ, of the free Hamiltonian H_0, when we make the coupling constant g approach to zero. (We shall say that Ψ_μ corresponds to Φ_μ in this case.)

It is not needed to assume the regularity of $\Psi_\mu(g)$ at the origin of complex g-plane. We only assume the existence of the limit of $\Psi_\mu(g)$ for $g\to+0$ along the real axis. The convergence of the state vector in this limit may be weak, that is, we do not assume the convergence of its norm.*)

Treating some models, we can get an information that the bound state corresponds in the above sense to the lowest state of the continuous spectrum of the free Hamiltonian.** In the nuclear physics we know that there are not two stable bound states with the same charge and mass numbers but with different energies. Accordingly it is reasonable to assume that

B) any stable bound state $\Psi_B^{(i)}$ corresponds to a lowest eigenstate $\Phi_0^{(i)}$ of H_0 belonging to the i-th class defined below. It must be remarked that in this case no other bound states can correspond to $\Phi_0^{(i)}$.

Since each eigenstate of H_0 can be characterized by the number of the particles present and the total momentum, the Hilbert space can be divided into a set of subspaces mutually orthogonal to each other. According to this division, the energy spectrum and the eigenstates are respectively classified into a set of classes of energy levels. The 'class' used in the above assumption should be read in this sense. For example the state with a proton and a neutron at rest belongs to the same class as that of a proton and a neutron with some kinetic energy and the zero total momentum, while the two-proton-state and the two-neutron-state belong to the different classes.

For the sake of simplicity we assume that the state vectors of total Hamiltonian consist of one class of eigenfunctions, $\{\Psi^{(1)}\}$ which corresponds to a class $\{\Phi^{(1)}\}$, and that the bound state $\Psi_B^{(1)}$ involved in $\{\Psi^{(1)}\}$ corresponds to the lowest state $\Phi_0^{(1)}$ belonging to $\{\Phi^{(1)}\}$. The total functional space made of all the eigenfunctions of H_0 is divided into two mutually orthogonal subspaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$; to the former the set $\{\Phi^{(1)}\}$ belongs, while the latter consists of the set of remaining eigenfunctions $\{\Phi^{(2)}\}$ of H_0. In other words, the whole space $\{\Psi^{(1)}\}$ spanned by the eigenfunctions of H weakly tends to $\mathcal{H}^{(1)}$ in the limit $g\to+0$, while $\mathcal{H}^{(2)}$ is also necessary, as will be shown later, to represent $\{\Psi^{(1)}\}$ in terms of the eigenfunctions of H_0 in case of $g\neq 0$. With regard to the continuous energies H, we shall use the notation $\Psi_\mu^{(1)}$ to represent a state with same energy E_μ as that of a state $\Phi_\mu^{(1)}$ to which the state $\Psi_\mu^{(1)}$ corresponds under the assumption that the energy-renormalization has been made in a suitable way.

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* Cf. eq. (2·4).

** Concerning the relation of these two states, there is Suura et al's discussion.[15)]

Let us introduce the following Hamiltonian,

$$H_0' = H_0 + \int d\mu\ \delta_{\mu 0}\ \Phi_\mu^{(1)}\rangle \Delta \langle \Phi_\mu^{(0)}, \tag{1·1}$$

$$V' = V - \int d\mu\ \delta_{\mu 0}\ \Phi_\mu^{(1)}\rangle \Delta \langle \Phi_\mu^{(1)} \tag{1·2}$$

where $-\Delta$ is the binding energy $E_0 - E_B$ (E_0 is the minimum value of the continuous spectrum $\{\Phi_\mu^{(1)}\}$) and $\int d\mu$ means the integration with respect to quantum numbers. Eigenvalues of H_0' are similar to those of the total Hamiltonian, while eigenvectors of H_0' are nothing but those of H_0. By using this Hamiltonian let us introduce the operator $\Omega(E)$ defined as a solution of the following integral equation

$$\Omega^\epsilon(E) = 1 + \frac{1}{E - H_0' + i\epsilon} V'\Omega^\epsilon(E) \quad \Omega(E) = \lim_{\epsilon \to +0} \Omega^\epsilon(E). \tag{1·3}$$

If the limit $\Omega(E_\mu)\Phi_\mu (\mu \fallingdotseq 0)$ exists and is not a zerovector, then this is an eigenvecor of H which corresponds to Φ_μ. The assumption that no eigenstates of H 'correspond to' $\Phi_\mu^{(2)}$ suggests that the limit $\Omega(E_\mu)\ \Phi_\mu^{(2)}$ does not exist or is a zerovector.*) On the other hand, the outgoing solution of the total Hamiltonian which 'corrresponds to' $\Phi_\mu^{(1)}$ is customarily written as $\Omega(E_\mu)\ \Phi_\mu^{(1)}$. It is very likely to assume that $\Omega(E_\mu)\ \Phi_\mu^{(1)}$ $(\mu \fallingdotseq 0)$ represents the non trivial solution. For the discrete energy $E = E_B$, however, we may encounter the difficulty as to the singularity of right-hand side of (1.3) in the limit $\epsilon \to +0$. In order to avoid this difficulty we shall add the subsidiary condition

$$(\Phi_0^{(1)} V'\Omega(E_B)\ \Phi_0^{(1)}) = 0(\epsilon^{1+\alpha}) \quad \alpha > 0.^{**)} \tag{1·4}$$

This subsidiary condition shows that $\Omega(E_B)\ \Phi_0^{(1)}$ is the eigenvector of H which belongs to eigenvalue E_B. This enables us not only to determine the binding energy of the state concerned but also to guarantee that the statevector $\Omega(E_B)\ \Phi_0^{(1)}$ actually represents a bound state, that is, particles are localized only in the restricted spatial region.[16)]

If we define the operator W as follows

$$\begin{aligned} (\Phi_\mu\ W\ \Phi_\nu) &= (\Phi_\mu\ \Omega\ (E_\nu)\ \Phi_\nu)\, c_\nu \quad \nu \fallingdotseq 0 \\ (\Phi_\mu\ W\ \Phi_0^{(1)}) &= (\Phi_\mu\ \Omega\ (E_B)\ \Phi_0^{(1)})\ c_0, \end{aligned} \tag{1·5}$$

where c_ν is a normalization constant, we can write the state vectors of the total Hamiltonian as

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* For example, see eq. (2·11).

** It seems sufficient to take $\alpha \geqslant 0$ in order to give the definite meaning to the right-hand side of equation (1.5). The proof that $\Omega(E_B)\ \Phi_0^{(1)}$ is an eigenvector of H belonging to energy E_B, however is, made by multiplying $\Omega(E_B)\Phi_0^{(1)}$ by the factor $(E_B - H_0 + i\epsilon)$ and by neglecting the term of order $O(\epsilon)$. For the $\Phi_0^{(1)}$-component the above procedure means to multiply by the factor $i\epsilon$. Therefore the neglected term must be of the order $O(\epsilon^{1+\alpha})$ $(\alpha > 0)$, that is $i\epsilon(\Phi_0^{(1)}, \Psi - \Phi_0^{(1)}) = O(\epsilon^{1+\alpha})$. The condition (1·4) introduced is in this way. We can see that if we take $\alpha = 0$ we cannot get a binding energy from (1·4) for Lee model.

$$\begin{aligned} \Psi^{(1)} &= W\Phi_\mu^{(1)}, \qquad \mu \neq 0, \\ \Psi_B &= W\Phi_0^{(1)}. \end{aligned} \tag{1·6}$$

Thus, we have obtained the operator which produces the eigenvector of H by operating to $\{\Phi^{(1)}\}$. If we assume the completeness of sets $\{\Psi_\mu^{(1)}, \Psi_B\}$ and $\{\Phi_\mu\}$, namely, if any vector of the one set can be represented by a linear combination of the vectors of the other, the following equations can be proved:

$$\begin{aligned} W^+ W &= 1 - P_{\mathcal{H}(2)} \\ W W^+ &= 1 \end{aligned} \tag{1·7}$$

where $P_{\mathcal{H}(2)}$ is a projection operator on the sub-space $\mathcal{H}^{(2)}$. Here, the existence of the limit $\Omega(E_\mu)\ \Phi_\mu$ is assumed. It is interesting to compare equations (1·7) with equations for the ordinary wave matrix Ω^0, that is,

$$\begin{aligned} \Omega^{0+}\ \Omega^0 &= 1 \\ \Omega^0\ \Omega^{0+} &= 1 - P_B \end{aligned} \tag{1·8}$$

where P_B is a projection operator on bound states. Corresponding to the ordinary incoming operator

$$O^{in}(t) = \Omega^0\ O(t)\ \Omega^{0+}, \tag{1·9}$$

we shall define the modified incoming operator

$$O'^{in}(t) = WO'(t)W^+. \tag{1·10}$$

Here $O'(t)$ means the operator in the intermediate representation, where the temporal change of the operator is generated by H_0'. The fact that the operator O'^{in} satisfies the same commutation relations as those of O' can be derived from the situation $W^+\mathcal{H} \subseteq \mathcal{H}^{(1)}$ and from the assumption $O'\mathcal{H}^{(1)} \subseteq \mathcal{H}^{(1)}$. We shall see in § 2 that such a restricted operator O' is enough to discuss the eigenvectors of H.

§ 2. Example

We shall illustrate the above mentioned general procedure, taking the Lee model as an example.[10] Here the relation

$$m_0\,(\mathit{fixed}) > m_N + \mu$$

is assumed, where m_0 is a bare mass of V-particle and is considered to be fixed, that is, independent of g. The eigenvectors in the subspace spanned by one bare V-particle states or $N+\theta$-states are well known, that is,

$$\begin{aligned} \Psi_k = \phi_k + \frac{g}{N\sqrt{V}} \frac{f(\omega)}{\sqrt{2\omega}} \frac{1}{m_N+\omega-m_0+i\epsilon} \Bigg\{ 1 - \frac{1}{m_N+\omega-m_0+i\epsilon} \frac{g^2}{N^2 V} \sum \frac{f^2(\omega')}{2\omega'} \\ \times \frac{1}{\omega-\omega'+i\epsilon} \Bigg\}^{-1} \Bigg\{ \phi_k + \frac{g}{N\sqrt{V}} \sum \frac{f(\omega'')}{\sqrt{2\omega''}} \frac{1}{\omega-\omega''+i\epsilon} \phi_{k''} \Bigg\}, \end{aligned} \tag{2·1}$$

$$\Psi=\left\{1+\frac{g^2}{N^2V}\sum\frac{f^2(\omega)}{2\omega(E-m_N-\omega)^2}\right\}^{-1/2}\left\{\phi_v+\frac{g}{N\sqrt{V}}\sum\frac{f(\omega')}{\sqrt{2\omega'}\ (m+\omega-E)}\phi_k.\right\}. \tag{2·2}$$

Here E is determined by the following equation ;

$$m_0-E=\left(\frac{g}{2\pi}\right)^2\frac{1}{N^2}\int\frac{f^2(\omega)\ k\ d\omega}{(m_N+\mu-E)}\equiv C(g^2, E). \tag{2·3}$$

We can easily see the following relations :

$$\lim_{g\to+0}\Psi_k^-=\Phi_k, \tag{2·4}$$

$$\lim_{g\to g_1-0}\Psi=\Psi_0^+(g_1),$$

$$\lim_{g\to g_1+0}E=E_0(=m_N+\mu), \tag{2·5}$$

where g_1 is the solution of the equation

$$m_0-m_N-\mu=C(g_1^2,\ m_N+\mu). \tag{2·6}$$

It must be noted that, in the present case, namely, in the case of m_0 being independent of g, the bound state Ψ of N and θ can be dissociated into a kind of scattering state with the energy $E_0=m_N+\mu$ before g vanishes. The coupling constant g_1 determined by $(2\cdot6)$ is just the critical magnitude of g mentioned above where the dissociation of N and θ begins. Therefore, in this example we can attribute Ψ_k^+, Ψ, Φ_k and Φ_v to $\{\Psi_\mu^{(1)}\}$, Ψ_B, $\{\Phi^{(1)}\}$ and $\{\Phi^{(2)}\}$ in § 2 respectively. The assumptions in § 2 are satisfied in this case.

Applying the general procedure to this example, we can get the following expected results. Firstly, the solution of eqs. $(1\cdot4)$ and $(1\cdot3)$ gives the usual binding energy $(2\cdot3)$. Secondly, we can derive the usual eigenvectors $(2\cdot1)$ and $(2\cdot2)$ by making use of the equation $(1\cdot6)$, that is,

$$\begin{aligned}\Psi_k^+&=W\ \Phi_k,\\ \Psi&=W\ \Phi_0.\end{aligned} \tag{2·7}$$

Thirdly, if we introduce modified incoming operators

$$O'^{in}(t)=WO'(t)W^+, \tag{2·8}$$

we can write $(2\cdot7)$ in such a form as

$$\begin{aligned}\Psi^+&=a_k'^{in*}\ \psi_N'^{in*}\ \Psi_{vac},\\ \Psi&=a_0'^{in*}\ \psi_N'^{in*}\ \Psi_{vac},\end{aligned} \tag{2·9}$$

where $*$ means the creation operators and Ψ_{vac} is defined by

$$\Psi_{vac}=W\ \Phi_{vac}. \tag{2·10}$$

These expressions for eigenvectors can be considered as an extension of the usual one written in terms of the incoming field-operators. It is worth while to mention that the operators a' and ψ_N' satisfy the condition $O'\mathcal{H}^{(1)}\subseteq\mathcal{H}^{(1)}$ but ψ_ν' does not. The algebra of $\psi_\nu'^{in}$ is not the same as ψ_ν', but it does not make our procedure difficult because we can construct all eigenvectors of H without using the operator $\psi_\nu'^{in}$ by virtue of the assumption $m_0>m_N+\mu$. As we expected in § 1, we see that $\Omega(E)\mathcal{H}^{(2)}$ actually represents a zero vector, that is,

$$W\Phi_\nu=0. \tag{2·11}$$

§ 3. Discussions

In Gupta's quantum electrodynamics,[17)-19)] he introduced the indefinite metric into the Hilbert space, by means of which he could succeed to get rid of the formal difficulty concerning the Lorentz condition at least in the case of the free radiation field. Recently Sunakawa[20)] extended Gupta's approach to the case of the system of charged particles interacting with the radiation field by using the so-called incoming field-representation under the assumption of the nonexistence of the bound states. Now from our viewpoint so far mentioned, it seems possible to apply Sunakawa's line of thought even to the system which involves the bound states.

As an example we shall consider a system of electrons and protons interacting with the radiation field. According to our classifications of the energy spectrum of H_0, the class which has the state with energy m_p+m_e as the lowest state is composed from states with one proton, one electron and n photons $(n=0,1,2,\cdots)$. It must be noted that the number of photons present is not effective for the classifications of the energy spectrum of H_0 owing to the zero rest mass of photons. It seems natural to consider that the ground state of a hydrogen atom corresponds to the lowest state of the class considered above, namely, to the state where one proton and one electron exist but no photons are present. Thus we see that the term of the energy renormalization $\int d\mu\delta_{\mu\nu}\,\Phi_\mu^{(1)}>\Delta<\Phi_\mu^{(1)}$ $(=H_0'-H_0)$ does not involve any photon operator. Accordingly the field equation for the operator A' in the intermediate representation is just the same one as that for the usual free operator. We now introduce the operator $W(t)$ by

$$\begin{aligned} W(t) &= e^{iHt}\ e^{-iH_0't} \\ &= 1+i\int_0^t W(t')\,V'(t')\,dt' \end{aligned} \tag{3·1}$$

where $V'(t)=e^{iH_0't}\,V'\,e^{-iH_0't}$. Let us take the limit $\lim\limits_{t\to-\infty} W(t)$ following Sunakawa's treatment for the usual wave matrix,[21)] then we get

$$W(-\infty)=1+i\int_0^{-\infty} W(t')\,V'(t')\,e^{\epsilon t'}\ dt'. \tag{3·2}$$

It is easily seen that $W(-\infty)$ is equal to our W. The following equations can be derived in similar manner as his proof:

$$\begin{aligned} A_\mu(x) &\equiv W(t)\,A_\mu'(x)\,W^+(t) \\ &= A_\mu'(x)-\int_0^t \Delta(x-x')\ j(x')\,dx' \end{aligned} \tag{3·3}$$

$$A_\mu'^{in}(x) \equiv W A_\mu'(x) W^{+}$$
$$= A_\mu'(x) + \int_{-\infty}^{0} \Delta(x-x')\, \boldsymbol{j}_\mu(x')\, dx' \tag{3·4}$$

and

$$\boldsymbol{A}_\mu(x) = A_\mu'^{in}(x) - \int_{-\infty}^{t} \Delta(x-x') \boldsymbol{j}_\mu(x')\, dx' \tag{3·5}$$

where we have used the fact that

$$W\, W^{+} = 1 \tag{3·6}$$

$$[A_\mu'(x),\ A_\nu'(x')] = i\delta_{\mu\nu}\Delta(x-x'). \tag{3·7}$$

Thus we see that Sunakawa's results can be completely reproduced even in the case of the system involving bound states with exceptions that in our case W is not unitary and A'^{in} or A' respectively appears in place of A^{in} or A in Sunakawa's articles. This situation is essentially due to the fact that in the derivation of eqs. (3·3)—(3·5), the relation

$$W^{+}\, W = 1 - P_{\mathcal{H}(e)}{}^{*)} \tag{3·8}$$

has not been used. Namely the expressions (3·3)—(3·5) do not need the unitarity of W but only the character (3·6). (3·7) is guaranteed by the fact that the field equation and the initial condition of our A' are the same as those of the usual free operator. By using the equation (3·5) and the adiabatic switching off procedures, the Lorentz condition is written as

$$(\partial A_\mu'^{in}/\partial x_\mu)^{(+)}\Psi = 0. \tag{3·9}$$

According to the present definition of the class of the energy spectrum of H_0, it is easy to see that $A'\mathcal{H}^{(i)} \subset \mathcal{H}^{(i)}$. Therefore from the remark stated in § 1, we see that the algebra $\{A'^{in}\}$ coincides with that of A'.**) Thus we arrive at the conclusion that Gupta's formalism can be applied even to the system with bound states, because we can obtain the same expression (3·9) for the Lorentz condition as that derived by Sunakawa in addition to the same commutation relations as the usual ones.***)

The creation operator for the ground state of the hydrogen atom is given as

$$W\psi_p'^{*}(p-k)\,\psi_e'^{*}(k)\,W^{\dagger}$$

where p is a total momentum of the ground state and k is a momentum which makes the energy $\sqrt{(p-k)^2+M_p^2}+\sqrt{k^2+M_e^2}$ minimal. The above expression makes it difficult to represent the ground state in the x-space because the equations for ψ_ν' and ψ_e' are different from that of the free fields owing to the fact that the term of the renormalization of the energy involves the operators of the material fields. It is due to the special character of the Lee model that the lowest state of the continuous spectrum can be written as $a_0' \psi_N'(p) \Phi_{vac}$ without regard to the total momentum p, and that $a_k'(k \doteqdot 0)$ satisfies the free equation because of the infinite mass of N-particle. According to this situation, we see that a_0' alone is responsible for making the bound state, while other $a_k'(k \neq 0)$'s can be transformed into $a'(x)$ in the x-representation which satisfies, of course, the free equation in x-space.

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* To any vector in the sub-space $\mathcal{H}^{(e)}$, there is no corresponding eigenvector of H.

** It seems natural to assume that, if there is an eigenvector of H which corresponds to one eigenvector of H_0 belonging to the class $\{\Phi^{(1)}\}$, there should be an eigenvector of H which corresponds to each eigenvector belonging to the class $\{\Phi^{(1)}\}$.

*** The same line of refinement of the proof in the case of the system involving bound states, can be made by using the Lorentz condition $(\partial A_\mu'^{in}/\partial x_\mu)\Psi = 0$ instead of (3·9) and the limiting process.[22]

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Nuclear Magnetic Relaxation in Liquid Hydrogen

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A theory of nuclear relaxation time T_1 in solid H_2 given by the present author and Motizuki is simply extended to the liquid case. In liquid, the distinctive feature is the diffusional motion of the molecules ($\tau_c \sim 10^{-12}$ sec.) which modulates the anisotropic intermolecular forces. The correlation functions of the rotational angular momentum, $\langle J_+(t) J_- \rangle$, etc...., which appear in the expression for T_1 are calculated by using the Kubo-Tomita formalism for motional narrowing adopting some simplifying assumptions. The calculated values of T_1 agree with Bloom's recent measurement in its order of magnitude and the qualitative character of the temperature and the concentration dependence.

§ 1. Introduction

Nuclear magnetic resonance in hydrogen molecules has been studied by several authors and much interesting information on the properties of hydrogen molecules, especially in their solid state, were obtained*. Recently, Bloom[1] measured precisely the relaxation time of the protons in hydrogen molecules related to their magnetic resonance absorption in the gaseous, liquid and solid states. The result in the gaseous state was analyzed by Schwinger's theory[1],[2] and the cross sections for collisions in which reorientations of molecules take place were obtained. As for the solid case, the experimental result seems to be well explained by the theory given by the present author and Motizuki in the preceding paper[3], referred to as I below. For the liquid case, however, the experimental result has not yet been fully explained, though we gave a simple qualitative explanation of it in the preceding paper I.

The most important nuclear relaxation mechanism in hydrogen molecules is due to the intra-molecular interactions, i.e., magnetic interactions between two nuclei and between the nuclei and the rotational magnetic moment, modulated by the reorientation of the molecule. In the gaseous state the reorientation of the molecules takes place by collisions while in the solid state it arises from the continuous interactions among the molecules. In the liquid state, the state of affairs is similar to that in the solid, though the inter-molecular interaction is modulated by the diffusional motion of the molecules. Bloom has calculated T_1 by using a correlation time for the reorientation of the molecules estimated from viscosity data following the B.P.P. theory[2]. The result was inconsistent with the measurements, i.e., the temperature and the concentration dependence were wrong**. This disagreement seems to be due to inadequate application of a simple classical model for H_2 molecules whose reorientation caused by intermolecular forces is much frequent without any effect of the diffusional motion or even in the solid state. For a correct treatment it is required to calculate quantum mechanically the correlation time for the reorientation of the molecules in terms of intermolecular forces. The object of the present note is to make such a calculation, simply extending our theory of relaxation time for solid hydrogen to the liquid case, and to make a comparison with Bloom's experiment.

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* For the bibliography see reference 3.

** The agreement in the order of magnitude seems to be due to the accidental fact that the anisotropic intermolecular interaction is of the order of $\hbar/\tau_c$.

Another relaxation mechanism due to inter-molecular dipole-dipole interactions will be negligible in the gaseous and the liquid state because of the very rapid diffusional motion of the molecules[1] ($\tau_c \leq 10^{-9}$ sec. in gas, $\tau_c \sim 10^{-12}$ sec. in liquid). This mechanism is important only in some cases in the solid state[3].

§ 2. Calculation of the relaxation time

A general expression for the nuclear relaxation time T_1 of H_2 molecule due to the intra-molecular interaction can be written as (see I)

$$\begin{aligned}\frac{1}{T_1}=\frac{1}{\hbar^2}\int_{-\infty}^{\infty}dt\, e^{-i\omega_0 t}\Big[&\frac{1}{4}(c-3d/2)^2\Big(\langle J_+(t)J_-\rangle+\langle J_-(t)J_+\rangle)\\ &+\frac{9}{8}d^2(\langle J_z(t)J_-(t)J_+J_z\rangle+\langle J_-(t)J_z(t)J_zJ_-\rangle\\ &\quad+\langle J_z(t)J_+(t)J_-J_z\rangle+\langle J_+(t)J_z(t)J_zJ_-\rangle\Big)\\ &+\frac{3}{8}d(c-3d/2)\Big(-\langle J_-(t)J_+J_z\rangle+\langle J_-(t)J_zJ_+\rangle-\langle J_z(t)J_-(t)J_+\rangle\\ &\quad+\langle J_-(t)J_z(t)J_+\rangle-\langle J_+(t)J_zJ_-\rangle+\langle J_+(t)J_-J_z\rangle\\ &\quad-\langle J_-(t)J_z(t)J_-\rangle+\langle J_z(t)J_-(t)J_-\rangle\Big)\\ &+\frac{9}{4}d^2\Big(\langle J_+^2(t)J_-^2\rangle+\langle J_-^2(t)J_+^2\rangle\Big)\Big],\end{aligned} \tag{1}$$

where $\boldsymbol{J}$ denotes the rotational angular momentum of a molecule, c and d the coupling constants for the $\boldsymbol{I}\cdot\boldsymbol{J}$ coupling and the dipole-dipole interaction between two protons in a molecule, respectively, and ω_0 the nuclear Larmor frequency.

In the case of solid we could calculate in I almost non-empirically the correlation functions appearing in (1) by using the intermolecular interaction Hamiltonian.

In the gaseous state, the reorientation of molecules takes place by collisions between two molecules. If we assume that all the correlation functions in (1) can be expressed by a unique correlation time τ_c in the form of $e^{-|t|/\tau_c}$, we get

$$\begin{aligned}\frac{1}{T_1}&=\frac{1}{\hbar^2}[(2c^2/3)+(15d^2/2)]\int_{-\infty}^{\infty}dt\, e^{-(|t|/\tau_c)-i\omega_0 t}\\ &=[(2c^2/3\hbar^2)+(15d^2/2\hbar^2)]\frac{2\tau_c}{1+\omega_0^2\tau_c^2}.\end{aligned} \tag{2}$$

This is the Schwinger formula.

In the case of liquid, with which we are now concerned, the anisotropic intermolecular forces, which give rise to the reorientations of the molecules, are reduced owing to the averaging effect of the translational diffusion of the molecules. As temperature rises the diffusional motion becomes more rapid, the reorientations of the molecules less frequent and T_1 shorter. This explains the experimental temperature dependence of T_1. In practice, we calculate the correlation functions by using the Kubo-Tomita formalism[4] for motional narrowing. The Hamiltonian for the rotational and the translational motions are written as $\mathfrak{H}_R$ and $\mathfrak{H}_T$, respectively. Then the correlation functions $\langle J_+(t)J_-\rangle$ etc.... become

$$\langle J_+(t)J_-\rangle=\langle J_+J_-\rangle e^{-\psi(t)}, \tag{3}$$

$$\psi(t)=\sigma^2\int_0^t (t-\tau)f(\tau)\,d\tau, \tag{4}$$

$$\sigma^2=\frac{\langle[[J_+,\mathfrak{H}_R],\mathfrak{H}_R]J_-\rangle}{\hbar^2\langle J_+J_-\rangle}, \tag{5}$$

$$f(t)=\frac{\langle[[J_+,\mathfrak{H}_R(t)],\mathfrak{H}_R(0)]J_-\rangle}{\langle[[J_+,\mathfrak{H}_R],\mathfrak{H}_R]J_-\rangle},\ \text{etc.}\ldots, \tag{6}$$

where

$$\mathfrak{H}_R(t)=e^{it\mathfrak{H}_T/\hbar}\mathfrak{H}_R e^{-it\mathfrak{H}_T/\hbar}. \tag{7}$$

We shall now put a simplifying assumption, as in the case of gas, that for all the correlation functions in (1) the values of σ are equal and $f(t)$ are also equal. In solid, $f(t)=1$ and the values of σ are almost equal for all the correlation functions[3], so that this approximation will also be good for liquid. Eq. (1) is then simplified as

$$\begin{aligned}1/T_1&=[(2c^2/3\hbar^2)+(15d^2/2\hbar^2)]\int_{-\infty}^{\infty}dt\,e^{-\psi(t)-i\omega_0 t}\\&=1.326\times10^{12}\int_{-\infty}^{\infty}dt\,e^{-\psi(t)-i\omega_0 t}.\end{aligned} \tag{8}$$

The numerical values of c and d are seen in I. We shall further assume simply

$$f(t)=e^{-|t|/\tau_0}, \tag{9}$$

and get

$$\psi(t)=\sigma^2\tau_0^2[\exp(-|t|/\tau_0)-1+|t|/\tau_0]. \tag{10}$$

Inserting (10) in (8), we get

$$1/T_1=3.324\times10^{12}F(\sigma\tau_0)/\sigma, \tag{11}$$

where

$$F(x)=\sqrt{\frac{2}{\pi}}e^{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n}}{x+(n/x)}. \tag{12}$$

Numerical values of $F(x)$ are shown in Fig. 1.

We have now only to evaluate σ and τ_0. τ_0 can be obtained from viscosity data using the relation $\tau_0 = \delta^3 \eta / 6kT^{5)}$, where η is the viscosity and δ the mean length of a jump of a molecule in the liquid. The temperature dependence of τ_0 is the same with that of η/T which is shown in Fig. 2.* Though the order of magnitude of τ_0 is estimated as $\sim 10^{-12}$ sec., it will be better to determine δ so as to make the calculated T_1 fit the experimental data. The value of σ will be of the same order of magnitude as in the case of solid. In the case of solid, σ^2 is proportional to the ortho-concentration K and T_1 to $\sqrt{K}$ as was seen in I. In the liquid case, we may write

$$\sigma^2 = \omega_{R0}^2 + \omega_R^2 K, \quad (13)$$

taking account of the intermolecular interaction common to the ortho-ortho and ortho-para interactions, which we neglected in the solid case because

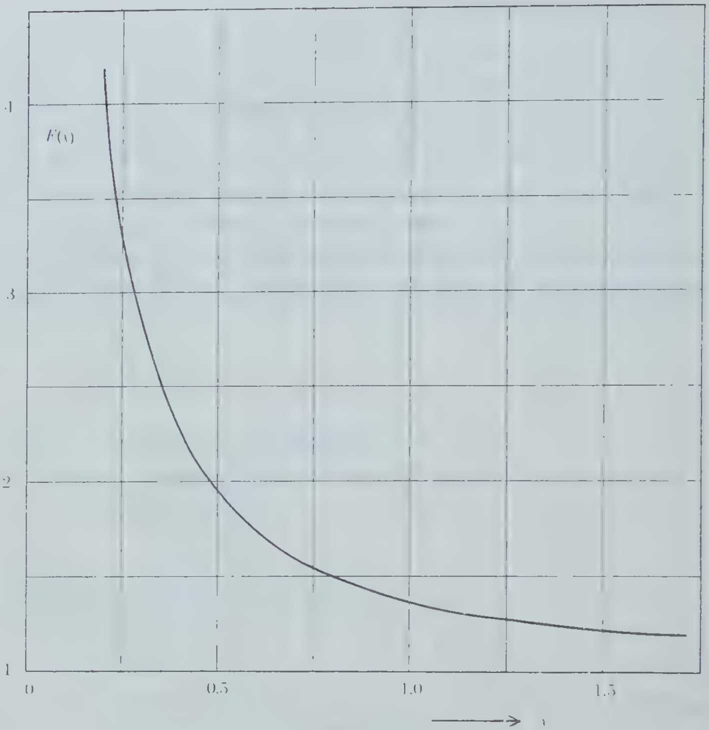

Fig. 1.

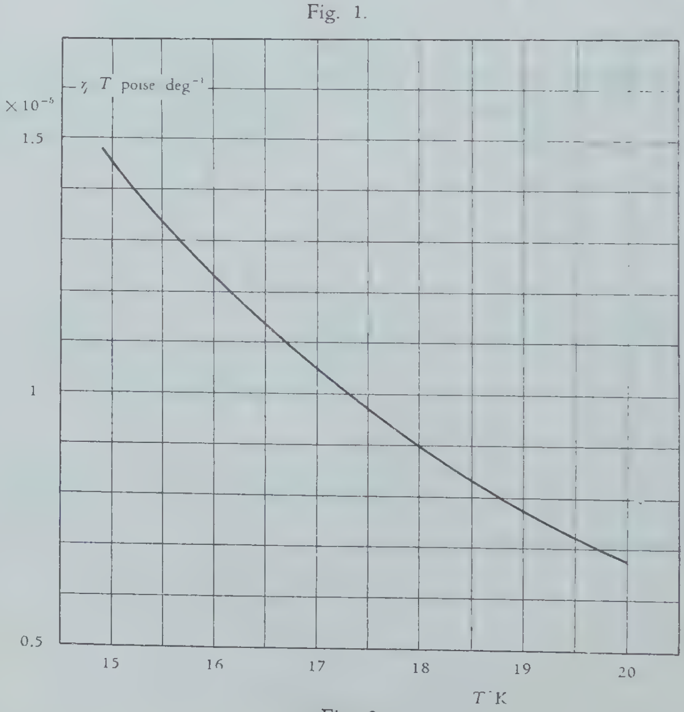

Fig. 2.

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* The values of viscosity were taken from the compilation of the National Bureau of Standards [Vol 41 (1948), 379].

of the cancellation due to the hexagonal close packed arrangement of the molecules (see I). ω_{R0} and ω_R can, in principle, be calculated adopting a proper model for liquid hydrogen. Actually, in solid we saw that $\omega_R \simeq 10^{12}$ by calculating it in terms of the intermolecular forces between ortho-molecules. In the present case we shall determine ω_{R0} and ω_R so as to make the calculated T_1 fit the experimental data within the reasonable range of their magnitudes.

We shall choose the following values

$$\tau_0 = 2.02 \times 10^{-12}\ \text{sec.}$$

$$\text{at } 15^\circ\text{K}, \ (\tau_0 \propto \eta/T),$$

$$\omega_R = 0.913 \times 10^{12},$$

$$\omega_{R0}^2\ \ \omega_R^2 = 0.058. \quad (14)$$

The calculated values of T_1 using the above values are shown in Fig. 3. The order of magnitude and the temperature and the concentration dependence of T_1 accord well with experiment considering that the theory is very simple. The values of the parameters seem to be reasonable.

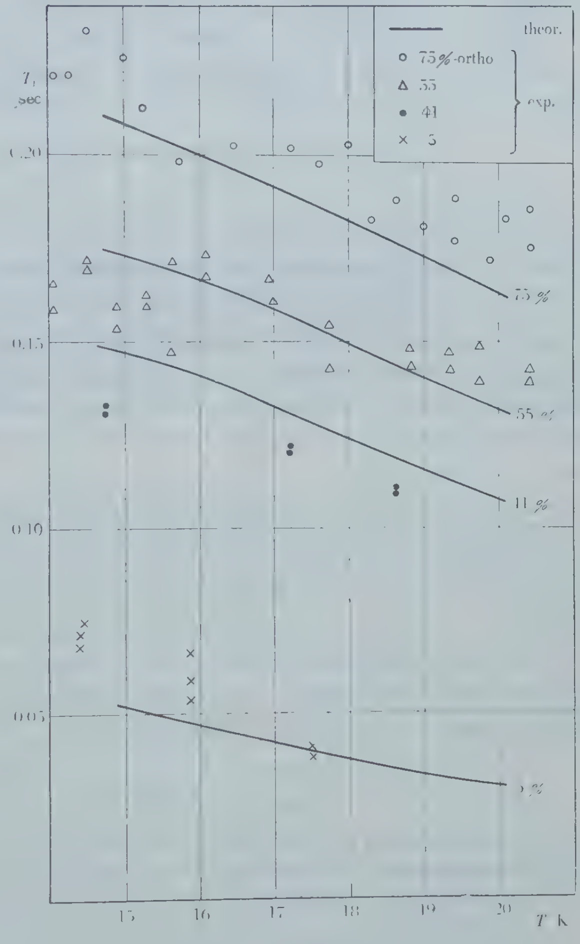

Fig. 3.

Though the values of T_1 change somewhat when we change the values of the parameters within their reasonable range, the order of magnitude and the qualitative characters of the temperature and the concentration dependence are not altered.

§ 3. Discussions

We shall in this section discuss on the rather small quantitative discrepancy between the calculated values and the experimental ones. The discrepancy is not much improved by any choice of the numerical values of the parameters. If we choose such numerical values of the parameters as to make better fit in the temperature dependence of T_1, the fit in the concentration dependence becomes worse, and vice versa. Bloom's data at 17.5°K show the concentration dependence like

$$T_1 = a + bK,$$

which makes us expect that $\sigma\tau_0 \ll 1$. However, when this is the case T_1 should be proportional to η/T, resulting in too strong temperature dependence of T_1.

To improve the agreement, it seems necessary to refine the theory by considering more precisely the state of liquid hydrogen, though we shall not here get into this direction. The simplest refinement of our assumption, $f(t) = e^{-|t|/\tau_0}$, will be to assume

$$\psi(t) = \int_0^t (t-\tau)\,(\omega_{R0}^2 e^{-|\tau|/\tau_1} + \omega_R^2 K e^{-|\tau|/\tau_0})\,d\tau\,,$$

where τ_0 and τ_1 are naturally different. When $\tau_1 < \tau_0$, the temperature dependence of T_1 at low ortho-concentrations will become stronger than is expected from the theory in § 2. Further improvement will be expected if we adopt a model of isotropic diffusion such as used by Torrey[6] in improving the B.P.P. theory for T_1 due to the translational diffusion. In practice, however, theory becomes complicated especially in treating the intermolecular interaction common to the ortho-ortho and ortho-para interactions, and the numerical calculation will be very much laborious. We shall here content ourselves with the simple theory in § 2.

Bloom measured also the relaxation times for protons in HD and for protons in H_2 in liquid sample containing approximately 94% HD, 3% H_2 and 3% D_2. The former has been discussed by Bloom and the latter seems to be not inconsistent with the present calculation.

The author should like to thank Dr. M. Bloom for informing him of his experimental data prior to publication. He is also indebted to Prof. R. Kubo and Prof. M. Shimizu for their kind interest and discussions.

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Description of Pauli Matter as a Continuous Assembly of Small Rotating Bodies*

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The method of Kramers to represent a two-component spinor is revised and developed into a consequent formulation of the quantum mechanical theory of a non-relativistic spinning particle (§ 4).

Furthermore the method is unified with the hydrodynamical representation of the Pauli electron theory established in the previous article, giving the picture of the Pauli field as an assembly of very small rotating bodies (distinct from usual rigid bodies) continuously distributed in space (§ 3). The formalistic and physical features of these new representations are considered.

Also some remarks are given on the characteristics of the former hydrodynamical formulation (§ 2). Particularly it is pointed out that Planck's constant enters this scheme only as the constant signifying the magnitude of the spin.

§ 1. Introduction

The quantum-mechanical theory of a non-relativistic particle with spin is ordinarily constructed by combining the general principles of quantum mechanics with the concept of a two-component spinor. On the other hand it was shown in our previous paper[1)] that the same theory can be constructed in quite a different realistic manner, i. e., we can formulate the theory in a form of a specified hydrodynamics of a fluid carrying intrinsic angular momentum to be called "spin". In this scheme, the assumptions of the general concepts of quantum mechanics and the properties of a spinor in the usual formulation are replaced by a coherent set of assumptions to define the physical properties of the underlying model. It is observed furthermore that Planck's constant is then introduced into the scheme only as the constant signifying twice the magnitude of the spin. The theory takes form of a classical one and is interpretable as such. It must, however, be emphasized that the model possesses certain novel characteristics different from any *ready-made* classical model, though they are in themselves consistent and coherent: The first is the condition which connects the vorticity of the hydrodynamical field to the gradients of the spin field in a specified manner, and the second is the fact that the adjacent spins interact in a manner of the "exchange interaction" type.

Now the main purpose of this paper is to give another "realistic" representations to describe the same theory (quantum mechanics of a Pauli particle). The new representation is the development of the Kramers' idea[2] of representing a two-component spinor with an orthogonal set of coordinate axes, and it can be unified with the above mentioned hydrodynamical scheme in a very natural way. Then the method depicts the Pauli field with the motion of an assembly of very small rotating bodies continuously distributed in space,* and thus amounts to giving a more concrete model to the "particles" constituting the fluid defined in the hydrodynamical representation. For instance, the "intrinsic" angular momentum of particle is now interpreted as the angular momentum of rotation of such body. It must be noted that by "rotating body" we only mean a certain body whose rotational orientation of internal rotation has a definite meaning and do not assume farther properties of the usual concept of rigidity. Really our rotating body has a novel property distinct from a customary rigid body: The direction of the angular momentum of rotation is fixed to the body, while the angular velocity of rotation points to the direction of the "effective magnetic field" acting on it and fluctuates over the body. (§ 5).

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* This work was completed in March 1957. Cf. Séminaire Louis de Broglie (1956/57), Exposé n° 3, p. 4.

Formally speaking the present theory is characterized by first the introduction of a new variable Λ which is conjugate to the density ρ and serves to secure the independency of the model of an arbitrary *gauge-transformation,* and second the resolution of the spin variables S_i, $(i=1\ 2\ 3)$, into *conjugate pairs* $\{a_i^{(1)},\ a_i^{(2)}\}$ such that

$$\boldsymbol{S}=\hbar/2\cdot[\boldsymbol{a}^{(1)}\times\boldsymbol{a}^{(2)}], \qquad \boldsymbol{a}^{(1)}\perp\boldsymbol{a}^{(2)}.$$

It is important that this theory realizes the new *canonical formalism* of the Pauli field keeping to the realistic representation. Thus it will allow us to make the second quantization of the theory in such form. The result will then represent the many particle system and would be useful for various practical problems.

§ 2. Remarks on hydrodynamical representation

a) *Brief review*

It is convenient first to recapitulate the hydrodynamical scheme previously established[1] for the sake of future references. The usual description of the quantum-mechanical behaviour of a single non-relativistic spinning particle is given by a two-component spinor function $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, which changes according to the Schrödinger-Pauli equation:

$$(\hbar/i\cdot\partial_0+\underline{H})\psi=0, \qquad (\partial_0\equiv\partial/\partial t) \tag{2·1}$$

with the hamiltonian

$$\underline{H}=\frac{1}{2m}\left(\underline{\boldsymbol{p}}-\frac{e}{c}\underline{\boldsymbol{A}}\right)^2+e\underline{A_0}-\frac{e\hbar}{2mc}\underline{\boldsymbol{H}}\sigma, \tag{2·2)**}$$

where $(\boldsymbol{A},\ A_0)$ is the external electromagnetic potential, $\boldsymbol{H}=\mathrm{curl}\,\boldsymbol{A}$ the magnetic field strength, and σ the Pauli matrices.

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* Similar picture was given by Bohm, Schiller and Tiomno[2], who investigated hydrodynamical aspect of the Pauli equation on the basis of the Euler angle representation of ψ.

** Underlined are q-numbers. We are considering, for simplicity, in the lowest non-relativistic approximation.

This theory can equivalently be grasped as a special hydrodynamics with intrinsic angular momentum, in which first the Pauli spinor ψ is replaced by a set of scalar and vector quantities, i. e., a scalar ρ, an axial vector $\boldsymbol{S}$, and a vector $\boldsymbol{v}$:

$$\{\rho, \boldsymbol{v}, \boldsymbol{S}\}. \tag{2·3}$$ *

They are all real and gauge-independent quantities, and are restricted by following subsidiary conditions:

$$\begin{cases} \boldsymbol{S}^2 = \hbar^2/4, & \text{(I)} \\ \operatorname{curl}_i \boldsymbol{v} = \dfrac{4}{m\hbar^2}\varepsilon_{lmn} S_l \partial_j S_m \cdot \partial_k S_n - \dfrac{e}{mc} H_i, (ijk) \sim (123). & \text{(II)}^{**} \end{cases}$$

Those quantities are regarded as defining a spinning hydrodynamical field; ρ, $\boldsymbol{v}$ and $\boldsymbol{S}$ meaning the density, velocity, and spin (i. e., intrinsic angular momentum in classical sense) of the hydrodynamical field respectively. The condition (I) simply means that the spin is of a constant magnitude $\hbar/2$ everywhere; while (II) shows that the vorticity of the orbital motion stems, apart from the "Larmor part" $-(e/mc)\boldsymbol{H}$ which is due to external magnetic field, only from the spatial variation of the spin field through a specified relation.

Corresponding to the evolution of ψ, (2·1), of the original Pauli-theory, the basic field quantities (2·3) obey the following set of equations of motion:

$$\begin{cases} \partial_0 \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0, & \text{(III)} \\ m\dot{v}_i = K_i + \dfrac{e}{mc}\partial_i H_k \cdot S_k - \partial_i \Pi^{(\rho)} - \dfrac{1}{m\rho}\partial_k(\rho \partial_i S_l \cdot \partial_k S_l), & \text{(IV)} \\ \dot{\boldsymbol{S}} = \dfrac{e}{mc}[\boldsymbol{S} \times \boldsymbol{H}] + \dfrac{1}{m\rho}[\boldsymbol{S} \times \partial_k(\rho \partial_k \boldsymbol{S})], & \text{(V)} \end{cases}$$

where

$$\cdot \equiv d/dt \equiv \partial_0 + \boldsymbol{v}\nabla \tag{2·4}$$

is the time derivative along the motion of each element of the fluid, and

$$\boldsymbol{K} = e\boldsymbol{E} + (e/c)[\boldsymbol{v} \times \boldsymbol{H}], \tag{2·5}$$

$$(\text{with } E_i = -(1/c)\partial_0 A_i - \partial_i A_0 = \text{electric field strength}) \tag{2·6}$$

is the Lorentz force acting on the element of the fluid. Also we used the abbreviation:

$$\Pi^{(\rho)} = \frac{\hbar^2}{4m}\left\{\frac{(\nabla\rho)^2}{2\rho^2} - \frac{\Delta\rho}{\rho}\right\} = -\frac{\hbar^2}{2m}\frac{\Delta(\rho^{1/2})}{\rho^{1/2}}. \tag{2·7}$$

The energy density of the hydrodynamical field is given by:

$$\mathscr{H} = \rho\left\{\frac{1}{2}m\boldsymbol{v}^2 + eA_0 - \frac{e}{mc}\boldsymbol{H}\boldsymbol{S} + \frac{\hbar^2}{8m}\frac{(\nabla\rho)^2}{\rho^2} + \frac{1}{2m}|\nabla\boldsymbol{S}|^2\right\}. \tag{VI}$$ ***

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* We use the same notations as in the previous paper[1] excepting ρ and $\boldsymbol{S}$ [appearing in eq. (2·9)] which were written formerly as P and (σ) respectively.

** ε_{lmn} denotes perfectly antisymmetrical symbol of the third order with $\varepsilon_{123}=1$; $(ijk) \sim (123)$ signifies that (ijk) is an even permutation of (123). Summation convention for repeated indices is understood unless otherwise stated. Finally $\partial_k = \partial/\partial x_k$.

*** $|\nabla\boldsymbol{S}|^2 \equiv \partial_k S_l \cdot \partial_k S_l$.

Eq. (III) is simply the equation of continuity. If we associate the magnetic moment,

$$\mu = (e/mc)\boldsymbol{S} \qquad (2\cdot 8)$$

to the spin $\boldsymbol{S}$ of each particle constituting the fluid, then the 2nd term of R. H. S. of (IV), the 1st term of R. H. S. of (V), and the 3rd term of R. H. S. of (VI), represent the usual effects due to the coupling between μ and $\boldsymbol{H}$; i. e., the Stern-Gerlach force, the torque causing the precession of μ by the magnetic field, and the potential energy, $-\mu\boldsymbol{H}$, respectively. On the other hand the 3rd and 4th terms in R. H. S. of (IV), the 2nd term in R. H. S. of (V), and the 4th and 5th terms in R. H. S. of (VI) correspond to quantum-mechanical effects separated from the purely classical effects, each of them being the order of $\hbar^2$. We have stated several different points of view to grasp them in the previous paper. Now the following part of this section is devoted to some further considerations to interpret the character of those effects.

b) *Internal potential and internal magnetic field*

Using the spin density,

$$\boldsymbol{s} = \rho \boldsymbol{S} \qquad (2\cdot 9)^{\dagger}$$

we introduce two quantities $\boldsymbol{H}'$ and Π^* by

$$\boxed{\begin{aligned} H_i' &\equiv \frac{c}{e}\partial_k\left(\frac{1}{\rho}\partial_k s_l\right), \\ \Pi^* &\equiv \frac{1}{2m\rho^2}|\nabla \boldsymbol{s}|^2. \end{aligned}} \qquad \begin{aligned} &(2\cdot 10) \\ &(2\cdot 11)^{\dagger\dagger} \end{aligned}$$

We also define $\boldsymbol{H}^*$ by:

$$\boldsymbol{H}^* = \boldsymbol{H} + \boldsymbol{H}'. \qquad (2\cdot 13)$$

Then eqs. (IV), (V), and (VI) are cast in more compact forms as follows:

$$\boxed{\begin{aligned} m\dot{v}_i &= K_i + \partial_i \Pi^* + \frac{e}{mc}\partial_i H_k^* \cdot S_k, \\ \dot{\boldsymbol{S}} &= \frac{e}{mc}[\boldsymbol{S}\times\boldsymbol{H}^*], \\ \mathcal{H} &= \rho\left(\frac{1}{2}m\boldsymbol{v}^2 + eA_0 - \frac{e}{mc}\boldsymbol{HS} + \Pi^*\right). \end{aligned}} \qquad \begin{aligned} &(\mathrm{IV}'') \\ &(\mathrm{V}'') \\ &(\mathrm{VI}') \end{aligned}$$

[†] This quantity of course satisfies the condition

$$\boldsymbol{s}^2 = (\hbar^2/4)\rho^2. \qquad (\mathrm{I}')$$

[††] It is easily verified that between $\boldsymbol{H}'$ and Π^* there holds the relation:

$$2\rho\Pi^* = -\frac{e}{mc}\boldsymbol{H}'\boldsymbol{s} + \frac{\hbar^2}{4m}\Delta\rho, \qquad (2\cdot 12)$$

due to (I′).

Furthermore, (VI′) can equally be expressed by the help of (2·12) as

$$\mathcal{H}=\mathcal{H}'+(\hbar^2/4m)\,\Delta\rho, \tag{2·14}$$

with

$$\mathcal{H}'=\rho\Big(\frac{1}{2}m\boldsymbol{v}^2+eA_0-\frac{e}{mc}\boldsymbol{H}^*\boldsymbol{S}-\Pi^*\Big), \tag{VI″}$$

but the term $(\hbar^2/4m)\,\Delta\rho$ in (2·14) does not contribute to the total energy $\overline{\mathcal{H}}=\int\mathcal{H}\,d\boldsymbol{x}$, and so we may take $\mathcal{H}'$, in place of $\mathcal{H}$, to be the energy density.

Now the forms, (IV″), (V″), and (VI″) exhibit that the motion of the fluid is conducted in such a way that each element of the fluid moves like a classical particle subjected, besides to the usual Lorentz force, to the actions of *internal scalar potential* $-\Pi^*$ and also of total *effective magnetic field* $\boldsymbol{H}^*$ which acts on the magnetic moment (2·8).

Furthermore it is important that Planck's constant $\hbar$ disappears in (2·10), (2·11), (IV″), (V″), (VI′), or (VI″), entirely. This means that the quantum effects are represented by the occurring of the internal potential Π^* and a kind of *internal magnetic field* $\boldsymbol{H}'$, which themselves are determined by the gradients of $\boldsymbol{s}$-field, as indicated in (2·10) and (2·11), and can be regarded as classical actions without involving $\hbar$. Planck's constant enters into this scheme only through the constraint (I), i. e., as the magnitude of spin. It is particularly a remarkable situation that in the hydrodynamical representation of a simpler Schrödinger particle (without spin) there appears the "quantum potential" $\Pi^{(\rho)}$ of the form (2·7) involving $\hbar$, whereas in the present case (Pauli particle) this $\Pi^{(\rho)}$ disappears, being absorbed into the classical quantity Π^*.

We here remark that the above separation of the quantum effects into the "internal potential" and the "internal magnetic field" is arbitrary to the following extent. Namely the equations (IV″), (V″), and (VI″) are invariant to the transformation from ($\boldsymbol{H}'$, Π^*) to ($\boldsymbol{H}_1'$, Π_1^*) by

$$\begin{cases}\boldsymbol{H}_1'=\boldsymbol{H}'+f\boldsymbol{S}, & (2·15)\\[1ex] \Pi_1^*=\Pi^*-\dfrac{\hbar^2}{4}\,\dfrac{e}{mc}f, & (2·16)\end{cases}$$

where f is an arbitrary scalar function which may depend on ρ, $\boldsymbol{v}$, and $\boldsymbol{S}$. If we choose f such that $f=c/e\cdot\{(\nabla\rho)^2/\rho^2-\Delta\rho/\rho\}$, then the resulting Π_1^*, $\boldsymbol{H}_1'$ and $\boldsymbol{H}_1^*=\boldsymbol{H}+\boldsymbol{H}_1'$ are exactly the internal potential, internal and effective magnetic fields which were adopted in the previous paper denoted as $-\Pi$, $\boldsymbol{H}^{\text{in}}$, and $\boldsymbol{H}^{\text{eff}}$ respectively:

$$\begin{cases}\Pi=\Pi^{(\rho)}-|\nabla\boldsymbol{S}|^2/(2m), & (2·17)\\[1ex] \boldsymbol{H}^{\text{in}}=\dfrac{c}{e\rho}\,\partial_k(\rho\partial_k\boldsymbol{S}), \qquad \boldsymbol{H}^{\text{eff}}=\boldsymbol{H}+\boldsymbol{H}^{\text{in}}. & (2·18)\end{cases}$$

We adopt, however, the form (2·10) and (2·11) which is the most suitable one to physical interpretations.

c) *Comparison with known interactions*

The interactions working inside the fluid have thus been represented by the quantities, internal scalar potential Π^* and internal magnetic field $\boldsymbol{H}'$.

On the other hand we have associated the magnetic moment (2·9) to each particle of the fluid. This must mean that the fluid has the *magnetization*

$$\vec{\mathcal{M}}=\rho\mu=\frac{e}{mc}\boldsymbol{s}. \tag{2·19}$$

Now, the classical electromagnetic theory tells us that generally a distribution of magnetization $\vec{\mathcal{M}}(\boldsymbol{x})$ produces the magnetic field,

$$\boldsymbol{H}_i^{(m)}(\boldsymbol{x})=\frac{1}{4\pi}\int\mathcal{M}_k(\boldsymbol{x}')\,\partial_i\partial_k\Big(\frac{1}{r}\Big)d\boldsymbol{x}', \tag{2·20}$$

$$=\frac{1}{4\pi}\int\partial_i'\partial_k'\mathcal{M}_k(\boldsymbol{x}')\cdot\frac{1}{r}\,d\boldsymbol{x}', \tag{2·20$'$}$$

$$(r\equiv|\boldsymbol{x}-\boldsymbol{x}'|,\ \partial_i'=\partial/\partial x_i')$$

and accordingly accumulates the total energy of magnetic dipole interaction

$$\overline{\mathcal{H}}^{(m)}=-\frac{1}{4\pi}\cdot\frac{1}{2}\iint\mathcal{M}_i(\boldsymbol{x})\,\mathcal{M}_k(\boldsymbol{x}')\,\partial_i\partial_k\Big(\frac{1}{r}\Big)d\boldsymbol{x}\,d\boldsymbol{x}'. \tag{2·21}$$

We see then that our internal magnetic field, which may be written :

$$H_i'=\frac{1}{r_0}\partial_k\Big(\frac{1}{\rho}\partial_k\mathcal{M}_i\Big), \tag{2·10$'$}$$

$$(r_0=e^2/mc^2=\text{classical electron radius})$$

in terms of $\vec{\mathcal{M}}$ of (2·19) has a dependence on $\vec{\mathcal{M}}$ different from (2·20′) especially as this is a locally determined quantity. In similar manner the energy density of our field due to the internal interaction,

$$\rho\Pi^*=\frac{1}{2r_0}\frac{1}{\rho}|\nabla\vec{\mathcal{M}}|^2, \tag{2·11$'$}$$

makes up a total internal energy differing from (2·21).

On the other hand our interaction represented by (2·10′) and (2·11′) have similar forms with that of the *exchange interaction*, which has an origin different from the dynamical coupling between magnetic dipoles, arising as the result of quantum-mechanical treatment of many electron systems. Herring and Kittel[4)] indeed introduced the interaction of the type of (2·11′) in their phenomenological approach to spin waves in ferromagnetic media.

Roughly speaking the features of our hydrodynamics are first the kinematical condition (II) which defines the relation between vorticity and spin, and second the fact that the neighbouring spins interact in the manner of exchange interaction type.

d) It would be better to insert here the following remark. In our hydrodynamical theory the main subject was to prove the equivalency between the *c*-number Pauli field and a certain hydrodynamical field having a definite physical structure and therefore the theory provides us with a new realistic picture of the behaviour of electron in nature to the extent that this may be reflected by the *c-number Pauli field*. It is straightforward to illustrate this fact for special examples such as scattering, interference, etc.

On the other hand the behaviour of a quantum-mechanical Pauli particle has aspects connected with the eigenvalue problems and the problem of measurement, there is also the problems related to the regularity conditions of the ψ-function.

The applicability of the hydrodynamical scheme has partly been examined in the previous paper,1) i. e., concerning the problem of expectation values of various physical quantities. More sufficient study may be carried through along the pattern which was given for the case of a Schrödinger particle.5) It is obvious, however; that it then comes out that the hydrodynamical field, for the representation of a single quantum-mechanical particle, must be taken in a somewhat fictitious sense (i. e., as the probability fluid), at least in so far as we keep to the usual interpretation of quantum mechanics.

On the other hand the hydrodynamical field will attain more real sense for the representation of many Pauli particle system, although in this case it becomes a field of operator and so is associated with quantum fluctuations.

§ 3. The Lagrangian and unified representation

Now we enter the subject of introducing new variables into our hydrodynamical scheme to make physical model more concrete, attaining at the same time the variational principle of the theory.

a) *Introduction of new variables*

In our hydrodynamical formalism the spinor ψ was represented by the set of scalar and vector quantities $(2\cdot3)$, which were related to ψ as bilinear quantities in ψ^* and ψ, meaning density, velocity, and spin. On the other hand Kramers2) pointed out that the Pauli spinor ψ can be represented by a complex zero-vector $\boldsymbol{\xi}$ which is related to ψ as a bilinear quantity in ψ and its contragradient,

$$\tilde{\psi}=(\psi^1,\ \psi^2)=(\psi_2,\ -\psi_1), \tag{3·1}$$

such that

$$\boldsymbol{\xi}=\frac{\hbar}{2}\tilde{\psi}\sigma\psi. \tag{3·2}$$

This $\vec{\xi}$ assignes an orthogonal coordinate system to each point in space. Namely, if we put

$$\boldsymbol{\xi}=\beta^{(1)}+i\beta^{(2)},\quad (\beta^{(1)},\ \beta^{(2)}:\text{real}) \tag{3·3}$$

then $\vec{\beta}^{(1)}$ and $\vec{\beta}^{(2)}$, together with the spin density vector s of $(2\cdot9)$ constitute a mutually orthogonal set of vectors, each having the same magnitude $(\hbar/2)\rho$. We can thus represent the spinor ψ by the orientation of that orthogonal system, together with ρ.

Kramers, however, could not go farther. We shall now develop this idea into a consistent formulation of the Pauli theory and also combine it with the hydrodynamical method

It is important first to revise the above definitions of $\boldsymbol{\beta}^{(1)}$ and $\boldsymbol{\beta}^{(2)}$, since these ar not gauge-independent quantities. For that purpose we put

$$\psi=\exp(i\Lambda/\hbar)\cdot\psi', \tag{3·4}$$

and regard that Λ and ψ' above transform, under a gauge-transformation

$$A_k\to A_k+\partial_k\lambda, \quad A_0\to A_0-\frac{1}{c}\partial_0\lambda, \tag{3·5)*}$$

like

$$\begin{cases} \Lambda\to\Lambda+\dfrac{e}{c}\lambda, \\ \psi' \text{ : invariant.} \end{cases} \tag{3·6}$$

We then define the corresponding quantities ($\boldsymbol{\xi}'$, $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\alpha}^{(2)}$) by the relations similar to (3·1), (3·2) and (3·3), using here ψ' in place of ψ, such that**

$$\tilde{\psi}'=(\psi_2', \ -\psi_1'), \tag{3·7}$$

$$\xi'=\frac{\hbar}{2}\tilde{\psi}'\sigma\psi' \tag{3·8}$$

$$=\boldsymbol{\alpha}^{(1)}+i\boldsymbol{\alpha}^{(2)}, \quad (\boldsymbol{\alpha}^{(1)}, \ \boldsymbol{\alpha}^{(2)} \text{ : real}) \tag{3·9}$$

The ψ is now represented by the set

$$\{\Lambda, \ \boldsymbol{\alpha}^{(1)}, \ \boldsymbol{\alpha}^{(2)}\}. \tag{3·11}$$

If we denote the unit vectors in the directions of $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\alpha}^{(2)}$ and $\mathbf{s}$, as $\mathbf{a}^{(\xi)}$, ($\xi=123$), these $a_k^{(\xi)}$, (ξ, $k=123$) signify the coefficients of the orthogonal transformation transferring the fixed laboratory coordinate system into that defined by ($\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\alpha}^{(2)}$, $\mathbf{s}$). Thus $a_k^{(\xi)}$ satisfy the following relations

$$\begin{cases} \boldsymbol{\alpha}^{(\xi)}=\dfrac{\hbar}{2}\rho\mathbf{a}^{(\xi)}, \quad (\xi=1, \ 2) & (3·12) \\ \mathbf{S}=\dfrac{\hbar}{2}\mathbf{a}^{(3)}, & (3·13) \end{cases}$$

$$\begin{cases} a_i^{(\xi)}a_j^{(\xi)}=\delta_{ij}, & (3·14)^{***} \\ a_i^{(\xi)}a_i^{(\eta)}=\delta_{\xi\eta}. & (3·15) \end{cases}$$

---

* (A_k, A_0) is of course the external electromagnetic potential acting on the particle.

** The new quantities are related to the former quantities by

$$\left.\begin{aligned} &\xi'=\exp(-2i\Lambda/\hbar)\cdot\boldsymbol{\xi}, \\ &\boldsymbol{\alpha}^{(1)}=\boldsymbol{\beta}^{(1)}\cos(2\Lambda/\hbar)+\boldsymbol{\beta}^{(2)}\sin(2\Lambda/\hbar), \\ &\boldsymbol{\alpha}^{(2)}=-{}^{(1)}\boldsymbol{\beta}\sin(2\Lambda/\hbar)+\boldsymbol{\beta}^{(2)}\cos(2\Lambda/\hbar). \end{aligned}\right\} \tag{3·10}$$

*** It is to be noted that ξ in $a_k^{(\xi)}$ is not a vector suffix. The summation convention is applied to this suffix also when repeated.

We also note the relation

$$\boldsymbol{s}=[\boldsymbol{a}^{(1)}\times\boldsymbol{a}^{(2)}]. \tag{3·16}$$

On account of (3·14), $(a_k^{(\xi)})$ implies 3 independent quantities, and we can say that ψ is represented by the set

$$\{\Lambda,\ \rho,\ a_k^{(\xi)}\},\quad (\xi,\ k=123). \tag{3·17}$$

In this method $a_k^{(\xi)}$ are gauge-independent, only Λ changing according to (3·6). Eq. (3·4) with the condition (3·6), however, does not define Λ in a unique manner. When we transform (3·17) by the relations :

$$\left.\begin{aligned}
&\Lambda\to\Lambda'=\Lambda+\frac{\hbar}{2}\gamma,\\
&\boldsymbol{a}^{(1)}\to\boldsymbol{a}^{(1)\prime}=\cos\gamma\cdot\boldsymbol{a}^{(1)}+\sin\gamma\cdot\boldsymbol{a}^{(2)},\\
&\boldsymbol{a}^{(2)}\to\boldsymbol{a}^{(2)\prime}=-\sin\gamma\cdot\boldsymbol{a}^{(1)}+\cos\gamma\cdot\boldsymbol{a}^{(2)},\\
&\qquad(\rho,\ \boldsymbol{a}^{(3)}\ \text{: invariant})
\end{aligned}\right\} \tag{3·18}$$

then $\{\Lambda',\ \rho,\ a_k^{(\xi)\prime}\}$ represents the same physical state as (3·17). Thus the values of $\{\Lambda,\ \boldsymbol{a}^{(1)},\ \boldsymbol{a}^{(2)}\}$ have the arbitrariness by one scalar function $\gamma(\boldsymbol{x},\ t)$, which means the angle of rotation of $\boldsymbol{a}^{(1)}$, $\boldsymbol{a}^{(2)}$ axes in the plane normal to $\boldsymbol{a}^{(3)}$. Every physical result must of course be invariant to the transformation (3·18).

Since we can thus represents a state of a quantum-mechanical spinning particle by (3·17), it must also be possible to represent the equation of motion in terms of these variables in a closed form. We shall really give such formalism in § 4, which we call as "rotator representation", contrasted to the "hydrodynamical representation" reviewed in § 2. Both are equivalent and transmutable to each other. It is furthermore natural to unify both representations. This *unified representation*, is especially suitable for the Lagrangian formalism,* and it is the latter standpoint that we begin with in the following. On the other hand, the rotator representation suits the canonical formalism of the theory. The introduction of the variable Λ is essential to making those new representations possible, as well as to making the corresponding model gauge-independent.

b) *Lagrangian*

Our purpose is, first of all, to reveal the realistic structure of the Pauli field in a physically interpretable sense. Our hydrodynamical representation has already met with that purpose, but we shall now combine the method of § 3, a) with it to make the model even more concrete.

In the hydrodynamical scheme we have regarded the Pauli field to be a fluid constituted of a continuous distribution of particles having intrinsic angular momentum **S**.

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* Lagrangian formalism is also possible in the pure hydrodynamical formalism, although we do not enter it here.

Now we consider that each of these particles is really a very small rotating body* and $\boldsymbol{S}$ means nothing but its *angular momentum of rotation* while $a_k^{(\xi)}$ just specify the *orientation* of the rotating body. Namely we consider that the orientation of such body is specified by an orthogonal coordinate frame fixed to the body just as in the case with a usual rigid body. We shall see later that our rotating body has a property distinct from a customary rigid body in respect to the relation between the angular velocity and angular momentum, but in other respects we may regard it as if it be a usual rigid body.

Formally speaking, the above picture corresponds to describing the Pauli field by the set of variables which comprises (2·3) and (3·17) ; i. e.

$$\{\Lambda,\ \rho,\ a_k^{(\xi)},\ \boldsymbol{v}\}. \tag{3·19}$$

Naturally such increasing of variables results in certain additional constraints among them. The most remarkable one is the relation (3·13), which means that the *angular momentum of each rotating body is fixed to that pseudo-rigid body** with the constant magnitude* $\hbar/2$.

We must now find the field equations satisfied by (3·19). To do this we may start from the original Pauli equation (2·1), but it is more profitable to achieve this by finding out the Lagrangian density $\mathcal{L}$ of such matter field by the aid of a more physical argument. First we consider the *angular velocity* $\boldsymbol{\omega}$ of each rotating body. It is derived from the rotational orientation of the body $a_k^{(\xi)}$ by the relation[6)]

$$\left.\begin{aligned}\omega_k\equiv\omega_{ij}&=\dot{a}_i^{(\xi)}\cdot a_j^{(\xi)},\\&=(\partial_0+v_k\partial_k)\,a_i^{(\xi)}\cdot a_j^{(\xi)},\\&(ijk)\sim(123)\end{aligned}\right\} \tag{3·20}$$

which satisfies of course

$$\left.\begin{aligned}\dot{a}_i^{(\xi)}a_j^{(\xi)}&=-\dot{a}_j^{(\xi)}a_i^{(\xi)},\\ \text{i. e.,}\qquad \omega_{ij}&=-\omega_{ji}\end{aligned}\right\} \tag{3·21}$$

on account of (3·14).

On the other hand the angular velocity $\boldsymbol{\Omega}$ in the coordinate system fixed to the body is given by

$$\begin{aligned}\Omega^{(3)}\equiv\Omega^{(1,2)}&=a_i^{(1)}a_j^{(2)}\,\omega_{ij}\\&=a_i^{(1)}\dot{a}_i^{(2)}=-\dot{a}_i^{(1)}a_i^{(2)}.\end{aligned} \tag{3·22}$$ ***

Now it is natural to assume that, corresponding to the rotational motion of the pseudo-rigid body, the Lagrangian density $\mathcal{L}$ contain the term

$$\mathcal{L}^{(r)}=-\rho\omega_k S_k=-\frac{\hbar}{2}\rho\omega_k a_k^{(3)}, \tag{3·23}$$

---

* We may regard the spatial dimension of such rotating body as infinitesimally small ; in that limit it becomes a "point rotator".

** And just in that direction was taken the third axis (ζ-axis) of the coordinate system fixed to the body.

*** Using formula (A·15).

which may also be written*

$$\mathcal{L}^{(\nu)} = -\rho\Omega^{(3)} \cdot \frac{\hbar}{2} = \frac{\hbar}{2}\rho\dot{a}_i^{(1)} a_i^{(2)}. \tag{3·24}$$

Next, $\mathcal{L}$ must involve also the part

$$\rho\left\{-(\partial_0 \Lambda + eA_0) + v_k\left(-\partial_k \Lambda + \frac{e}{c} A_k\right)\right\}. \tag{3·25}$$

Such Λ-dependent part is required for $\mathcal{L}$ to derive the equation of continuity (III); and the above combination of the derivatives of Λ (which we now identify with the Λ introduced in § 3, a)) with A_0 and A_k is necessary to secure the gauge-invariancy of the expression.

Furthermore we recognize from (VI′) that $\mathcal{L}$ must have the term $-\rho\Pi^*$ corresponding to the characteristic internal actions working inside the fluid. Adding purely classical terms and the part to secure the constraint (3·14), we finally anticipate the Lagrangian density to be of the form

$$\begin{aligned}\mathcal{L} = \rho\Big\{&\left(-\dot{\Lambda} - eA_0 + \frac{e}{c}\boldsymbol{v}\boldsymbol{A}\right)\\ &+ \frac{m}{2}\boldsymbol{v}^2 + \frac{\hbar}{2}\dot{\boldsymbol{a}}^{(1)}\boldsymbol{a}^{(2)} + \frac{e\hbar}{2mc}\boldsymbol{a}^{(3)}\boldsymbol{H}\Big\}\\ &- \frac{\hbar^2}{8m\rho}\{\partial_k(\rho a_i^{(3)})\}^2 + \lambda_{ik}(a_i^{(\xi)} a_k^{(\xi)} - \delta_{ik}),\end{aligned} \tag{3·26}$$

where the arguments are (3·19) together with λ_{ik}:

$$\{\Lambda,\ \rho,\ a_k^{(\xi)},\ v_k,\ \lambda_{ik}\}, \tag{3·27}$$

λ_{ik} being a symmetrical tensor

$$\lambda_{ik} = \lambda_{ki}. \tag{3·28}$$

c) *Derivation of the field equations from the Lagrangian*

Now we shall examine the field equations resulting from the Lagrangian (3·26), and then prove that they are indeed equivalent with the hydrodynamical representation stated in § 2.

First, taking variation of $\mathcal{L}$ in respect to Λ, we get (III). The variation in v_k gives

$$mv_k = \partial_k \Lambda - \frac{e}{c} A_k - \frac{\hbar}{2}\partial_k a_l^{(1)} \cdot a_l^{(2)}, \tag{3·29}$$

while the variation in ρ gives

---

* Using formula (A·16).

$$\dot{\Lambda} = -eA_0 + \frac{e}{c}\boldsymbol{v}\boldsymbol{A} + \frac{m}{2}\boldsymbol{v}^2 + \frac{\hbar}{2}\dot{\boldsymbol{a}}^{(1)}\boldsymbol{a}^{(2)} + \frac{e\hbar}{2mc}\boldsymbol{H}\boldsymbol{a}^{(3)} - \frac{\hbar^2}{8m\rho^2}\{\partial_k(\rho a_i^{(3)})\}^2 + \frac{\hbar^2}{4m}\frac{\Delta\rho}{\rho}. \tag{3·30}†$$

The variation in $a_i^{(\xi)}\,(\xi=1,\ 2)$, yields

$$\rho\dot{a}_i^{(\xi)} = -\frac{4}{\hbar}\varepsilon_{\xi\eta}\lambda_{ij}a_j^{(\eta)}, \quad (\xi=1,\ 2) \tag{3·33}$$

where $\varepsilon_{\xi\eta}$ is the antisymmetric symbol of the 2nd order :

$$(\varepsilon_{\xi\eta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3·34}$$

Finally the variation in $a_i^{(3)}$ results in

$$\frac{e}{mc}\rho H_i^{\mathrm{eff}} = -\frac{4}{\hbar}\lambda_{ij}a_j^{(3)}, \tag{3·35}$$

with $H_i^{\mathrm{eff}} \equiv H_i + \dfrac{\hbar c}{2e\rho}\partial_k(\rho\partial_k a_i^{(3)})$, [see (2·18)].

The set of eqs. (3·33) and (3·35) is also written in the following form :

$$\rho(\dot{a}_i^{(1)}a_j^{(2)} - \dot{a}_i^{(2)}a_j^{(1)}) = -\frac{e}{mc}\rho a_j^{(3)}H_i^{\mathrm{eff}} - \frac{4}{\hbar}\lambda_{ij}. \tag{3·36}$$

The antisymmetrical part of this equation gives*

$$\dot{a}_k^{(3)} = \frac{e}{mc}\varepsilon_{kij}a_i^{(3)}H_j^{\mathrm{eff}}. \tag{3·37}$$

Incidentally, we note another relation following from* (3·33) and (3·35) :

$$\dot{a}_i^{(1)}\partial_k a_i^{(2)} - \dot{a}_i^{(2)}\partial_k a_i^{(1)} = -\partial_k a_i^{(3)}\cdot\frac{e}{mc}H_i^{\mathrm{eff}}. \tag{3·38}$$

Here we insert the following remark : We have explained in § 3, a) that in the choice of $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ there is an arbitrariness by a scalar function $\gamma(\boldsymbol{x},\ t)$ as seen in (3·18). The condition to fix the coordinate axes $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ to each rotating body is evidently expressed by :

$$d\gamma/dt = 0. \tag{3·39}$$

---

† By use of this we know that for a permissible motion

$$\mathcal{L} = -(\hbar^2/4m)\Delta\rho, \tag{3·31}$$

and therefore

$$\overline{\mathcal{L}} = \int \mathcal{L}\,dx = 0. \tag{3·32}$$

* We use the formula (A·4). By the way the symmetrical part of (3·36) serves only to solve λ_{ij} in other quantities.

* We employ the symmetry of λ_{ij} and antisymmetry of $a_j^{(\xi)}\partial a_i^{(\xi)}$ in i and j. [see (A·2)].

In fact, however, throughout the treatment of this section we have not assumed this condition, and therefore $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ can be any axes* such as orthogonal mutually and at the same time orthogonal to **S**. It is easily verified that, under the transformation of (3·18), $\dot{\Lambda}-(\hbar/2)\dot{\boldsymbol{a}}^{(1)}\boldsymbol{a}^{(2)}$ is invariant, but $\dot{\Lambda}$ is not, nor $\dot{\boldsymbol{a}}^{(1)}$, $\dot{\boldsymbol{a}}^{(2)}$ separately. Only when we restrict the transformation to those satisfying (3·39), $\dot{\Lambda}$ and $\Omega^{(3)}=-\dot{\boldsymbol{a}}^{(1)}\boldsymbol{a}^{(2)}$ become invariant separately.

d) *Transition to the purely hydrodynamical representation*

The equations of c) derived from the Lagrangian are written in the variables (3·27). We shall now show that when we eliminate the excess variables

$$\{\lambda_{ij},\ \Lambda,\ \boldsymbol{a}^{(1)},\ \boldsymbol{a}^{(2)}\} \tag{3·40}$$

from those equations, we are exactly led to the hydrodynamical representation stated in § 2.

We have already obtained eq. (3·37) not containing the variables (3·40), but this equation is nothing but the former equation (V). Next, we take d/dt of eq. (3·29) and $\partial/\partial x_k$ of (3·30), and then add the results. Using there again (3·29) to eliminate $\partial_k\Lambda$, and also (3·38) to eliminate $\dot{\boldsymbol{a}}^{(1)}\partial_k\boldsymbol{a}^{(2)}-\dot{\boldsymbol{a}}^{(2)}\partial_k\boldsymbol{a}^{(1)}$, then we reach eq. (IV). Finally, we rewrite (3·29) (by (A·4)) in the form

$$mv_k=\partial_k\Lambda-\frac{e}{c}A_k+\frac{\hbar}{4}\sum_{\xi=1,2}\varepsilon_{lmn}\partial_k a_l^{(\xi)}\cdot a_m^{(\xi)}\cdot a_n^{(3)}, \tag{3·41}$$

and then take the curl of both sides. Then we get

$$m\partial_{[j}v_{k]}=-\frac{e}{c}H_i+\frac{\hbar}{4}\sum_{\xi=1,2}\varepsilon_{lmn}\Big\{(\partial_k a_l^{(\xi)}\cdot\partial_j a_m^{(\xi)}\cdot a_n^{(3)}$$

$$+\partial_k a_l^{(\xi)}\cdot a_m^{(\xi)}\cdot\partial_j a_n^{(3)})-[j,\ k]\Big\}, \qquad (3·42)^{**}$$

$$(ijk)\sim(123).$$

Taking account of (A·9) and (A·11), this becomes

$$m\partial_{[j}v_{k]}=-\frac{e}{c}H_i+\frac{\hbar}{2}\varepsilon_{lmn}a_l^{(3)}\partial_j a_m^{(3)}\partial_k a_n^{(3)}, \tag{3·43}$$

which is exactly the condition (II) of § 2. We have thus derived all of the fundamental equations of § 2.

§ 4. Canonical formalism and the rotator representation

From the Lagrangian (3·26), we can at once obtain the canonical conjugate $P[Q]$ for each variable Q, as follows:

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* On the other hand in the treatment of § 4 we shall put the condition (3·39).

** Here $[j,\ k]$ signifies the expression resulting from the preceding one by commuting suffixes j and k.

$$\begin{cases} P[\Lambda]=-\rho, \quad P[a_i^{(1)}]=\dfrac{\hbar}{2}\rho\, a_i^{(2)}=\alpha_i^{(2)}, & (4\cdot 1) \\ P[\rho]=P[v_i]=P[a_i^{(2)}]=P[a_i^{(3)}]=P[\lambda_{ij}]=0\,. & (4\cdot 2) \end{cases}$$

Energy density function $\mathcal{H}$ is also calculated from $\mathcal{L}'$ by the standard formula. Transforming this result by use of (3·29) we obtain

$$\mathcal{H}=\rho e A_0+\frac{m}{2}\rho\boldsymbol{v}^2-\frac{e}{mc}\boldsymbol{H}\boldsymbol{s}+\frac{1}{2m\rho}|\nabla\boldsymbol{s}|^2, \tag{4·3}$$

which is nothing but the equation (VI′).

We can overcome the difficulty of the vanishing of the conjugate variables for ρ, $\boldsymbol{v}$, $\boldsymbol{a}^{(2)}$, $\boldsymbol{a}^{(3)}$, and λ_{ij} by the following procedures: we take (4·3) as the hamiltonian density of the system, which is now regarded as the functional of the variables

$$\{\Lambda,\ P[\Lambda]=-\rho,\ a_i^{(1)},\ P[a_i^{(1)}]=\alpha_i^{(2)}\} \tag{4·4}$$

only; the quantities $\boldsymbol{v}$ and $\boldsymbol{s}$ appearing in $\mathcal{H}$ are taken as dependent variables expressed in terms of the basic variables (4·4), such that

$$\begin{cases} mv_k=\partial_k\Lambda-\dfrac{e}{c}A_k-\dfrac{1}{\rho}\partial_k a_l^{(1)}\cdot\alpha_l^{(2)}, & (4\cdot 5) \\ \boldsymbol{s}=[\boldsymbol{a}^{(1)}\times\boldsymbol{\alpha}^{(2)}]. & (4\cdot 6) \end{cases}$$

Eq. (4·5) is the same thing with (3·29), while (4·6) with (3·16).

We can really verify that (4·3) is the hamiltonian of the system under consideration: We calculate the field equations to be derived from this hamiltonian, i.e.,

$$\begin{cases} \partial_0\Lambda=-\partial'\mathcal{H}/\partial'\rho, \quad \partial_0\rho=\partial'\mathcal{H}/\partial'\Lambda, & (4\cdot 7),\ (4\cdot 8) \\ \partial_0 a_i^{(1)}=\partial'\mathcal{H}/\partial'\alpha_i^{(2)}, \quad \partial_0\alpha_i^{(2)}=-\partial'\mathcal{H}/\partial' a_i^{(1)} & (4\cdot 9),\ (4\cdot 10) \end{cases}$$

where

$$\partial'\mathcal{H}/\partial' Q\equiv\partial\mathcal{H}/\partial Q-\partial_k(\partial\mathcal{H}/\partial(\partial_k Q))\,. \tag{4·11}$$

First we find that (4·7) takes the form

$$\partial_0\Lambda=-\frac{m}{2}\boldsymbol{v}^2-eA_0-\frac{1}{\rho}v_k\partial_k a_l^{(1)}\cdot\alpha_l^{(2)}+\frac{1}{2m\rho^2}(\partial_k s_l)^2, \tag{4·12}$$

which is rewritten

$$\dot{\Lambda}=\frac{m}{2}\boldsymbol{v}^2-eA_0+\frac{e}{c}\boldsymbol{v}\boldsymbol{A}+\Pi^*, \tag{4·12′}$$

by using (4·5). Eq. (4·8) simply gives (III).

On the other hand (4·9) and (4·10) become:

$$\dot{a}_i^{(\xi)}=\varepsilon_{ilm}a_l^{(\xi)}\left\{\frac{e}{mc}H_m+\frac{1}{m}\partial_k\left(\frac{1}{\rho}\partial_k s_m\right)\right\}, \tag{4·13}$$

$$(\xi=1,\ 2)$$

where $\xi=1$ corresponds to (4·9) and $\xi=2$ corresponds to (4·10). From (4·13) we can get the similar relation for $a_i^{(3)}$, too, owing to the relation $\boldsymbol{a}^{(3)}=[\boldsymbol{a}^{(1)}\times\boldsymbol{a}^{(2)}]$. If we use the quantity $\boldsymbol{H}^*=\boldsymbol{H}+\boldsymbol{H}'$ defined by (2·10) and (2·13), all of them are put in a single form:

$$\boxed{\dot{\boldsymbol{a}}^{(\xi)}=\frac{e}{mc}[\boldsymbol{a}^{(\xi)}\times\boldsymbol{H}^*]. \quad (\xi=123)} \tag{4·14}$$

We see the above results of the hamiltonian formalism are simpler than the corresponding results in § 3 c) of the Lagrangian method. In the present case the basic variables (4·4) are just the same with the set (3·11), and so the above representation means that *we have established the formalism in terms of the modified Kramers' variables.*

In this formalism the equations of motion for Λ, $\boldsymbol{a}^{(1)}$, and $\boldsymbol{a}^{(2)}$ do not contain arbitrariness any longer. This corresponds to the fact that they are invariant to the transformation (3·18) only under the restriction (3·39). In other words $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ are "fixed" to each rotating body. Also it is important that in this representation Planck's constant does not appear in the formalism except in the single relation:

$$\boxed{(\boldsymbol{a}^{(2)})^2=\hbar^2/4.} \tag{4·15}$$

We also remark that the canonical formalism for the spinning hydrodynamics have been made feasible only through the introduction of the variable Λ and also factorizing the spin vector $\boldsymbol{s}$ into $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ in the manner (4·6).

Various Poisson brackets are calculated in this formalism, especially we see that

$$[s_i,\ s_j]=s_k, \quad (ijk)\sim(123) \tag{4·16}$$

conforming to the condition for angular momentum.

Now the fact that the present formalism be equivalent with the hydrodynamical formalism is easy to ascertain. The equation (V) was already obtained, while the equation of motion (IV) is derived from (4·5) and (4·12′) in the similar manner as was carried out for the case of § 3 c), but in a simpler way. The condition (II) is naturally obtainable from (4·5).

§ 5. Further remarks on the features of the model

a) *Velocity field*

The velocity field of our theory was written as (3·29), which shows that Λ, $a_i^{(1)}$, $a_i^{(2)}$ play the role of Clebsch parameters for the velocity variable. It must be noted, however, that this velocity field does not correspond to a most general vortical field, on account of the fact that $a_k^{(\xi)}$'s are restricted by the constraints (3·14).

b) *Angular velocity*

The equation of motion (4·14) exactly shows that each rotating body has angular velocity of rotation

$$\boldsymbol{\omega}=\frac{e}{mc}\boldsymbol{H}^{*}=\frac{e}{mc}\boldsymbol{H}+\frac{1}{m}\partial_{k}\left(\frac{1}{\rho}\partial_{k}\boldsymbol{s}\right). \tag{5·1}$$

Namely, the angular velocity points to the direction of the effective magnetic field, which is not collinear to the angular momentum **S**, nor is fixed to the body. Thus in our model, the angular velocity and angular momentum have disparate characters, since they are not connected through constant moments of inertia as in the case with a customary rigid body motion.

In conclusion the authors would like to express their sincere gratitude to Professor L. de Broglie for his kind interest taken in this work.

Appendix

a) *Formulas on the coefficients of orthogonal transformations*

Since our theory deals with the motion of *a continuous assembly* of orthogonal coordinate systems each attached to each point of space, we write here some formulas concerned with it. They are simply mathematical relations following from the definition of orthogonal transformation, (3·14) and (3·15), where $a_i^{(\xi)}$'s are functions of space and time coordinates. We take the right-handed system so that

$$\varepsilon_{ijk}\,a_i^{(1)}\,a_j^{(2)}\,a_k^{(3)}=+1. \tag{A·1}$$

First it is clear from (3·14) and (3·15) that

$$\begin{cases}\delta a_i^{(\xi)}\cdot a_j^{(\xi)}=-\delta a_j^{(\xi)}\cdot a_i^{(\xi)}=\text{antisymmetric in } i \text{ and } j, & \text{(A·2)}\\ \delta a_i^{(\xi)}\cdot a_i^{(\eta)}=-\delta a_i^{(\eta)}\cdot a_i^{(\xi)}=\text{antisymmetric in } \xi \text{ and } \eta, & \text{(A·3)}\end{cases}$$

where δ means ∂_0, ∂_k, or d/dt.

Next, we have the well-known relation that

$$\boldsymbol{a}^{(\xi)}=[\boldsymbol{a}^{(\eta)}\times\boldsymbol{a}^{(\zeta)}],\quad \text{i.e.,}\quad a_i^{(\xi)}=\varepsilon_{ijk}\,a_j^{(\eta)}\,a_k^{(\zeta)}, \qquad \text{(A·4), (A·4')}$$

$$(\xi\eta\zeta)\sim(123).$$

Similarly we have

$$a_i^{(\xi)}=\varepsilon_{\xi\eta\zeta}\,a_j^{(\eta)}\,a_k^{(\zeta)},\quad (ijk)\sim(123). \tag{A·5}$$

(A·4') and (A·5) are also expressed in the form

$$\begin{cases}\varepsilon_{ijk}\,a_j^{(\xi)}\,a_k^{(\eta)}=\varepsilon_{\xi\eta\zeta}\,a_i^{(\zeta)}, & \text{(A·6)}\\ \varepsilon_{\xi\eta\zeta}\,a_i^{(\eta)}\,a_j^{(\zeta)}=\varepsilon_{ijk}\,a_k^{(\xi)}. & \text{(A·7)}\end{cases}$$

Also it is easy to show that

$$\varepsilon_{lmn}\partial_j a_m^{(\xi)}\cdot a_n^{(3)}=-(\partial_j a_m^{(1)}\cdot a_m^{(2)})\,a_l^{(\xi)}, \tag{A·8}$$

$$\varepsilon_{lmn}\partial_j a_l^{(\xi)}\cdot\partial_k a_m^{(\xi)}\cdot a_n^{(3)}=0, \tag{A·9}$$

$$\text{both for}\quad \xi=1,\ 2,$$

and

$$\varepsilon_{lmn}\delta a_l^{(1)}\cdot a_m^{(1)}a_n^{(3)}=\varepsilon_{lmn}\delta a_l^{(2)}\cdot a_m^{(2)}a_n^{(3)}=-\delta a_l^{(1)}\cdot a_l^{(2)}. \tag{A·10}$$

Next we prove the relation

$$\sum_{\xi=1,2}\varepsilon_{lmn}\partial_k a_l^{(\xi)}\cdot a_m^{(\xi)}\partial_j a_n^{(3)}=\varepsilon_{lmn}a_l^{(3)}\partial_j a_m^{(3)}\partial_k a_n^{(3)}. \tag{A·11}$$

For that purpose first we express the left side in the determinant form

$$\sum_{\xi=1,2}\begin{vmatrix}\partial_k a_1^{(\xi)} & \partial_k a_2^{(\xi)} & \partial_k a_3^{(\xi)}\\ a_1^{(\xi)} & a_2^{(\xi)} & a_3^{(\xi)}\\ \partial_j a_1^{(3)} & \partial_j a_2^{(3)} & \partial_j a_3^{(3)}\end{vmatrix}, \tag{A·12}$$

and multiply each column by $a_1^{(3)}$, $a_2^{(3)}$, $a_3^{(3)}$ respectively, and add, then (A·12) becomes

$$\sum_{\xi=1,2}\frac{1}{a_{31}}\begin{vmatrix}a_l^{(3)}\partial_k a_l^{(\xi)} & \partial_k a_2^{(\xi)} & \partial_k a_3^{(\xi)}\\ 0 & a_2^{(\xi)} & a_3^{(\xi)}\\ 0 & \partial_j a_2^{(3)} & \partial_j a_3^{(3)}\end{vmatrix}. \tag{A·13}$$

Noting (A·3) this equals

$$-\sum_{\xi=1,2}\frac{\partial_k a_l^{(3)}\cdot a_l^{(\xi)}}{a_1^{(3)}}\begin{vmatrix}a_2^{(\xi)} & a_3^{(\xi)}\\ \partial_j a_2^{(3)} & \partial_j a_3^{(3)}\end{vmatrix}=-\frac{1}{a_1^{(3)}}\begin{vmatrix}\partial_k a_2^{(3)} & \partial_k a_3^{(3)}\\ \partial_j a_2^{(3)} & \partial_j a_3^{(3)}\end{vmatrix}. \tag{A·14}$$

On the other hand it is easy to see that the right side of (A·11) equals (A·14).

Finally we note two simple formulas concerned with ω_{ij} defined by (3·20):

$$\left\{\begin{aligned}&\omega_{ij}a_j^{(\xi)}=\dot{a}_i^{(\xi)}, &&\text{(A·15)}\\ &\varepsilon_{ijk}\omega_{ij}a_k^{(3)}=-2\dot{a}_l^{(1)}a_l^{(2)}. &&\text{(A·16)}\end{aligned}\right.$$

b) *Euler angles*

We have employed the set of quantities $\boldsymbol{a}^{(\xi)}$ which served to specify the orientation of the rotating body. In place of them we may use the corresponding Eulerian angles (θ, φ, χ). Then the relations between both sets of quantities are given, as is well known, by

$$(a_k^{\xi})=\begin{pmatrix}-(\cos\theta\cos\varphi\cos\chi+\sin\varphi\sin\chi), & -\cos\theta\sin\varphi\cos\chi+\cos\varphi\sin\chi, & \sin\theta\cos\chi\\ -\cos\theta\cos\varphi\sin\chi+\sin\varphi\cos\chi, & -\cos\theta\sin\varphi\sin\chi+\cos\varphi\cos\chi, & \sin\theta\sin\chi\\ \sin\theta\cos\varphi, & \sin\theta\sin\varphi, & \cos\theta\end{pmatrix}. \tag{A·17}$$

These Eulerian angles are related to the spinor ψ' as follows,

$$\begin{cases} \psi_1' = \rho^{1/2}\cos\dfrac{\theta}{2}\exp\left[\dfrac{i}{2}(\chi-\varphi)\right], \\ \psi_2' = \rho^{1/2}\sin\dfrac{\theta}{2}\exp\left[\dfrac{i}{2}(\chi+\varphi)\right]. \end{cases} \tag{A·18}$$

In other words $(\psi_1', \psi_2')/\rho^{1/2}$ are just the Cayley-Klein parameters for (θ, φ, χ).

In terms of Eulerian angles, the velocity field, for instance, is represented as

$$mv_k = \partial_k \Lambda - \frac{e}{c}A_k + \frac{\hbar}{2}(\partial_k\chi - \cos\theta\cdot\partial_k\varphi). \tag{A·19}$$

In this form, $\Lambda + (\hbar/2)\chi$, $\cos\theta$ and φ play the role of Clebsch-parameters.

c) *A generalization of Pauli field*

The fact that the Pauli hydrodynamics corresponds to quasi-irrotational mode of flows is a remarkable feature common to any hydrodynamics representing a quantum wave equation.

Now, in order to generalize the hydrodynamics so as to include general vortical mode of flows, we are to introduce new degrees of freedom ξ and η and add the term $\xi\partial_k\eta$ to the right side of (3·29). This corresponds to adding the term

$$-\rho\xi(d\eta/dt) = -\rho\xi(\partial\eta/\partial t + v_l\partial_l\eta), \tag{A·20}$$

to the Lagrangian density (3·26). When we derive the field equations from this modified Lagrangian, and then go to the hydrodynamical formalism by eliminating the excess variables, we reach the following theory : The basic equations (I), (III), (IV), (V) do not change at all, while in the right side of (II) there appears one additive term

$$(1/m)\partial_{[i}\xi\cdot\partial_{j]}\eta\,; \tag{A·21}$$

and besides them we have the equations of motion for ξ and η

$$d\xi/dt = d\eta/dt = 0.$$

This means that the theory is the generalization of the Pauli hydrodynamics so as to include general vortical flows without affecting the equations of motion and the condition (I). This hydrodynamics involves two more degrees of freedom than in the Pauli theory and cannot be described by a spinor ψ alone.

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Multiple Production of Particles and Hydrodynamical Aspect of Quantum Theory of Fields*

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The theory of irreversible processes based on quantum statistical mechanics is formulated in the framework of the quantum field theory, for the purpose to investigate whether the hydrodynamical description is applicable to the meson cloud as considered in Landau's theory of the multiple production of particles. The hydrodynamical quantities are defined from the field-theoretical operators by means of the averaging procedures in quantum statistical mechanics. Through the discussions about the relation between the equation of state of meson-nucleon fluid and the feature of interactions, it seems possible to conclude that at extremely high energy the equation of state assumed by Landau holds in the case of the first kind of interactions (having the dimensionless coupling constants), while it is not the case in the systems with the second kind of interactions (having the coupling constants with dimension). In particular, the violation of Landau's equation would be serious in the case of interactions with derivatives. Next the formulae are presented to calculate the phenomenological coefficients to be considered as the representatives of the properties of the fluid. Such a formulation includes the calculations of the relaxation time, the time for a system to reach the local equilibrium. There the methods are also prepared to estimate deviations from the macroscopic values of physical quantities predicted by the hydrodynamics due to fluctuations.

§ 1. Introduction

From recent experiments of high energy cosmic rays, we could derive the following conclusions: (i) the sharp angular distribution of the energy flow of secondary particles in air showers, (ii) a small ratio of the number of heavy unstable particles and anti-nucleons to that of pions compared with that predicted by Fermi's theory[1)], (iii) the multiplicity proportional to $E_0^{1/4\sim 1/3}$, E_0 being the energy of an incident particle in laboratory system, and (iv) the value $0.2\sim 0.7$ of the inelasticity. The qualitative behaviors of the above characteristic features seem to be interpreted most successfully by Landau's theory[2)]*** using a hydrodynamical model for the excited meson cloud formed in a collision, although it may be possible to explain the phenomena by other models and the above experimental results are not definite yet.

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** This article was completed while this author was at the Tokyo University of Education.

*** In Landau's model so far considered, one cannot calculate the inelasticity because of the implicit assumption that the entire initial energy is released from the colliding particles into their clouds. However, there may be a possibility to calculate the inelasticity in the framework of Landau's theory with some modifications.

Along Landau's line of thought, we consider the phenomena in such a way that the hydrodynamical expansion will begin after the particles (or the proto-type of particles) have been rapidly produced in the initial excited cloud. That is, the phenomena are to be understood in the following three steps. (i) Directly after the collision, the meson-nucleon cloud surrounding two colliding particles is suddenly excited into a furious thermal motion and goes rapidly into a local equilibrium because of strong interactions in high density. At the end of this period the cloud would be considered to be a system with the volume $\Omega \simeq \Omega_0/E$ in the state of an approximate equilibrium, where $\Omega_0 = (4\pi/3)R^3$ (R=Compton wave length of pion) and E is the incident energy in the centre-of-mass system. The effects of interactions in this period are taken into account only through the initial condition that demands one-dimensional motion of the cloud in a subsequent stage and the formation of a local equilibrium. (ii) The next period is characterized by the hydrodynamical expansion of the cloud, in which Landau has used the assumptions of the perfect fluid and the equation of state $p=\frac{1}{3}\epsilon$ for the fluid, p and ϵ being the pressure and the energy density, respectively. One may consider that the final interactions among diverging particles would be taken into account in the hydrodynamical equations. (iii) Since owing to the expansion the cloud becomes cold and dilute, there occurs the particle character and the cloud splits into free particles after cooled down to the critical temperature (equivalent to meson mass).

It seems true that the above line of consideration would be a correct way to analyse the phenomena, because the experiments of the multiple production of particles in nucleon-nucleon and nucleon-nucleus collisions may be interpreted by Landau's theory.[2)3)] His theory is, however, only a phenomenological one, whose relations to the present quantum field theory cannot be clarified by itself. In the present paper we shall search the procedure to formulate the hydrodynamical equations in the framework of the quantum field theory and to investigate whether the experiments of the multiple production can be considered as a tool for testing the present field theory. Our main interest may lie in the following problems: (i) Does the present quantum field theory permit the hydrodynamical description for the motion of the meson-nucleon cloud? (ii) Are the initial boundary conditions acceptable? (iii) What kind of equation of state and phenomenological equations should one assume, i.e., either the perfect fluid or the more complicated viscous one? (iv) How much deviations from the values predicted by Landau's theory are caused by fluctuations?

Now one handles only a few quantities such as the local velocity, the pressure and the energy density, while the many quantum numbers must be required for description of the systems participated by many particles. This situation comes from the fact that the hydrodynamics is associated with the averaged motion of the system. Thus it needs the statistical mechanical methods for the irreversible processes as in the molecular theory of the ordinary fluids.

We shall firstly define the hydrodynamical quantities from the field-theoretical operators on the analogy to Kirkwood's classical theory[4] for the systems of non-relativistic particles and look over the types of equations of state for several systems with various interactions (§ 2). Next we shall obtain the formulae to calculate the phenomenological coefficients, such as the thermal conductivity and the viscosity coefficients, and the relaxation times (§ 3). Finally brief discussions will be given for estimation of the deviations due to fluctuations (§ 4). In Appendix the macroscopic theory of the relativistic hydrodynamics is outlined together with the thermodynamics for irreversible processes.

§ 2. Definitions of hydrodynamical quantities and the equation of state

As stated in Appendix, in order to describe the motions of the meson cloud hydrodynamically, we must define the energy-momentum density four-tensor $T_{\mu\nu}$ and the mass flow four-vector J_μ in such a way that they have reasonable physical meanings. On the other hand the field theory is completely determined by the Lagrangian density $\boldsymbol{L}$ which is a function of field operators; hence it is necessary to define $T_{\mu\nu}$ and J_μ in terms of the quantum operators* derived from $\boldsymbol{L}$. As the physical meanings of $T_{\mu\nu}$ and J_μ are well understood in the classical theory, we shall consider first the relativistic classical theory which is a relativistic extension of Irving and Kirkwood's formalism[4]. Next we shall go on setting up a quantum mechanical analog of the classical theory.

2·1 Classical systems

We consider a classical system consisting of N particles, the motion of which is described by relativistic mechanics. For the sake of simplicity we treat a free particle system without external fields and mutual interactions as the limiting case of very weak interaction, though the system in the complete absence of interactions never becomes the fluid.

Now a state of the N-particle system can be represented by a point in the $6N$-dimensional phase space with $3N$ coordinates $\vec{x}_1, \vec{x}_2, \cdots \vec{x}_N$ and $3N$ momenta $\vec{p}_1, \vec{p}_2, \cdots \vec{p}_N$. The probability distribution function in this phase space $f(\vec{x}_1, \cdots \vec{x}_N; \vec{p}_1 \cdots \vec{p}_N; t)$ obeys the Liouville equation

$$\partial f/\partial t + \sum_{k=1}^{N} (\vec{p}_k/E_k) \cdot \vec{\nabla}_{x_k} f = 0 \qquad (2\cdot1)^{**}$$

and satisfies the invariant normalization condition

$$\int \cdots \int d^3\vec{x}_1 \cdots d^3\vec{x}_N d^3\vec{p}_1 \cdots d^3\vec{p}_N f(\vec{x}_1 \cdots \vec{x}_N; \vec{p}_1 \cdots \vec{p}_N; t) = 1, \qquad (2\cdot2)$$

where $E_k = \sqrt{\vec{p}_k^{\,2} + m_k^2}$. The transformation property of f is determined by the condition

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* In what follows we shall denote the quantum operators by the Gothic type, while the arrows stand for the three dimensional vectors.

** Throughout this paper we use the unit $\hbar = c = 1$.

(2·2). The expectation value $\langle\alpha\,;\,f\rangle$ of a dynamical variable $\alpha(\vec{x}_1\cdots\vec{x}_N\,;\,\vec{p}_1\cdots\vec{p}_N)$ for the distribution f is defined by

$$\langle\alpha\,;\,f\rangle=\int\cdots\int d\vec{x}_1\cdots d\vec{x}_N d\vec{p}_1\cdots d\vec{p}_N\alpha(\vec{x}_1\cdots\vec{x}_N\,;\,\vec{p}_1\cdots\vec{p}_N)\times$$
$$f(\vec{x}_1\cdots\vec{x}_N\,;\,\vec{p}_1\cdots\vec{p}_N\,;\,t)\,. \tag{2·3}$$

The mass density $J_0(\vec{x},\,t)$ and the mass flow $\vec{J}(\vec{x},\,t)$ at $\vec{x}$ and t of the system are given by

$$J_0(\vec{x},\,t)=\sum_{k=1}^{N}\langle m_k\delta(\vec{x}-\vec{x}_k)\,;\,f\rangle, \tag{2·4}$$

$$\vec{J}(\vec{x},\,t)=\sum_{k=1}^{N}\left\langle m_k\frac{\vec{p}_k}{E_k}\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle, \tag{2·5}$$

or they are unified to a four-vector

$$J_\mu(\vec{x},\,t)=\sum_{k=1}^{N}\left\langle m_k\frac{(p_k)_\mu}{E_k}\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle, \tag{2·6}$$

where m_k is the mass of the k-th particle and $p_\mu=(\vec{p},\,iE)$. Using the Liouville equation, we easily prove the equation of continuity of mass

$$\Gamma_\mu J_\mu=0. \tag{2·7}$$

As the quantity $\left\langle\frac{1}{E_k}\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle$ is invariant, it is evident that J_μ transforms like a four-vector. With this mass flow one can define the local velocity $\vec{u}(\vec{x},\,t)$ at $\vec{x}$ and t.

$$\begin{aligned}\vec{u}(\vec{x},\,t)&=\vec{J}(\vec{x},\,t)/J_0(\vec{x},\,t)\\&=\sum_{k=1}^{N}\left\langle m_k\frac{\vec{p}_k}{E_k}\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle\bigg/\left\langle\sum_{k=1}^{N}m_k\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle,\end{aligned} \tag{2·8}$$

as prescribed in Appendix. The right-hand side of (2·8) explains that $\vec{u}(\vec{x},\,t)$ is the velocity of the center of mass of the system in a small volume surrounding the point $\vec{x}$ at the time t.

The energy density $g_0(\vec{x},\,t)$ and the momentum density $\vec{g}(\vec{x},\,t)$ at $\vec{x}$ and t are respectively defined by

$$g_0(\vec{x},\,t)=\sum_{k=1}^{N}\langle E_k\delta(\vec{x}-\vec{x}_k);\,f\rangle, \tag{2·9}$$

$$\vec{g}(\vec{x},\,t)=\sum_{k=1}^{N}\langle\vec{p}_k\delta(\vec{x}-\vec{x}_k);\,f\rangle. \tag{2·10}$$

By using the Liouville equation we get

$$\frac{\partial\vec{g}(\vec{x},\,t)}{\partial t}+\vec{\Gamma}_x\cdot\sum_{k=1}^{N}\left\langle\frac{\vec{p}_k\vec{p}_k}{E_k}\delta(\vec{x}-\vec{x}_k);\,f\right\rangle=0,$$

$$\partial g_0(\vec{x},\ t)/\partial t + \vec{\nabla}_k \cdot \vec{g}(\vec{x},\ t) = 0,$$

which can be collected in the equation

$$\nabla_\nu T_{\mu\nu} = 0, \qquad (2\cdot 11)$$

where $T_{\mu\nu}$ is the energy-momentum density tensor defined by

$$T_{\mu\nu} = \sum_{k=1}^{N} \left\langle \frac{(p_k)_\mu (p_k)_\nu}{E_k} \delta(\vec{x} - \vec{x}_k)\ ;\ f \right\rangle. \qquad (2\cdot 12)$$

$T_{\mu\nu}$ transforms like a four-tensor as is easily seen.

Following the prescription (A·12) and (A·13) given in Appendix, we can derive from the above energy-momentum density tensor the heat flow and the internal stress

$$q_\mu(\vec{x},\ t) = \sum_{k=1}^{N} \left\langle \frac{[(p_k)_\mu - m_k U_\mu]_\perp}{E_k} \frac{1}{2m_k} [(p_k)_\nu - m_k U_\nu]^2 \delta(\vec{x} - \vec{x}_k)\ ;\ f \right\rangle, \qquad (2\cdot 13)$$

$$t_{\mu\nu}(\vec{x},\ t) = \sum_{k=1}^{N} \left\langle \frac{1}{E_k} [(p_k)_\mu - m_k U_\mu]_\perp [(p_k)_\nu - m_k U_\nu]_\perp \delta(\vec{x} - \vec{x}_k)\ ;\ f \right\rangle, \qquad (2\cdot 14)$$

respectively, where $(A_\nu)_\perp = A_\nu \Delta_{\mu\nu}$ denotes the space-like component* of the vector A_μ perpendicular to U_μ. Here we have used the relations

$$\Delta_{\mu\nu}(p_k)_\nu = \Delta_{\mu\nu}[(p_k)_\nu - m_k U_\nu] \equiv [(p_k)_\mu - m_k U_\mu]_\perp,$$

$$-(p_k)_\lambda U_\lambda = \frac{1}{2m_k}[(p_k)_\lambda - m_k U_\lambda]^2 + m_k,$$

and the equation of continuity, besides the trivial equation $(p_k)_\lambda^{\ 2} = -m_k^{\ 2}$. Since the right-hand side of (2·13) represents the average of the invariant relative kinetic energy density $\frac{1}{2m_k}[(p_k)_\nu - m_k U_\nu]^2 \cdot \delta(\vec{x} - \vec{x}_k)$ carried with the relative velocity $[(p_k)_\mu - m_k U_\mu]_\perp / E_k$ to the fluid motion, we may just interpret it as the heat flow. For (2·14) $t_{\mu\nu}$ can be regarded as the internal stress tensor. As emphasized in Appendix, these quantities are designed to have the most definite meanings in the local rest system, that is, in the system $\vec{u} = 0$ we get

$$\left.\begin{aligned} &\vec{q}(\vec{x},\ t) = \sum_{k=1}^{N} \left\langle \frac{\vec{p}_k}{E_k} (E_k - m_k) \delta(\vec{x} - \vec{x}_k)\ ;\ f \right\rangle \\ &q_0(\vec{x},\ t) = 0 \end{aligned}\right\} \qquad (2\cdot 13')$$

$$\left\{\begin{aligned} &t_{ij}(\vec{x},\ t) = \sum_{k=1}^{N} \left\langle \frac{(p_k)_i (p_k)_j}{E_k} \delta(\vec{x} - \vec{x}_k)\ ;\ f \right\rangle \text{ for } i,\ j = 1,\ 2,\ 3. \\ &t_{4\mu}(\vec{x},\ t) = 0. \end{aligned}\right. \qquad (2\cdot 14')$$

On the other hand, in the non-relativistic case ($|u| \ll 1$) we get

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* $\Delta_{\mu\nu} = \delta_{\mu\nu} + U_\mu U_\nu$, see Appendix.

$$\begin{cases} \vec{q}\,(\vec{x},\,t)=\sum_{k=1}^{N}\left\langle\left(\dfrac{\vec{p}_k}{m_k}-\vec{u}\right)\dfrac{1}{2m_k}(\vec{p}_k-m_k\vec{u})^2\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle \\ q_0(\vec{x},\,t)=0 \end{cases} \tag{2·13″}$$

$$\begin{cases} t_{ij}(\vec{x},\,t)=\sum_{k=1}^{N}\left\langle m_k\left(\dfrac{\vec{p}_k}{m_k}-\vec{u}\right)_i\left(\dfrac{\vec{p}_k}{m_k}-\vec{u}\right)_j\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i,\ j=1,\ 2,\ 3 \\ t_{4\mu}(\vec{x},\,t)=0 \end{cases} \tag{2·14″}$$

which coincide with the formulae given by Irving and Kirkwood$^{4)}$ in the non-interacting case. Thus we see that our method is a proper generalization of Irving and Kirkwood's procedure to the relativistic system.

Now one can derive from $t_{\mu\nu}$, to be interpreted as the internal stress causing the transport of momentum, the expression of pressure p which turns to the thermodynamical equation of state in an equilibrium state. Calculating $\mathrm{Tr}(T_{\mu\nu})$ by (2·12), (A14) becomes

$$3p=\epsilon-\sum_{k=1}^{N}\left\langle\frac{m_k^{\,2}}{E_k}\delta(\vec{x}-\vec{x}_k)\,;\,f\right\rangle\leq\epsilon. \tag{2·15}$$

In the case of the extremely high energy the second term of the middle part of this equation can be neglected in comparison with ϵ. Consequently we get

$$3p=\epsilon \tag{2·16}$$

which is nothing but the equation assumed by Landau.$^{2)}$

2·2. Free quantized fields

In order to define the hydrodynamical quantities $T_{\mu\nu}$ and J_μ in the framework of the quantum theory of fields, it is favourable to utilize the correspondence of quantum quantities to classical ones.* For simplicity we consider the free neutral meson field as an illustration.

Let the operator $(1/\sqrt{2})\,\phi^{\pm}(\vec{x})$ be the plus or minus frequency part of the quantized free meson field $\phi(\vec{x})$, which is connected with the creation or annihilation operator $A^{\pm}(\vec{x})$ as follows:

$$\phi^{\pm}(\vec{x})=\frac{1}{\sqrt{E_p}}A^{\pm}(\vec{x}),$$

where $E_p=\sqrt{m^2-\vec{\nabla}^2}$ and $\sqrt{E_p}$ is an operator defined by means of the Fourier transformation.

After subtraction of the zero point terms, it is natural to define the total mass operator and the total mass flow operator by

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* Irving and Zwanzig$^{4)}$ have made use of Wigner's distribution function for the purpose to extend Irving and Kirkwood's method to the case of quantum theory. Hewever, it is so complicated that it is not useful in the relativistic field.

$$\begin{aligned} &\int A^+(\vec{x})\, mA^-(\vec{x})\, d^3\vec{x} = \int \phi^+(\vec{x})\, mE_p\phi^-(\vec{x})\, d^3\vec{x} \\ &\int A^+(\vec{x})\, m\frac{\vec{p}}{E_p}A^-(\vec{x})\, d^3\vec{x} = \int \phi^+(\vec{x})\, m\vec{p}\phi^-(\vec{x})\, d^3\vec{x} \end{aligned} \qquad (2\cdot17)$$

where $\vec{p} = \frac{1}{i}\vec{\nabla}$, and the total energy and the total momentum by operators

$$\begin{aligned} &\int A^+(\vec{x})\, E_pA^-(\vec{x})\, d^3\vec{x} = \int \phi^+(\vec{x})\, E_p{}^2\phi^-(\vec{x})\, d^3\vec{x} \\ &\int A^+(\vec{x})\, \vec{p}A^-(\vec{x})\, d^3\vec{x} = \int \phi^+(\vec{x})\, E_p\vec{p}\phi^-(\vec{x})\, d^3\vec{x}. \end{aligned} \qquad (2\cdot18)$$

The density operators are, therefore, defined by the corresponding integrands, provided the appropriate symmetrization is performed.* Thus the mass flow operator is given by

$$\boldsymbol{J}_\mu(x) = \phi^+(x)\, m\overleftrightarrow{p}_\mu\phi^-(x), \qquad (2\cdot19)$$

where $\overleftrightarrow{p}_\mu = (1/2i)(\overrightarrow{\nabla}_\mu - \overleftarrow{\nabla}_\mu)$. Note that here the arrow on ∇_μ does not denote the vector but the direction of differentiation (right or left). The mass flow in hydrodynamics is defined by

$$J_\mu(x) = \langle \boldsymbol{J}_\mu(x) \rangle, \qquad (2\cdot20)$$

where the symbol $\langle \boldsymbol{A} \rangle$ stands for the quantum statistical average of an operator $\boldsymbol{A}$, given by

$$\langle \boldsymbol{A} \rangle = \mathrm{Tr}(\boldsymbol{\rho}\boldsymbol{A}), \qquad (2\cdot21)$$

$\boldsymbol{\rho}$ being the density matrix of the state in the corresponding transport process. The explicit form of $\boldsymbol{\rho}$ will be discussed in § 3. The energy-momentum density tensor is obtained by

$$T_{\mu\nu} = \langle \phi^+(x)\, \overleftrightarrow{p}_\mu\overleftrightarrow{p}_\nu\phi^-(x) \rangle. \qquad (2\cdot22)$$

Using such definitions of $T_{\mu\nu}$ and J_μ and the prescription of Appendix, we can write the expressions of $\vec{u}$, q_μ and $t_{\mu\nu}$ as follows :

$$u_i(\vec{x}, t) = \langle \phi^+(\vec{x}, t)\, m\overleftrightarrow{p}_i\phi^-(\vec{x}, t) \rangle / \langle \phi^+(\vec{x}, t)\, m\overleftrightarrow{p}_0\phi^-(\vec{x}, t) \rangle \qquad (2\cdot23)$$

for $i = 1, 2, 3$

$$q_\mu(\vec{x}, t) = \langle \phi^+(\vec{x}, t)\, (\overleftrightarrow{p}_\mu - mU_\mu)_\perp (1/2m)\, (\overleftrightarrow{p}_\lambda - mU_\lambda)^2\phi^-(\vec{x}, t) \rangle \qquad (2\cdot24)$$

$$t_{\mu\nu}(\vec{x}, t) = \langle \phi^+(\vec{x}, t)\, (\overleftrightarrow{p}_\mu - mU_\mu)_\perp (\overleftrightarrow{p}_\nu - mU_\nu)_\perp \phi(\vec{x}, t) \rangle, \qquad (2\cdot25)$$

where the differential operator $\overleftrightarrow{p}_\mu$ should not be applied to U_μ. From the above forms of the quantities, it is evident that they have the correct transformation character and the proper physical meanings to be expected. Here the quantity $\langle \phi^+(\vec{x}, t)\phi^-(\vec{x}, t)\rangle$ is to be compared with $\left\langle \sum_k \frac{1}{E_k}\delta(\vec{x}-\vec{x}_k); f \right\rangle$ in the classical theory.

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* Since the operators written in terms of $\phi^\pm$ do not contain $1/E_p$, the symmetrization is easily performed if E_p is replaced by the differentiation with respect to time. On the other hand, in Irving and Zwanzig's method, one must symmetrize the operators written in terms of $A^\pm$, as the description is given by Wigner's distribution function.

After a few calculations, one can get the equation of state

$$3p=\epsilon-m^2\langle \boldsymbol{\phi}^+(x)\boldsymbol{\phi}^-(x)\rangle \leq \epsilon, \tag{2·26}*$$

which reduces to (2·16) at extremely high energy.

Now one may say in general that the prescription to obtain the hydrodynamical $T_{\mu\nu}$ is the quantum statistical average of the symmetric energy-momentum tensor, subtracted by its vacuum expectation value, derived from the Lagrangian. Consequently we can easily obtain $T_{\mu\nu}$ in the case of the co-existing system of several fields. The mass flow is the sum of those of all fields. For example, the co-existing system of nucleon and meson fields has the mass flow

$$J_\mu(x)=\langle \boldsymbol{\phi}^+(x)\,m\overleftrightarrow{\boldsymbol{p}}_\mu\boldsymbol{\phi}^-(x)\rangle+\tfrac{1}{2}\langle \bar{\boldsymbol{\psi}}^-(x)\,M\gamma_\mu\boldsymbol{\psi}^-(x)\rangle+\tfrac{1}{2}\langle \bar{\boldsymbol{\psi}}_C^-(x)\,M\gamma_\mu\boldsymbol{\psi}_C^-(x)\rangle, \tag{2·20'}$$

where $\boldsymbol{\psi}$ is the nucleon field with mass M and $\boldsymbol{\psi}_C$ is the charge-conjugate field.

2·3. Interacting quantized fields

In the present subsection we treat the interacting fields with the Lagrangian density $\boldsymbol{L}$. According to the above considerations, it seems natural to define the hydrodynamical energy-momentum density tensor $T_{\mu\nu}$ by

$$T_{\mu\nu}(x)=\langle \boldsymbol{\theta}_{\mu\nu}(x)\rangle-\langle \boldsymbol{\theta}_{\mu\nu}(x)\rangle_{\text{vac}}, \tag{2·27}**$$

where $\boldsymbol{\theta}_{\mu\nu}$ is the symmetric energy-momentum tensor derived from the Lagrangian density $\boldsymbol{L}$ by means of the well-known technique. The definition of $\langle \boldsymbol{A}\rangle$ is given by (2·21), but $\langle \boldsymbol{A}\rangle_{\text{vac}}$ is the expectation value for the true vacuum. On the other hand, unfortunately, it seems that one does not obtain so useful definition of the mass flow for the cloud in early stages of the multiple production as the classical or free field case. However, we know that the operator

$$\boldsymbol{J}_\mu(x)=T(\boldsymbol{S}\boldsymbol{J}_\mu^{(0)}(x)) \tag{2·28}$$

represents asymptotically the mass flow associated with real particles[5], where $\boldsymbol{J}_\mu^{(0)}$ is the mass flow operator without interactions, $\boldsymbol{S}$ the $\boldsymbol{S}$-matrix and T the chronological ordering operator. $\boldsymbol{J}_\mu$ may be also interpreted as the mass flow existing when the interaction is switched off at the time t. Thus we shall tentatively define the mass flow by

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* In derivation of (2·26) we have dropped the unphysical terms which are to be canceled by adding an appropriate term ($T_{\mu\nu}'$ satisfying $\nabla_\nu T_{\mu\nu}'=0$ and $\int T_{\mu4}' d^3\vec{x}=0$,) to the energy-momentum tensor. Such a term has the form proportional to $(\nabla_\mu\nabla_\nu-\delta_{\mu\nu}\Box)\langle\phi^+\phi^-\rangle$.

** As in Irving and Kirkwood's paper[4], all the densities might have to be averaged over a small volume arround the point x. However, it is taken into account by means of the statistical averages with the density matrix as given in § 3, provided that $A_n(x)$'s can be chosen to be slowly varying functions.

$$\begin{aligned} J_\mu(x) &= \langle \boldsymbol{J}_\mu(x) \rangle \\ \text{or} \quad &= \langle \boldsymbol{S}^{-1}(\sigma)\, \boldsymbol{J}_\mu{}^{(0)}(x)\, \boldsymbol{S}(\sigma) \rangle, \end{aligned} \tag{2·29}$$*

where $\boldsymbol{S}(\sigma)$ is the transformation operator obeying the equation

$$i\,\frac{\delta \boldsymbol{S}(\sigma)}{\delta\sigma(x)} = \boldsymbol{H}_{\text{int}}(x)\, \boldsymbol{S}(\sigma),$$

$\boldsymbol{H}_{\text{int}}$ being the interaction Hamiltonian density. For the meson field, (2·29) may be rewritten as

$$J_\mu(x) = \langle \boldsymbol{\phi}^+(x, \sigma)\, m \overleftrightarrow{p}_\mu \boldsymbol{\phi}^-(x, \sigma) \rangle_{x/\sigma}, \tag{2·30}$$

in which $\boldsymbol{\phi}^\pm(x, \sigma) = \boldsymbol{S}^{-1}(\sigma)\, \boldsymbol{\phi}^\pm{}_{(0)}(x)\, \boldsymbol{S}(\sigma)$ is the plus or minus frequency part of the operator defined by Yang and Feldman[5)6)] and x must be put on σ after the differentiations. Now the mass flow (2·29) satisfies the equation

$$\nabla_\mu J_\mu(x) = i \langle [\boldsymbol{J}(x), \int \boldsymbol{H}_{\text{int}}(x')\, d\sigma_\mu{}'] \rangle_{x/\sigma}. \tag{2·31}$$

Although there may be some ambiguities in the definition of the mass flow, the discussions below will not suffer essential alterations.

In the remainder of this section we shall discuss the equation of state for the fields with various interactions.

(i) The neutral meson field with Schiff's type of interaction.

From the Lagrangian density

$$\boldsymbol{L} = -\tfrac{1}{2}\left\{ (\nabla_\mu \boldsymbol{\phi})^2 + m^2 \boldsymbol{\phi}^2 + \frac{\lambda}{2} \boldsymbol{\phi}^4 \right\}, \tag{2·32}$$

one can easily derive the symmetric energy-momentum density tensor

$$\boldsymbol{\Theta}_{\mu\nu} = \tfrac{1}{2}\{\nabla_\mu \boldsymbol{\phi} \nabla_\nu \boldsymbol{\phi} + \nabla_\nu \boldsymbol{\phi} \nabla_\mu \boldsymbol{\phi}\} + \boldsymbol{L}\delta_{\mu\nu}.$$

Using the equation of motion and dropping the unphysical term proportional to $(\nabla_\mu \nabla_\nu - \delta_{\mu\nu}\Box)\boldsymbol{\phi}^2$, we get

$$T_{\mu\nu} = \left\langle \boldsymbol{\phi} \left(\overleftrightarrow{p}_\mu \overleftrightarrow{p}_\nu + \frac{\lambda}{4} \delta_{\mu\nu} \boldsymbol{\phi}^2 \right) \boldsymbol{\phi} \right\rangle \tag{2·33}$$

where the vacuum expectation term is suppressed.** From $T_{\mu\nu}$, the equation of state

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* $J_\mu(x)$ has the σ-dependence, since the right-hand side of

$$\delta J_\mu(x)/\delta\sigma(x') = -i \langle \boldsymbol{S}^{-1}(\sigma)\, [\boldsymbol{J}_\mu{}^{(0)}(x),\ \boldsymbol{H}_{\text{int}}(x')]\, \boldsymbol{S}(\sigma) \rangle$$

never vanishes unless the distance between x and x' is over $(1/m)$, even in the case the vector $x_\mu - x_\mu{}'$ is space-like. However the effective correlation distance between x and x' might be reduced to a smaller length in very short time interval passed in the expansion of the cloud. Otherwise, the σ-independence of J_μ would have to be considered asymptotically. The authors are indebted to Mr. Z. Maki, Mr. Y. Ônuki and the other members of Nagoya University for interesting discussions about this point.

** In what follows, we will always suppress the vacuum terms and disregard the unphysical terms.

can be derived in the form

$$3p=\epsilon-m^2\langle\phi^2\rangle, \tag{2·34}$$

where the equation of motion has been used again.

(ii) The nucleon-pseudoscalar (or scalar) meson field with the pseudoscalar (or scalar) coupling.

The Lagrangian density is given by

$$\begin{aligned} \boldsymbol{L}=-\tfrac{1}{2}\{&(\nabla_\mu\phi_\alpha)^2+m^2\phi_\alpha{}^2+\bar{\psi}(\gamma_\mu\overrightarrow{\nabla}_\mu+M+g_\alpha\phi_\alpha 0\tau_\alpha)\psi\\ &+\bar{\psi}(-\gamma_\mu\overleftarrow{\nabla}_\mu+M+g_\alpha\phi_\alpha O\tau_\alpha)\psi\}, \end{aligned} \tag{2·35}$$

from which we get the hydrodynamical energy-momentum tensor

$$T_{\mu\nu}=\left\langle\frac{i}{2}\bar{\psi}(\gamma_\mu\overleftrightarrow{p}_\nu+\gamma_\nu\overleftrightarrow{p}_\mu)\psi+\phi_\alpha\overleftrightarrow{p}_\mu\overleftrightarrow{p}_\nu\phi_\alpha+\tfrac{1}{2}\delta_{\mu\nu}g_\alpha\bar{\psi}O\tau_\alpha\psi\phi_\alpha\right\rangle, \tag{2·36}$$

where α stands for the isotopic index. In this case the equation of state becomes

$$3p=\epsilon-M\langle\bar{\psi}\psi\rangle-m^2\langle\phi_\alpha{}^2\rangle. \tag{2·37}$$

In spite of the existence of the interaction, in these two cases the equation of state has the same form as that of the free field. Consequently one may expect that at extremely high energy the equation would reduce to the simple form (2·16) assumed by Landau. It is, however, noted that in the case of the flowing fluid the pressure p must, in general, contain the viscous pressure besides the hydrostatic one (see Appendix). Thus, in a local equilibrium the pressure p in our equation, $3p=\epsilon$, does not necessarily coincide with that in the equation used by Landau, because the latter contains the static pressure alone. Nevertheless it is surprising that simple equation, $3p=\epsilon$, holds in an equilibrium.

Now, let the interactions in the above two case alter into

(i′) $\dfrac{\lambda}{n}\phi^n\,(n>4)$, in the case of (i)

(ii′) $g_\alpha\bar{\psi}O\tau_\alpha\psi_\alpha\phi^n\,(n>1)$, in the case of (ii),

so the terms

$$\lambda\langle\phi^n\rangle\left(1-\frac{4}{n}\right)\ \text{and}\ g_\alpha\langle\bar{\psi}O\tau_\alpha\psi\phi_\alpha{}^n\rangle(n-1) \tag{2·38}$$

must be added to (2·34) and (2·37), respectively. Hence we see the explicit appearance of the interaction terms in the equation of state. Such terms become serious in the region with the high density of mesons, but it seems possible to neglect them at extremely high energy if the cloud has expanded to some extent. Thus one may again expect the equation, $3p=\epsilon$, in this case.

However, the situations are quite different if the interaction contains derivatives of field quantities.

(iii) The nucleon-pseudoscalar (or scalar) meson field with the pseudovector (or vector) coupling.

From the Lagrangian density

$$L=-\tfrac{1}{2}\{(\nabla_\mu\phi_\alpha)^2+m^2\phi_\alpha^2+\bar{\psi}(\gamma_\mu\overrightarrow{\nabla}_\mu+M+g_\alpha(\nabla_\mu\phi_\alpha)O_\mu\tau_\alpha)\psi$$
$$+\bar{\psi}(-\gamma_\mu\overleftarrow{\nabla}_\mu+M+g_\alpha(\nabla_\mu\phi_\alpha)O_\mu\tau_\alpha)\psi\},\qquad(2\cdot39)$$

we get

$$T_{\mu\nu}=\Big\langle\frac{i}{2}\bar{\psi}(\gamma_\mu\overleftrightarrow{p}_\nu+\gamma_\nu\overleftrightarrow{p}_\mu)\psi+\phi_\alpha\overleftrightarrow{p}_\mu\overleftrightarrow{p}_\nu\phi_\alpha+\tfrac{1}{2}g_\alpha\bar{\psi}O_\mu\tau_\alpha\psi(\nabla_\nu\phi_\alpha)$$
$$+\tfrac{1}{2}g_\alpha\bar{\psi}O_\nu\tau_\alpha\psi(\nabla_\mu\phi_\alpha)+\delta_{\mu\nu}\tfrac{1}{2}g_\alpha\phi_\alpha\nabla_\mu(\bar{\psi}O_\mu\tau_\alpha\psi)\Big\rangle.\qquad(2\cdot40)$$

After some calculations, the equation of state becomes

$$3p=\epsilon-m^2\langle\phi_\alpha^2\rangle-M\langle\bar{\psi}\psi\rangle-m^2g_\alpha\langle\phi_\alpha\nabla_\mu(\bar{\psi}O_\mu\tau_\alpha\psi)\rangle.\qquad(2\cdot41)$$

The last term in the right-hand side of (2·41) would be, in general, of same order of magnitude as that of ϵ even at extremely high energy.* Consequently the violation of the equation assumed by Landau is serious.

From the above discussions, it seems possible to conclude that the equation, $3p=\epsilon$, holds in the case of the fields with the first kind of interactions (with coupling constants having no dimension), while it is not the case for the second kind of interactions (with coupling constants having the dimension)**. The cases (i) and (ii) are examples for the former, while (i′), (ii′) and (iii) for the latter. In particular, it is remarked that the equations of state for the fields with the derivative couplings would become quite different from (2·16). Since such conclusions are not altered in the more general cases, it may be possible to obtain some information about the type of interactions by means of comparing the results inferred from the equation of state with the experimental ones.***

§ 3. Irreversible processes in meson-nucleon clouds and phenomenological relations

In the preceding section, we have formulated the procedures deriving the general scheme of hydrodynamics from the relativistic quantum field theory. Now we shall go on to the next problem to study whether the meson-nucleon clouds appearing in the multiple productions of mesons can be regarded as fluid and, if it is the case, what kind of fluid they are.

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* Note an exceptional case of the vector coupling between the nucleon and the neutral meson, where the last term in (2·41) vanishes because of the equation of motion.

** The same conclusion has been obtained also by Ezawa, Tomozawa and Umezawa[7] in their theory of static fluid.

*** However, one may say that, if the higher energy parts of the interaction were cut off in the future theory, the equation, $3p=\epsilon$, would still hold at the extremely high energy even in the case of derivative couplings.

It may be expected that, because of strong interactions in high density, the initial excited cloud would go rapidly into a local equilibrium which is determined by some thermodynamical parameters such as temperature but is irrelevant to the details of the initial condition. In order that the cloud can be regarded as fluid, it is necessary that the time spent till the formation of local equilibrium, namely, the relaxation time in the terminology of statistical mechanics is very short in comparison with the diffusion time of the substantial parts of the cloud. Furthermore, the rate of change from the local equilibrium to the macroscopic one is closely related to the phenomenological coefficients, such as the viscosity coefficients and the thermal conductivity, together with the above relaxation times. Thus some essential parts of the problems presented in § 1 would be able to be handled in the framework of the quantum statistical mechanics of irreversible processes developed by Kubo, Nakano, Nakajima and others[8)].

3·1 *Outline of the statistical mechanics of irreversible processes.*

Following Nakajima's formalism[8)], we start from the density matrix

$$\boldsymbol{\rho}(t)=\exp\{\beta\Omega-\int\beta\mathbf{g}_0(\vec{x})\,d^3\vec{x}+\sum_i\int(\beta\mu^{(i)}/m^{(i)})\,\boldsymbol{J}_0^{(i)}(\vec{x})\,d^3\vec{x}$$

$$+\sum_n\int A_n(\vec{x},t)\,\alpha_n(\vec{x})\,d^3\vec{x}\} \qquad (3\cdot1)$$

for the local equilibrium, where Ω is the free energy determined by the condition $\mathrm{Tr}(\boldsymbol{\rho})=1$ and $\beta=(1/kT)$. $A_n(\vec{x},t)$, which is considered to be a measure of deviation of the state from the equilibrium, is assumed to vary with time to enable us to obtain the expectation value of an operator α_n at time t

$$\langle\alpha_n\rangle_t\equiv\mathrm{Tr}(\boldsymbol{\rho}(t)\,\alpha_n)$$

$$=\mathrm{Tr}[\boldsymbol{\rho}(0)\exp\{it\int\mathbf{g}_0(\vec{x})\,d^3\vec{x}\}\,\alpha_n\exp\{-it\int\mathbf{g}_0(\vec{x})\,d^3\vec{x}\}]$$

and tends to zero as the state turns to the equilibrium. If deviations from the equilibrum are not too large, it would be sufficient to maintain only the terms linear in A_n in the average values of physical quantities. In this approximation, the rate of increase of the entropy, $\Sigma=-k\mathrm{Tr}(\boldsymbol{\rho}\log\boldsymbol{\rho})$, due to irreversible processes is written as

$$[d\Sigma/dt]_{\mathrm{irr}}=\int\sum_n\frac{1}{T}J_n(\vec{x},t)\,X_n(\vec{x},t)\,d^3\vec{x}, \qquad (3\cdot2)$$

where X_n and J_n are the thermodynamical driving force and flow, respectively, defined by

$$\left.\begin{aligned}X_n(\vec{x},t)&=-kTA_n(\vec{x},t)\\J_n(\vec{x},t)&=d\langle\alpha_n(\vec{x},t)\rangle_t/dt.\end{aligned}\right\} \qquad (3\cdot3)$$

(3·2) is to be compared with the integral of (A·20). (Note that $(d\Sigma/dt)_{\mathrm{irr}}=\int(\nabla_\mu S_\mu)\,d^3\vec{x}$). The average value of α_n is given by

$$\langle \alpha_n(\vec{x}) \rangle_{t+s} = \sum_m \int d^3\vec{x}' A_m(\vec{x}', t)\, \Phi_{mn}(\vec{x}', \vec{x}\,;\, s), \tag{3·4}$$

where

$$\Phi_{mn}(\vec{x}', \vec{x}\,;\, s) = \int_0^\beta \frac{d\lambda}{\beta} \langle \alpha_m(\vec{x}')\, \alpha_n(\vec{x}, s+i\lambda) \rangle_0, \tag{3·5}$$

in which $\langle \cdots \rangle_0$ stands for the average value in equilibrium. From (3·4) we obtain

$$\frac{d}{dt} \langle \alpha_n(\vec{x}) \rangle_t = -\sum_n \int \frac{1}{\beta} G_{nm}(\vec{x}, \vec{x}')\, A_m(\vec{x}', t)\, d^3\vec{x}', \tag{3·6}$$

where

$$G_{nm}(\vec{x}, \vec{x}') = \beta \int_0^\infty \Phi_{\dot{m}\dot{n}}(\vec{x}', \vec{x}\,;\, t)\, dt \tag{3·7}$$

$$\Phi_{\dot{m}\dot{n}}(\vec{x}_,', \vec{x}\,;\, t) = \int_0^\beta \frac{d\lambda}{\beta} \langle \dot{\alpha}_m(\vec{x}')\, \dot{\alpha}_n(\vec{x}, t+i\lambda) \rangle_0. \tag{3·8}$$

In derivation of (3·6), we have used the assumptions that $\Phi_{\dot{n}\dot{m}}(t)$ damps, for example, as $\exp(-t/\tau_0)$ and $\Phi_{\dot{n}\dot{m}}(0)=0$, and that there is a sufficient long time interval, $\tau \gg s \gg \tau_0$, where τ is the time passed till the macroscopic equilibrium revives, namely,

$$\tau = \beta \Phi_{nm}(0) / G_{mn}\,. \tag{3·9}$$

By the definitions (3·3), (3·6) can be rewritten in the form

$$J_n(\vec{x}, t) = \sum_n \int G_{nm}(\vec{x}, \vec{x}')\, X_m(\vec{x}', t)\, d^3\vec{x}', \tag{3·10}$$

which is to be compared with (A·24). Here it is noted that we have chosen the operators, α_n's, with the vanishing average values in the equilibrium.

3.2. *Phenomenological relations for the meson-nucleon cloud.*

In the present subsection, we shall derive the phenomenological relations as those given in Appendix by means of the prescription outlined in the preceding subsection. Since the coefficients contained in them are invariant as shown in Appendix, we have only to calculate them in each local rest system at each point.

Here we shall derive the formulae of the thermal conductivity and the viscosity coefficients in the absence of mass action. In the rest system, the heat flow becomes $\vec{q} = \langle \vec{g} \rangle$ and is related to the temperature gradient as

$$q_k(\vec{x}) = \int K(\vec{x} - \vec{x}')\, \frac{1}{\beta} \vec{\nabla}' \beta(\vec{x}')\, d^3\vec{x}', \tag{3·11}^*$$

while the internal stress is given by $t_{ik} = \langle \boldsymbol{T}_{ik} \rangle$ and

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* In the absence of external fields, it is easily shown that the functions K and Y depend only on the variable $\vec{x} - \vec{x}'$.

$$t_{ik}(\vec{x})=p_s(\vec{x})\,\delta_{ik}-\int Y_{(s)}(\vec{x}-\vec{x}')\big[(\nabla_i'u_k(\vec{x}')+\nabla_k'u_i(\vec{x}'))$$

$$-\frac{2}{3}\delta_{ik}(\vec{\nabla}'\cdot\vec{u}(\vec{x}'))\big]d\vec{x}'-\int Y_{(v)}(\vec{x}-\vec{x}')\,\delta_{ik}(\vec{\nabla}'\cdot\vec{u}(\vec{x}'))\,d^3\vec{x}', \qquad (3\cdot 12)$$

Here it is noted that these equations are to first order of the temperature gradient or the deformation velocities. Strictly speaking, we should investigate the order of magnitude of their higher order terms, but we shall not enter in this point, although Nakajima's formalism perhaps permits us to calculate such non-linear terms. If the temperature gradient and the deformation velocities are constant arround a point, we have the so-called conductivity and viscosity coefficients such as $\kappa=\int K(\vec{x}-'\vec{x})\,d^3\vec{x}'$ and $\eta=\int Y(\vec{x}-\vec{x}')\,d^3\vec{x}'$, respectively.

In a small region around the point in consideration, the state of the meson-nucleon cloud is considered to be in a local equilibrium deviating from an equilibrium with the definite temperature $1/\beta$ and the zero local velocity by $\Delta\beta(\vec{x})$ in reciprocal temperature and by $\Delta\vec{u}(\vec{x})$ in velocity. Hence the calculations have to be performed only in a considerably large system which have the same variations in temperature and velocity as in above small region of the cloud. For the sake of simplicity we first neglect the cross effect between the heat flow and the mass flow. In this case we must take the choice of A_n and α_n in the preceding subsection as follows:

$$\sum_n\int A_n\alpha_n d^3\vec{x}=\int\left[-\Delta\beta(\vec{x})\,\mathbf{g}_0(\vec{x})+\beta\Delta u(\vec{x})\cdot\vec{\mathbf{g}}(\vec{x})\right]d^3\vec{x}\,.$$

Thus we get

$$\left[\frac{d\Sigma}{dt}\right]_{\text{irr}}=-k\int\left[-\Delta\beta(\vec{x})\,\frac{d}{dt}\langle\mathbf{g}_0(\vec{x})\rangle_t+\beta\Delta\vec{u}(\vec{x})\cdot\frac{d}{dt}\langle\vec{\mathbf{g}}(\vec{x})\rangle_t\right]d^3\vec{x}$$

$$=\int\frac{1}{T}\left\{\langle\vec{\mathbf{g}}(\vec{x})\rangle_t\cdot\frac{1}{\beta}\vec{\nabla}\beta(\vec{x})-\langle\overset{\Rightarrow}{\mathbf{T}}(\vec{x})\rangle_t:(\vec{\nabla}\vec{u}(\vec{x}))\right\}d^3\vec{x}, \qquad (3\cdot 13)^*$$

in which the last equation has been derived with the aid of

$$\left.\begin{aligned}&\frac{d}{dt}\langle\mathbf{g}_0(\vec{x})\rangle_t+\vec{\nabla}\cdot\langle\vec{\mathbf{g}}(\vec{x})\rangle_t=0\\&\frac{d}{dt}\langle\vec{\mathbf{g}}(\vec{x})\rangle_t+\vec{\nabla}\cdot\langle\overset{\Rightarrow}{\mathbf{T}}(x)\rangle_t=0\end{aligned}\right\} \qquad (3\cdot 14)$$

and the integration by parts. The prescription given in 3·1 yields the equation

$$\frac{d}{dt}\langle\mathbf{g}_0(\vec{x})\rangle_t=-\int G(\vec{x}-\vec{x}')\,(-\Delta\beta(\vec{x}'))\,d^3\vec{x}',$$

where

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* The double arrow denotes the second rank tensor, while the double dot stands for the contraction of two tensor indices.

$$G(\vec{x}-\vec{x}')=\int_0^\infty dt'\int_0^\beta \frac{d\lambda}{\beta}\langle \boldsymbol{g}_0(\vec{x}')\,\boldsymbol{g}_0(\vec{x},\,t+i\lambda)\rangle_0.$$

By making use of (3·14) and the integration by parts, the above equation turns to (3·11), namely,

$$\langle\vec{\boldsymbol{g}}(\vec{x})\rangle_t=\int K(\vec{x}-\vec{x}')\,\frac{1}{\beta}\vec{\nabla}'\beta(\vec{x}')\,d^3\vec{x}', \tag{3·15}$$

where

$$K(\vec{x}-\vec{x}')=\int_0^\infty dt\int_0^\beta d\lambda\langle \boldsymbol{g}_1(\vec{x}')\,\boldsymbol{g}_1(\vec{x},\,t+i\lambda)\rangle_0. \tag{3·16}$$

Here we have used the condition $\langle\vec{\boldsymbol{g}}\rangle_0=0$ when $\vec{\nabla}\beta=0$ and the isotropy of the fluid. In the same fashion the equation of the internal stress becomes

$$\langle \boldsymbol{T}_{ik}(\vec{x})\rangle_t=\langle \boldsymbol{T}_{ik}(\vec{x})\rangle_0-\sum_{lm}\int Y_{ik,lm}(\vec{x}-\vec{x}')\,\nabla_l'u_m(\vec{x}')\,d^3x', \tag{3·17}$$

where

$$Y_{ik,lm}(\vec{x}-\vec{x}')=\int_0^\infty dt\int_0^\beta d\lambda\langle \boldsymbol{T}_{lm}(\vec{x}')\,\boldsymbol{T}_{ik}(\vec{x},\,t+i\lambda)\rangle_0. \tag{3·18}$$

The first term in the right-hand side of (3·17) turns to the hydro-static pressure $p_s\,\delta_{ik}$ on account of the isotropy of the fluid. Taking the same property of the fluid into account, we can easily show that the last term does not contain the anti-symmetric part of the tensor $\nabla_l u_m$ and that the tensor $Y_{ik\ lm}$ is reduced to the sum of two scalars corresponding to the decomposition of the symmetric part of $\nabla_l u_m$ such as

$$\text{Sym.}(\nabla_l u_m)=\tfrac{1}{2}[(\nabla_l u_m+\nabla_m u_l)-\tfrac{2}{3}\delta_{lm}\vec{\nabla}\cdot\vec{u}]+\tfrac{2}{3}\delta_{lm}\vec{\nabla}\cdot\vec{u}.$$

Consequently, one can obtain the desired form (3·12) with

$$Y_{(s)}(\vec{x}-\vec{x}')=\tfrac{1}{2}\int_0^\infty dt\int_0^\beta d\lambda\langle \boldsymbol{T}_{12}(\vec{x}')\,\boldsymbol{T}_{12}(\vec{x},\,t+i\lambda)\rangle_0 \tag{3·19}$$

$$Y_{(v)}(\vec{x}-\vec{x}')=\tfrac{1}{3}\int_0^\infty dt\int_0^\beta d\lambda\langle \boldsymbol{T}_{11}(\vec{x}')\,\boldsymbol{T}_{11}(\vec{x},\,t+i\lambda)\rangle_0\,. \tag{3·20}$$

In order to obtain the phenomenological relations with the more general form as given in Appendix, we shall mention below a straightforward but somewhat formal method. Although A_n and α_n are usually chosen as

$$\int\sum_n A_n\alpha_n d^3\vec{x}=-\int \Delta\beta(\vec{x})\,\boldsymbol{g}_0(\vec{x})\,d^3\vec{x}+\int\beta\Delta\vec{u}(\vec{x})\cdot\vec{\boldsymbol{g}}(\vec{x})\,d^3\vec{x}$$
$$+\sum_i\int\Delta(\beta\mu_i^{(i)}(\vec{x}))\,(\boldsymbol{J}_0^{(i)}(\vec{x})/m^{(i)})\,d^3\vec{x},$$

it is convenient to make use of the form

$$\int \sum A_n \alpha_n d^3\vec{x} = -\int \vec{\nabla}\beta(\vec{x}) \cdot \int^t \vec{g}(\vec{x})\, dt d^3\vec{x} + \int \beta\vec{\nabla}\vec{u}(\vec{x}) : \int^t \vec{\vec{T}}(\vec{x})\, dt d^3\vec{x}$$
$$+\sum_i \int \vec{\nabla}(\beta\mu^{(i)}(\vec{x})) \cdot \int^t (\vec{J}^{(i)}(\vec{x})/m^{(i)})\, dt d^3\vec{x}$$
$$+\sum_i \int \Delta(\beta\mu^{(i)}(\vec{x})) \int^t (A^{(i)}(\vec{x})/m^{(i)})\, dt d^3\vec{x}. \qquad (3\cdot21)$$

Of course this choice (3·21) is equivalent to the former if we take into account the formal integration with respect to t of the equations

$$\partial\vec{g}_0/\partial t + \vec{\nabla}\cdot\vec{g} = 0, \qquad \partial\vec{g}/\partial t + \vec{\nabla}\cdot\vec{\vec{T}} = 0 \qquad \partial J_0^{(i)}/\partial t + \vec{\nabla}\cdot\vec{J}^{(i)} = A^{(i)} \qquad (3\cdot22)^*$$

and the integration by parts. We shall not consider the mass action, because we do not know whether the phenomena occurring in the meson cloud can be represented by the chemical reaction as mentioned in Appendix. Such a discussion will be given elsewhere. Thus our choice of A_n and α_n is as expressed in Table I, where X_n and J_n are also tabulated. By making use of such choice, the formula (3·10) allows us to obtain (3·12),

$$q_k(\vec{x}) = \int K(\vec{x}-\vec{x}')\frac{1}{\beta}\vec{\nabla}'\beta(\vec{x}')\, d^3\vec{x}' - \sum_i \int \Lambda^{(i)}(\vec{x}-\vec{x}')\frac{1}{\beta}\vec{\nabla}'(\beta\mu^{(i)}(\vec{x}'))\, d^3\vec{x}', \qquad (3\cdot23)$$

Table I. Choice of variables.

A_n	α_n	X_n	J_n
$-\vec{\nabla}\beta$	$\int^t \vec{g}dt$	$\frac{1}{\beta}\vec{\nabla}\beta$	$\langle\vec{g}\rangle_t$
$\beta\vec{\nabla}\vec{u}$	$\int^t \vec{\vec{T}}dt$	$-\vec{\nabla}\vec{u}$	$\langle\vec{\vec{T}}\rangle_t$
$\vec{\nabla}(\beta\mu^{(i)})$	$\int^t (\vec{J}^{(i)}/m^{(i)})dt$	$-\frac{1}{\beta}\vec{\nabla}(\beta\mu^{(i)})$	$\langle\vec{J}^{(i)}/m^{(i)}\rangle_t$

and

$$\langle[\vec{J}^{(i)}(\vec{x})/m^{(i)}]\rangle_t = \sum_j \int \Xi^{(i)(j)}(\vec{x}-\vec{x}')\vec{\nabla}'(\beta\mu^{(j)}(\vec{x}'))\, d^3\vec{x}'$$
$$+\int \Lambda'^{(i)}(\vec{x}-\vec{x}')\ \frac{1}{\beta}\vec{\nabla}'\beta(\vec{x}')\, d^3\vec{x}, \qquad (3\cdot24)$$

where the coefficients are given by (3·16), (3·19), (3·20),

$$\Lambda^{(i)}(\vec{x}-\vec{x}') = \int_0^\infty dt \int_0^\beta d\lambda \langle[J_1^{(i)}(\vec{x}')/m^{(i)}]g_1(\vec{x}, t+i\lambda)\rangle_0 . \qquad (3\cdot25)$$

The expressions of $\Xi^{(i)(j)}$ and $\Lambda'^{(i)}$ are not given here but can be obtained easily.

Now the relaxation time can be calculated in the above formulation as the damping

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[*] The last equation is the operator form of the equation given by (2·31). Here the total flow must be divided into the flows of their constituents.

time of the integrand of the function

$$K(\vec{x}-\vec{x}')=\beta\int_0^\infty dt\langle\{\mathbf{g}_1(\vec{x}'),\ \mathbf{g}_1(\vec{x},\ t)\}\rangle_0, \tag{3·26}$$

where $\{A,\ B\}=\frac{1}{2}(AB+BA)$. The absence of the λ-integral as in (3·26) is always proven in the case of the correlation functions between two same quantities as pointed out by Nakano.[8] The relaxation of the internal stress is represented by the damping of the corresponding correlation functions in the integrands of (3·19) and (3·20). The relaxation of other modes can be treated in a quite similar fashion to the above, but the correlation functions of the cross effects are never reduced to the simple form as (3·26).

§ 4. Discussions

In the previous sections we have shown the procedures how to formulate the hydrodynamical aspect of the meson-nucleon cloud in the framework of the present quantum field theory and the statistical mechanics of irreversible processes. Besides these, in order to be able to regard the meson-nucleon cloud as real fluid, the following three conditions must be satisfied:

(i) As we mentioned in § 3, it is necessary that the relaxation time is very short in comparision with the diffusion time of the substantial parts of the cloud, that is, the inequality $\tau\gg s\gg\tau_0$ (p. 603) is to be satisfied.

(ii) The fluctuation around the average value of a quantity must be smaller than the average value: Since the hydrodynamics is constructed on the average values of some quantities, it is true that the fluctuations cannot be estimated in the formulation so far considered, but the estimation of the fluctuations is to be taken into account in the derivation of the phenomenological equations. For simplicity, consider only one variable, so that the phenomenogical equation becomes simply

$$\langle\dot{\alpha}\rangle_t=\langle\dot{\alpha}\rangle_0+GX. \tag{4·1}$$

We restrict ourselves to first order of the thermodynamical force X causing the flow $\langle\alpha\rangle_t$, and

$$G=\beta\int_0^\infty dt\langle\{\dot{\alpha}(0),\ \dot{\alpha}(t)\}\rangle_0. \tag{4·2}$$

Under the special situation that the correlation function damps with increasing time in proportion to the factor $\exp(-t/\tau_0)$, the phenomenological equation (4·1) is rewritten as

$$\langle\dot{\alpha}\rangle_t=\langle\dot{\alpha}\rangle_0+\tau_0\langle(\dot{\alpha}(0))^2\rangle_0 X. \tag{4·3}$$

If the higher order terms in X were maintained in (4·1), we should consider the average values of the higher moments of $\dot{\alpha}^2$. Thus it is necessary that such average values are small enough to justify the neglect of the non-linear terms of X in (4·1). The calculation of these average values is nothing but estimation of the fluctuations. Consequently, we must utilize the phenomenological equations after justifying* the smallness of the fluctuations.

(iii) The quantum fluctuation around the average value of a quantity in a small volume must be smaller than the average value: we need to take an average procedure over a small volume around the point $\vec{x}$ such as $\langle g_0(\vec{x})\rangle$, as pointed out in the footnote for (2·27). Since in the case of the multiple production such a volume must be small in comparison with the dimension of the meson cloud, the fluctuations of the quantized fields localized in it might become so large that we could not permit the hydrodynamical scheme. Optimistically speaking, however, one may believe that there is an appropriate averaging volume with the dimention acceptable in the hydrodynamical aspect, because the high density of the meson-nucleon cloud would reduce the quantum fluctuations considerably. In other wards, we may choose the functions $A_n(x)$'s for the density matrix (see the footnote for (2·27)) which are slowly varying in such a volume and compatible with condition $\tau \gg s \gg \tau_0$.

The detailed calculation will be done in a succeeding paper.

Finally we remark the semi-phenomenological procedures to follow the temporal change of the fluctuations by means of Onsager's idea.[9] It starts from the assumption that the fluctuations of the quantity $\dot{\alpha}-\langle\dot{\alpha}\rangle_0$ is described by the c-number random variable $d\xi\, dt$, obeying a sort of the Langevin equation

$$\frac{d\xi}{dt}-G(X+F)=0. \tag{4·4}$$

Here the fluctuation force F is tentatively characterized by the following statistical natures:

$$\left.\begin{array}{l}\langle F(t)\rangle=0\\ \langle F(t)F(s)\rangle=\langle F^2\rangle\delta(t-s)\\ \text{Probability for the appearance of } F\propto\exp(-\gamma F^2),\end{array}\right\} \tag{4·5}$$

where $\langle F^2\rangle$ is independent of time and γ a constant determined by the temperature. Such a stochastic process is a stationary Marcoff one with the Gauss distribution. Consequently the temporal development of the probability for finding a value of ξ can be described by the so-called Fokker-Planck equation derived from (4·4) and (4·5)[10]. Discussions along this line may be useful for the study of the deviations from the hydrodynamical results for the case in which the cloud splits into free particles before the equilibrium.

The detailed properties of the meson cloud will be discussed in the succeeding paper.

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* One may expect the smallness of the higher order moments from that of the second moment.

Appendix. Phenomenological theory of relativistic hydrodynamics

The macroscopic theory of a continuous system starts from the densities of energy, momentum and mass flows defined by an appropriate prescription. In order to make such densities obey the well-known transport equations, we must introduce the internal stress as a force transmitting the momentum and the heat flows, the latter being the residual part of the energy transport subtracted by the mechanical work. The general transport equations are the fundamental laws themselves (that is, the conservation law of energy, the equation of motion and the equation of continuity) irrespective of the detailed properties of matter. In order to take the detailed macroscopic properties of the matter into account, we must require the phenomenological relations, for example, Fourier's law of heat conduction. The fluid state is to be also represented by such relations. The transport equations supplemented by such relations become a solvable system of equations, for exaple, Navier-Stokes' equation.

The thermodynamical quantities (temperature, pressure and so on) of the fluid may be defined through the following measurement. Let the small volume of the fluid surrounding the point in consideration be into contact with a system in thermal equilibrium. Then, if no thermodynamical change takes place, the temperature and the pressure of this system are considered to be those of the fluid at this point. (When such a measurement has the definite meaning, the fluid is, of course, in a local equilibrium.) Consequently, the physical meanings of the quantities in the relativistic hydrodynamics could be properly interpreted in the local rest system.

Now, in the relativistic hydrodynamics, the energy density g_0 and the momentum density $\vec{g}$ (or the energy flow $\vec{S}$) are unified into the energy-momentum density four-tensor $T_{\mu\nu}$ by the following connections:

$$\begin{aligned} &T_{44}=-g_0, \quad T_{i4}=icg_i=\frac{i}{c}S_i \quad (i=1,\ 2,\ 3) \\ &S_\mu=\frac{c}{i}T_{4\mu}=(\vec{S},\ icg_0) \quad (\mu=1,\ 2,\ 3,\ 4). \end{aligned} \tag{A$\cdot$1}$$

The energy-momentum density tensor $T_{\mu\nu}$ should be a symmetric one $(T_{\mu\nu}=T_{\nu\mu})$ because of the relativistic requirement that the energy flow is equivalent to the momentum flow. In the closed system the energy-momentum transport equation is simply written as the four-divergenceless condition of $T_{\mu\nu}$,

$$\nabla_\nu T_{\mu\nu}=0. \tag{A$\cdot$2}$$

The mass flow four-vector $J_\mu=(\vec{J},\ icJ_0)$ consists of the density J_0 and the current density $\vec{J}$ of the (rest) mass distribution and obeys the generalized equation of continuity

$$\nabla_\mu J_\mu=A, \tag{A$\cdot$3}$$

where A is the rate of increase (or decrease) of the mass. For the mixed fluid, denoting the mass flow of the i-th constituent of the fluid by $J_\mu^{(i)}$, one may write

$$\nabla_\mu J_\mu^{(i)}=A^{(i)},\quad J_\mu=\sum_i J_\mu^{(i)},\quad A=\sum_i A^{(i)},\tag{A·4}$$

where $A^{(i)}$ is the rate of increase of the mass for the i-th constituent.

In hydrodynamics an important quantity derived from $T_{\mu\nu}$ or J_μ is the local velocity of the fluid. Though its definition is trivial in the non-relativistic theory, it is not the case in the relativistic one. Since in the relativistic case the energy flow causes the momentum transport, the momentum of the system never vanishes even in the absence of movement of particles if the pure heat flow exists. Consequently, there are two ways to define the local velocity of the fluid, that is, (i) by the mass flow[11] and (ii) by the momentum density[12]. On the former standpoint one must permit the local rest system with non-vanishing momentum, while on the latter standpoint there is the movement of the substances in the local rest system. In the present paper we define the local velocity $\vec{u}$ by the mass flow, that is,

$$\vec{u}(\vec{x},t)=\vec{J}(\vec{x},t)/J_0(\vec{x},t)\tag{A·5}$$

which is interpreted as the center-of-mass velocity of the small volume surrounding the point $\vec{x}$ at time t. Hence one obtains the local four-velocity U_μ,

$$U_\mu=(\vec{u}/\sqrt{1-(u/c)^2},\quad ic/\sqrt{1-(u/c)^2})=cJ_\mu/\sqrt{-J_\mu^2}\,.\tag{A·6}$$

Since $U_\mu^2=-c^2$ and U_μ is a four-vector, the velocity $\vec{u}$ is transformed like a particle velocity under the Lorentz transformation. The relation between J_μ and U_μ is also rewritten as

$$J_\mu=\rho U_\mu=(1/v)\,U_\mu\,,\tag{A·7}$$

where $\rho=(1/c)\sqrt{-J_\mu^2}$ and $\rho=(1/v)$ are the invariant density of the mass distribution and the specific volume of the fluid, respectively.

Using the local rest system, we can determine at each point in the fluid the local rest system, in which all physical quantities are to be defined through the correspondence to the non-relativistic theory. First we define the concentration $c^{(i)}$ and the relative flow $I_\mu^{(i)}$ of the i-th constituent by

$$c^{(i)}=-U_\mu J_\mu^{(i)}/\rho c^2\tag{A·8}$$

$$I_\mu^{(i)}=\Delta_{\mu\nu}J_\nu^{(i)}=J_\mu^{(i)}-c^{(i)}J_\mu,\tag{A·9}$$

where $\Delta_{\mu\nu}=\delta_{\mu\nu}+U_\mu U_\nu/c^2$ is the projection tensor to the spacelike direction perpendicular to U_μ. In the rest system they reduce to $c^{(i)}=J_0^{(i)}/J_0$, $\vec{I}^{(i)}=\vec{J}^{(i)}$ and $I_0^{(i)}=0$. Note that $\sum_i c^{(i)}=1$, $U_\mu I_\mu^{(i)}=0$ and $\sum_i I_\mu^{(i)}=0$.

Since in our rest system the energy flow remains as a heat flow, it holds, in general, that

$$T_{\mu\nu}U_\nu=-\epsilon U_\mu-q_\mu,\tag{A·10}$$

where ϵ is the invariant energy density, becoming $-T_{44}$ in the rest system, and q_μ the heat flow, turning to $(\vec{q},0)$ in the rest system. They are explicitly defined by

$$\epsilon = (1/c^2)\, U_\mu T_{\mu\nu} U_\nu \tag{A·11}$$

$$q_\mu = -\Delta_{\mu\nu} T_{\nu\lambda} U_\lambda . \tag{A·12}$$

The internal stress $t_{\mu\nu}$ is formed as the space-like component of $T_{\mu\nu}$, that is,

$$t_{\mu\nu} = \Delta_{\mu\tau} T_{\tau\lambda} \Delta_{\lambda\nu} , \tag{A·13}$$

which becomes the spatial part of $T_{\mu\nu}$ in the rest system. This is interpreted as a transmitting force of the momentum transport. The pressure p is defined by

$$p = \tfrac{1}{3}\mathrm{Tr}\,(t_{\mu\nu}) = \tfrac{1}{3}\epsilon + \tfrac{1}{3}\mathrm{Tr}\,(T_{\mu\nu}) , \tag{A·14}$$

which is regarded as the equation of state when the fluid is in an equilibrium. Divide $t_{\mu\nu}$ into two parts as

$$t_{\mu\nu} = p_s \Delta_{\mu\nu} - P_{\mu\nu} , \tag{A·15}$$

where the hydrostatic pressure $p_s \Delta_{\mu\nu}$ is independent of the deformation velocities $\nabla_\mu U_\nu$, while the viscous stress $P_{\mu\nu}$ is dependent on $\nabla_\mu U_\nu$. The form $p_s \Delta_{\mu\nu}$ is due to the isotropy of the fluid. Thus the energy-momentum density tensor $T_{\mu\nu}$ is rewritten in terms of ϵ, U_μ, q_μ and $t_{\mu\nu}$ as

$$T_{\mu\nu} = (1/c^2)\, \epsilon U_\mu U_\nu + (1/c^2)\, (U_\mu q_\nu + U_\nu q_\mu) + t_{\mu\nu} , \tag{A·16}$$

or

$$\left.\begin{aligned} T_{\mu\nu} &= T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} \\ T^{(1)}_{\mu\nu} &= p_s \delta_{\mu\nu} + (1/c^2)\, (p_s + \epsilon)\, U_\mu U_\nu \\ T^{(2)}_{\mu\nu} &= (1/c^2)\, (U_\mu q_\nu + U_\nu q_\mu) - P_{\mu\nu} , \end{aligned}\right\} \tag{A·17}$$

where $T_{\mu\nu}{}^{(1)}$ is nothing but the energy-momentum tensor of the perfect fluid.

If we start from another definition of the local velocity using the momentum density, then we must write

$$T_{\mu\nu} U_\nu = -\epsilon U_\mu \tag{A·10'}$$

in the place of (A·10), where ϵ is defined by the same equation as (A·11). In such a fashion the local velocity U_μ is rather well defined by (A·10') and (A·11). Equations (A·13) to (A·17) hold with the same forms except for dropping the heat flow. On the other hand the relation between the local velocity and the mass flow should be changed as

$$J_\mu = \rho' U_\mu + q'_\mu , \qquad \rho' = -\frac{1}{c^2} U_\mu J_\mu , \tag{A·7'}$$

where ρ' is the invariant mass density, while q_μ' has a close relation to the heat flow.

Next let us derive some important laws of thermodynamics. Rearranging the equation $U_\mu \nabla_\nu T_{\mu\nu} = 0$, one can obtain the *first law* in the form

$$\Gamma_\mu[(\epsilon+p_s)U_\mu+q_\mu]=-\frac{1}{c^2}q_\mu DU_\mu+P_{\mu\nu}\Gamma_\nu U_\mu, \tag{A·18}$$

or

$$\rho(D\epsilon_v+p_sDv)=-(\epsilon_v+p_sv)A$$
$$-\left(\Gamma_\mu q_\mu+\frac{1}{c^2}q_\mu DU_\mu\right)+P_{\mu\nu}\Gamma_\nu U_\mu, \tag{A·18′}$$

where $\epsilon_v=\epsilon v$ and $D=U_\alpha\Gamma_\alpha$. This equation represents evidently the balance of energy. For the *second law* we have Gibbs' relation

$$D\epsilon_v+p_sDv-\sum_i\mu^{(i)}Dc^{(i)}=TDs_v, \tag{A·19}$$

where T is the temperature, $\mu^{(i)}$ the chemical potential of the i-th constituent and $s_v=sv$ the entropy in the specific volume, s being the invariant entropy density. Substituting (A·18′) and (A·8) in the term $D\epsilon_v+p_sDv$ and $c^{(i)}$ of the left-hand side of (A·19) and separating the entropy terms, one obtains

$$\rho Ds_v+s_vA+\Gamma_\mu I_\mu^{(s)}=\frac{1}{T}[q_\mu X_\mu^{(h)}+\sum_i I_\mu^{(i)}X_\mu^{(i)}+P_{\mu\nu}\Gamma_\nu U_\mu-\sum_i A^i\mu^{(i)}], \tag{A·20}$$

where we have made use of the relations $\mu=\sum_i\mu^{(i)}c^{(i)}$ and $v(\epsilon+p_s-\mu)=Ts_v$. The thermodynamical driving forces in the right-hand side of (A·20) are defined by

$$\begin{aligned}X_\mu^{(h)}&=-\left[\frac{1}{T}\Gamma_\mu T+\frac{1}{c^2}DU_\mu\right]\\ X_\mu^{(i)}&=-T\Gamma_\mu\left(\frac{\mu^{(i)}}{T}\right)\end{aligned} \tag{A·21}$$

and the conduction current of entropy by

$$I_\mu^{(s)}=\frac{1}{T}[q_\mu-\sum_i\mu^{(i)}I_\mu^{(i)}]. \tag{A·22}$$

By making use of the definition of the entropy flow

$$S_\mu=sU_\mu+I_\mu^{(s)}, \tag{A·23}$$

the left-hand side of (A·20) can be reduced to the simple form $\Gamma_\mu S_\mu$. It is easily seen that (A·20) represents the balance of entropy because its right-hand side is interpreted as the entropy increment produced in irreversible processes.

Finally we write the phenomenological equations, representing the properties of the fluid as the relations btween the flows and the driving forces,

$$\begin{aligned}q_\mu&=\kappa X_\mu^{(h)}+\sum_i\lambda^{(i)}X_\mu^{(i)}\\ I_\mu^{(i)}&=\sum_j\xi^{(i)(j)}X_\mu^{(i)}+\lambda'^{(i)}X_\mu^{(i)}\end{aligned}$$

$$\begin{aligned}
&\Pi \equiv \tfrac{1}{3}\mathrm{Tr}\,(P_{\mu\nu}) = \eta_{(v)}\Gamma_{\mu}U_{\mu} + \sum_{i} L^{(i)}_{(p)(c)}\mu^{(i)} \\
&A^{(i)} = L^{(i)}_{(c)(p)}\Gamma_{\mu}U_{\mu} + \sum_{j} L^{(i)(j)}\mu^{(j)} \qquad\qquad (\mathrm{A}\cdot 24) \\
&\bar{P}_{\mu\nu} \equiv P_{\mu\nu} - \Pi\Delta_{\mu\nu} = \eta_{(s)}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta}(\Gamma_{\alpha}U_{\beta} + \Gamma_{\beta}U_{\alpha}) - \tfrac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\Gamma_{\beta}U_{\alpha}\right].
\end{aligned}$$

Strictly speaking, such equations may have to be written in the integral forms as given in § 3. It is noted that the phenomenological coefficients such as κ, $\lambda^{(i)}$, η's and so forth are invariant under the Lorentz transformation. Furthermore, we remark the equation

$$p = p_s - \eta_{(v)}U_{\alpha}U_{\alpha} - \sum_{i} L^{(i)}_{(p)(c)}\mu^{(i)} \qquad (\mathrm{A}\cdot 25)$$

obtained by (A·24) and (A·15).

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On the Theory of the Unstable Particle, II

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A model which is more convenient than the Lee model for studying the theory of the unstable particle is proposed. In this model, the decayed particles are distinguished from the parental ones. The production and decay of an unstable particle is investigated by the stationary treatment. The conclusion we reached is that it is possible to interpret the renormalization constant as the probability in the same way as in the case of the stable particles.

Recently, many authors[1)2)] investigated the theory of the unstable particle by using the Lee model[3)]. It seems, however, that we have not yet found any satisfactory way of judging whether we can interpret the renormalization constant as representing the probability for finding the bare state in a physical unstable particle. The cause which makes this question complicated is due to the finite lifetime of the unstable particle and the uncertainties originated from this lifetime. It is worth while to make a research of the properties of physical unstable particles in order to answer the above mentioned question. In the previous paper of the present author, some arguments were made concerning this problem by using the Lee model. This model was, however, inconvenient for studying this problem because of the fact that the parental particles which produce the unstable V-particles are identical with the particles to which the V's decay.[2)] In the stationary treatment of the decay process, if the plural particles are produced by some collision, we can discuss the stability of the produced particles by introducing the relative co-ordinate of these particles, and by investigating the behaviour of the wave amplitude of the unstable particle without taking into account the decayed products. In this case, it is easily seen that the final particles produced from the unstable one should necessarily be different from the first particles from which the plural particles (at least one of them is unstable) were produced.

In order to distinguish the decayed particles from the parental ones, we shall consider, following Lee, the four kinds of fictitious fermions, N_1, N_2, N_3, N_4, together with three kinds of bosons, θ_1, θ_2 and θ_3, with the mutual interaction defined by the total Hamiltonian :

$$H=H_1+H_2\,,$$

where

$$H_1=\sum_{i=1}^{i=4} m_i\int d\boldsymbol{p}\psi_i^*(\boldsymbol{p})\,\psi_i(\boldsymbol{p})+\sum_{i=1}^{i=3}\int d\boldsymbol{k}\omega_k a_i^*(\boldsymbol{k})\,a_i(\boldsymbol{k})$$

and

$$H_2=(2\pi)^{-3/2}\int d\boldsymbol{p}d\boldsymbol{k}\,\frac{1}{\sqrt{2\omega_k}}\Big(g_1 f_1(\omega_k)\,\psi_2{}^*(\boldsymbol{p})\,\psi_1(\boldsymbol{p}-\boldsymbol{k})\,a_1(\boldsymbol{k})$$

$$+g_2 f_2(\omega_k)\psi_2^*(\boldsymbol{p})\psi_3(\boldsymbol{p}-\boldsymbol{k})a_2(\boldsymbol{k})+g_3 f_3(\omega_k)\psi_3^*(\boldsymbol{p})\psi_4(\boldsymbol{p}-\boldsymbol{k})a_3(\boldsymbol{k})\Big)$$

$$+\text{Hermite conjugate.}$$

Here, m_i is the unrenormalized mass of the i-th fermion and the masses of bosons are, for simplicity, taken to be equal to each other. The magnitude of the mass of each physical particle is assumed in such a way that N_3 is unstable[2] and can decay into $N_4+\theta_3$ while others are stable ones.

In the present paper, we shall investigate the behaviour of N_3 associated with θ_2 produced by an $N_1-\theta_1$ collision and discuss the relationship between the renormalization constant and the probability for finding the bare N_3 in its physical state in a similar way to that used in the case of stable particles.

The model stated above has the following six constants concerning the numbers of particles:

$$\begin{aligned}
\boldsymbol{N}_1+\boldsymbol{N}_2+\boldsymbol{N}_3+\boldsymbol{N}_4&=\text{constant},\\
\boldsymbol{\theta}_1+\boldsymbol{N}_2+\boldsymbol{N}_3+\boldsymbol{N}_4&=\text{constant},\\
\boldsymbol{N}_1+\boldsymbol{N}_2+\boldsymbol{N}_3+\boldsymbol{\theta}_3&=\text{constant},\\
\boldsymbol{\theta}_1+\boldsymbol{N}_2+\boldsymbol{N}_3+\boldsymbol{\theta}_3&=\text{constant},\\
\boldsymbol{N}_1+\boldsymbol{N}_2+\boldsymbol{\theta}_2&=\text{constant},\\
\boldsymbol{\theta}_1+\boldsymbol{N}_2+\boldsymbol{\theta}_2&=\text{constant},
\end{aligned}$$

where $\boldsymbol{N}_i$ or $\boldsymbol{\theta}_i$ is the total number of N_i- or θ_i-particle. Accordingly, the stationary state describing the $N_1-\theta_1$ collision can be expressed as

$$|Z\rangle=\Big(\int d\boldsymbol{k}\alpha(\boldsymbol{k})\psi_1^*(\boldsymbol{p}-\boldsymbol{k})a_1^*(\boldsymbol{k})+\beta\psi_2^*(\boldsymbol{p})+\int d\boldsymbol{k}\gamma(\boldsymbol{k})\psi_3^*(\boldsymbol{p}-\boldsymbol{k})a_2^*(\boldsymbol{k})$$
$$+\int d\boldsymbol{k}d\boldsymbol{k}'\lambda(\boldsymbol{k},\boldsymbol{k}')\psi_4^*(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{k}')a_2^*(\boldsymbol{k})a_3^*(\boldsymbol{k}')\Big)|0\rangle. \tag{1}$$

The eigenvalue E of this state is given by

$$\left.\begin{aligned}
&(m_1+\omega_k-E)\alpha(\boldsymbol{k})+(2\pi)^{-3/2}\frac{g_1 f_1(\omega_k)}{\sqrt{2\omega_k}}\beta=0,\\
&(m_2-E)\beta+(2\pi)^{-3/2}\int d\boldsymbol{k}\frac{g_1 f_1(\omega_k)}{\sqrt{2\omega_k}}\alpha(\boldsymbol{k})+(2\pi)^{-3/2}\int d\boldsymbol{k}\frac{g_2 f_2(\omega_k)}{\sqrt{2\omega_k}}\gamma(\boldsymbol{k})=0,\\
&(m_3+\omega_k-E)\gamma(\boldsymbol{k})+(2\pi)^{-3/2}\frac{g_2 f_2(\omega_k)}{\sqrt{2\omega_k}}\beta\\
&\qquad\qquad+(2\pi)^{-3/2}\int d\boldsymbol{k}'\frac{g_3 f_3(\omega_{k'})}{\sqrt{2\omega_{k'}}}\lambda(\boldsymbol{k},\boldsymbol{k}')=0
\end{aligned}\right\} \tag{2}$$

and $(m_4+\omega_k+\omega_{k'}-E)\lambda(\boldsymbol{k},\boldsymbol{k}')+(2\pi)^{-3/2}\dfrac{g_3 f_3(\omega_{k'})}{\sqrt{2\omega_{k'}}}\gamma(\boldsymbol{k})=0.$

We solve eqs. (2) with the boundary condition that $\alpha(\boldsymbol{k})$ represents the incident plane wave plus the outgoing spherical wave and $\lambda(\boldsymbol{k}, \boldsymbol{k}')$ the outgoing one. After the straightforward calculations, we get

$$\lambda(\boldsymbol{k}, \boldsymbol{k}') = -(2\pi)^{-3/2}\frac{g_3 f_3(\omega_{k'})}{\sqrt{2\omega_{k'}}}\gamma(\boldsymbol{k})\,(m_4+\omega_k+\omega_{k'}-E-i\epsilon)^{-1}, \tag{3}$$

$$\gamma(\boldsymbol{k}) = -(2\pi)^{-3/2}\frac{g_2 f_2(\omega_k)}{\sqrt{2\omega_k}}\beta\cdot\Big(m_3-y\,(E-\omega_k+i\epsilon)\Big)^{-1}, \tag{4}$$

$$\alpha(\boldsymbol{k}) = \delta(\boldsymbol{k}-\boldsymbol{k}_0) - (2\pi)^{-3/2}\frac{g_1 f_1(\omega_k)}{\sqrt{2\omega_k}}\beta\cdot(m_1+\omega_k-E-i\epsilon)^{-1} \tag{5}$$

and

$$\beta = -(2\pi)^{-3/2}\frac{g_1 f_1(\omega_{k_0})}{\sqrt{2\omega_{k_0}}}\Bigg[m_2-E-(2\pi)^{-3}\int d\boldsymbol{k}\,\frac{g_1^{\,2} f_1^{\,2}(\omega_k)}{2\omega_k}\,(m_1+\omega_k-E-i\epsilon)^{-1}$$

$$-(2\pi)^{-3}\int d\boldsymbol{k}\,\frac{g_2^{\,2} f_2^{\,2}(\omega_k)}{2\omega_k}\Big(m_3-y(E-\omega_k+i\epsilon)\Big)^{-1}\Bigg]^{-1}, \tag{6}$$

where

$$y(z) = z + (2\pi)^{-3}\int d\boldsymbol{k}\,\frac{g_3^{\,2} f_3^{\,2}(\omega_k)}{2\omega_k}\,\frac{1}{m_4+\omega_k-z} \tag{7}$$

and

$$m_1+\omega_{k_0}-E=0.$$

When N_3 is unstable, $m_3-y(E-\omega_k+i\epsilon)$ has no zero point in the real value of $E-\omega_k$ (cf. Fig. 1)[2].

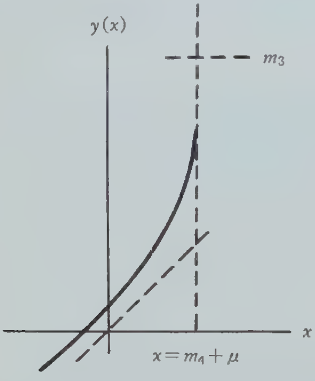

Fig. 1. The graph of $y(x)$

Therefore in solving the last two equations in (2) with respect to $\gamma(\boldsymbol{k})$, there occurs no ambiguity in dividing the both sides of the above equations by the factor $m_3-y(E-\omega_k+i\epsilon)$, namely, we need no boundary condition for determining the $\gamma(\boldsymbol{k})$ provided that the $\lambda(\boldsymbol{k}, \boldsymbol{k}')$ is determined by its boundary condition. This situation is due to the fact that $|\gamma(\boldsymbol{k})|^2$ represents the probability of finding the state of N_3 and θ_2 traveling away from the source and the wave function in the co-ordinate space corresponding to the above $\gamma(\boldsymbol{k})$ rapidly vanishes in the remote point from the source as it should be in the case of the unstable state. On the contrary, in the case of stable particle, since the factor $m_3-y(E-\omega_k+i\epsilon)$ has a real zero point, we need a boundary condition for removing the ambiguity occuring in the formal solution of $\gamma(\boldsymbol{k})$.

To investigate the production of the unstable N_3, let us examine the behaviour of $\gamma(\boldsymbol{x}) = \int d\boldsymbol{k}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\gamma(\boldsymbol{k})$ for the sufficiently large $r=|\boldsymbol{x}|$. A short calculation leads to

$$\gamma(\boldsymbol{x}) = (2\pi)^{-1/2}\beta\cdot g_2 r^{-1} i\int_{-\infty}^{\infty} k\,dk\,e^{ikr}\,\frac{f_2(\omega_k)}{\sqrt{2\omega_k}}\Big(m_3-y(E-\omega_k+i\epsilon)\Big)^{-1}. \tag{8}$$

The path of this integral is shown in Fig. 2.

In order to discuss the behaviour of $\gamma(x)$, it is of use to mention about the functional properties of $(m_3-y(z))^{-1}$. The function $(m_3-y(z))^{-1}$ has one branch point on the real axis of z-plane corresponding to the creation of the θ_3-meson. This situation forces us to put a cut on the z-plane from the branch point z_0 to ∞ or alternatively a cut on the complex k-plane from $-k_0$ to $+k_0$ along the real axis, where $z=E-\sqrt{k^2+\mu^2}$. Besides these branch points, according to the paper of Araki *et al.*, if E is sufficiently large, the function $(m_3-y(E-\omega_k+i\epsilon))^{-1}$ (the imaginary number $i\epsilon$ is necessary for determining the Riemanian plane) has a pole $k_1+ik_2(k_1>0,\ k_2>0)$ and another $-k_1-ik_2$ on the k-plane where the former can be connected with the latter by a curve crossing the cut (cf. Fig. 2).

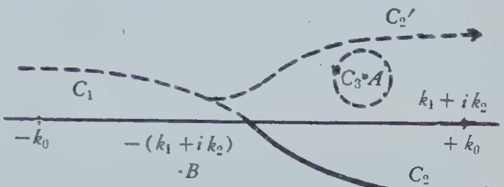

Fig. 2. The path of integral (8) The dotted line indicates that the path belongs to the lower Riemanian plane. Similarly the pole A and B are on the lower and upper Riemanian plane respectively.

The pole on the $E-\omega_k$ plane corresponding to the above mentioned poles is $M-i\Gamma/2$ $(M>0,\ \Gamma>0)$ which is given by

$$M-\frac{i}{2}\Gamma+(2\pi)^{-3}\int dk\,\frac{g_3^{\,2}f_3^{\,2}(\omega_k)}{2\omega_k}\,\frac{1}{m_4+\omega_k-M+(i\Gamma/2)}$$

$$+\frac{i}{2\pi}\sqrt{(M-(i\Gamma/2)-m_4)^2-\mu^2}\;g_3^{\,2}f_3^{\,2}(M-(i\Gamma/2)-m_4)=m_3. \quad (9)$$

We assume that the magnitude of Γ is much smaller than M, since M and Γ are the mass and the reciprocal of the lifetime of the N_3-particle respectively.

For a sufficiently large r, owing to the rapid oscillation of $\exp(ikr)$, the integrand for a large k has no contribution to $\gamma(x)$. In the vicinity of the pole k_1+ik_2, the integrand approximately has a form as

$$\frac{\chi}{k-(k_1+ik_2)}\,e^{ikr}=\frac{\chi}{\sqrt{(k-k_1)^2+k_2^{\,2}}}\,e^{i(kr+\phi)},$$

where

$$\chi=\sqrt{\frac{E-M+i\Gamma/2}{2}}\,f_2\Big(E-M+\frac{i}{2}\Gamma\Big)\Big[1+(2\pi)^{-3}\int dk\,\frac{g_3^{\,2}f_3^{\,2}(\omega_k)}{2\omega_k}\,\frac{1}{(m_4+\omega_k-M+i\Gamma/2)^2}$$

$$+\frac{i}{2\pi}\frac{\partial}{\partial M}\sqrt{(M-i\Gamma/2-m_4)^2-\mu^2}\;g_3^{\,2}f_3^{\,2}\Big(M-\frac{i}{2}\Gamma-m_4\Big)\Big]^{-1}$$

and

$$\tan\phi=k_2/(k-k_1).$$

The phase change of the above expression at the point $k\approx k_1$ is approximately

$$\Delta k \cdot r + \Delta\phi = \Delta k\left(r - \frac{1}{k_2}\right)$$

where Δk is the variation of k at the point $k \approx k_1$. In order to get a non-vanishing contribution to $\gamma(\boldsymbol{x})$, the phase change should be much smaller than unity. Accordingly, from the above expression of $\Delta\phi$, we see that $\gamma(\boldsymbol{x}) \neq 0$ if $r \lesssim k_2^{-1}$ and $\Delta k \lesssim k_2$, while if $\Delta k \gg k_2$ or $r \gg k_2^{-1}$ the strong cancellation of the integration occurs by virtue of the rapid oscillation of the phase factor. Therefore the path of the integral can be deformed in such a way as

$$c_1 + c_2 \to c_1 + c_2' + c_3$$

if $k_0 \gg \Delta k \approx k_2$ (cf. Fig. 2). It is easily seen that the contribution from the path $c_1 + c_2'$ vanishes. Thus we have

$$\gamma(\boldsymbol{x}) \approx -\sqrt{2\pi}\,\beta g_2 \chi \frac{e^{ik_1 r}\, e^{-k_2 r}}{r} \tag{10}$$

for a large r. In the same way, the asymptotic expression of $\lambda(\boldsymbol{x}, \boldsymbol{k})$ is

$$\begin{aligned}\lambda(\boldsymbol{x}, \boldsymbol{k}) \approx &-\pi^{1/2} \frac{g_3 f_3(\omega_k)}{\sqrt{\omega_k}} \omega_{k_3} \gamma(\boldsymbol{k}_3) \frac{e^{ik_3 r}}{r} \\ &-\gamma(\boldsymbol{x})\,(2\pi)^{-3/2} \frac{g_3 f_3(\omega_k)}{\sqrt{2\omega_k}} \left(m_4 + \omega_k - M + \frac{i}{2}\Gamma\right)^{-1},\end{aligned} \tag{11}$$

where

$$\lambda(\boldsymbol{x}, \boldsymbol{k}) = \int d\boldsymbol{k}' \exp(i\boldsymbol{k}', \cdot \boldsymbol{x})\, \lambda(\boldsymbol{k}', \boldsymbol{k})$$

and k_3 is the positive root of $m_4 + \omega_k + \omega_{k_3} - E = 0$.

In the case of the stable N_3, where the damping factor $\exp(-k_2 r)$ does not occur, it is obvious how to interpret the each term of the asymptotic expressions of γ and λ. Namely, the $\gamma(\boldsymbol{x})$ describes the state of the bare N_3 associated with θ_2, while the second term in (11) that of the dressed N_3, the sum of these corresponds to the state of the physical N_3 plus θ_2. The first term of (11) stands for the outgoing wave of θ_2, θ_3 and N_4. It may be quite reasonable to assume that the completely similar interpretation is also possible in the case of the unstable N_3. From this interpretation the probability of finding the θ_2-meson at the point $\boldsymbol{x}\,(r \gtrsim k_2^{-1})$ and at the same time N_3 at the origin is given by

$$P(\boldsymbol{x}) = \left(1 + (2\pi)^{-3} \int d\boldsymbol{k} \frac{g_3^2 f_3^2(\omega_k)}{2\omega_k} \frac{1}{(m_4 + \omega_k - M)^2 + (\Gamma^2/4)}\right) |\gamma(\boldsymbol{x})|^2. \tag{12}$$

Owing to the damping factor in $\gamma(\boldsymbol{x})$, the probability (12) decrease for increasing r as we expect in the case of the unstable particle.

The probability lost from $P(\boldsymbol{x})$ as the point $\boldsymbol{x}$ goes away may be gained by the state of θ_2, θ_3 and N_4. That this is the case is seen as follows. Let us put

$$\lambda(\boldsymbol{x}, \boldsymbol{k}) = \lambda_1(\boldsymbol{x}, \boldsymbol{k}) - \gamma(\boldsymbol{x})\,(2\pi)^{-3/2}\frac{g_3 f_3(\omega_k)}{\sqrt{2\omega_k}}\,\frac{1}{m_4+\omega_k-M+(i\Gamma/2)}$$

and consider the expression

$$|\gamma(\boldsymbol{x})|^2 + \int d\boldsymbol{k}|\lambda(\boldsymbol{x}, \boldsymbol{k})|^2 = \int d\boldsymbol{k}|\lambda_1(\boldsymbol{x}, \boldsymbol{k})|^2$$

$$-2\mathrm{Re}\int d\boldsymbol{k}\left(\lambda_1(\boldsymbol{x}, \boldsymbol{k})\,(2\pi)^{-3/2}\frac{g_3 f_3(\omega_k)}{\sqrt{2\omega_k}}\,\frac{1}{m_4+\omega_k-M-(i\Gamma/2)}\gamma^*(\boldsymbol{x})\right)+P(\boldsymbol{x}). \tag{13}$$

Now $\lambda_1(\boldsymbol{x}, \boldsymbol{k})$ can be rewritten as

$$-\pi^{1/2}\frac{g_3 f_3(\omega_k)}{\sqrt{\omega_k}}\int_0^\infty dk'\,k'\gamma(k')\,\frac{e^{ik'r}}{r}\delta(\omega_k+\omega_k'+m_4-E).$$

Inserting this expression into the second term of (13), we have

the second term of (13)

$$= (2\pi)^{-1}\mathrm{Re}\left(\int dk\int_0^\infty dk'\,\frac{g_3^2 f_3^2(\omega_k)}{\omega_k}\,\frac{k'\gamma(k')\,\delta(\omega_k+\omega_k'+m_1-E)}{m_4+\omega_k-M-(i\Gamma/2)}\,\frac{e^{ik'r}}{r}\gamma^*(\boldsymbol{x})\right)$$

$$= 2\mathrm{Re}\left(\int_0^a k'dk'\gamma(k')\,\frac{g_3^2 f_3^2(E-m_4-\omega_{k'})\sqrt{(E-m_4-\omega_{k'})^3-\mu^2}}{E-\omega_{k'}-M-(i\Gamma/2)}\,\frac{e^{ik'r}}{r}\gamma^*(\boldsymbol{x})\right), \tag{14}$$

where $a=\sqrt{(E-m_4-\mu)^2-\mu^2}$. Now recalling the strong cancellation occurred in the evaluation of (10), we see that the domain of integration in (14) can be extended from $-\infty$ to $+\infty$ without causing any change of the value of (14), if a pole of the integrand does not exist in the vicinity of the path of integral added. In the case of the stable N_3, since the integrand of (14) has no pole in the domain $(0, a)$ we see that (14) vanishishes owing to the above mentioned cancellation without extending the domain of integral. On the other hand, if N_3 is unstable, we have a pole in the domoin $(0, a)$ which give the result

the second term of (13)

$$= -\frac{2}{\Gamma}\mathrm{Re}\left(\frac{1}{2\pi}g_3^2 f_3^2(M-(i\Gamma/2)-m_4)\sqrt{(M-m_4-(i\Gamma/2)^2-\mu^2}\right)|\gamma(\boldsymbol{x})|^2 \tag{15}$$

by following the line of reasoning stated in (10).

Now the imaginary part of (9) leads to

$$\frac{\Gamma}{2}\left(1+(2\pi)^{-3}\int d\boldsymbol{k}\frac{g_3^2 f_3^2(\omega_k)}{2\omega_k}\,\frac{1}{(m_4+\omega_k-M)^2+(\Gamma^2/4)}\right)$$

$$= \mathrm{Re}\left(\frac{1}{2\pi}g_3^2 f_3^2(M-(i\Gamma/2)-m_4)\sqrt{(M-(i\Gamma/2)-m_4)^2-\mu^2}\right). \tag{16}$$

Inserting (12), (15) and (16) into (13), we see that the second and third terms in

(13) cancel each other. This cancellation is nothing but the conservation of the total probability.

From (12), the probability of finding the bare N_3-state in the physical N_3-particle is written as

$$N^2=\left(1+(2\pi)^{-3}\int dk\,\frac{g_3^{\,2}f_3^{\,2}(\omega_k)}{2\omega_k}\,\frac{1}{(m_4+\omega_k-M)^2+(\Gamma^2/4)}\right)^{-1}. \qquad (17)$$

We can easily confirm that N is taken as the renormalization constant for g_3. The expression (17) essentially depends on the method of approximation in evaluating $\gamma(x)$. Accordingly a slight difference in the expression N or Γ in the behaviour of $\gamma(\boldsymbol{x})$ can be expected when another method of approximation would be used, but we can say that the difference may be of the order of Γ/M and impossible to detect (cf. Araki *et al.*). Anyhow, it must be noted that the final expressions for N and $P(x)$ have the type (17) and (12) respectively without regard to the kinds of approximations owing to the equation (3). Therefore, we arrive at the conclusion that, even in the case of the unstable particle, we can interpret the renormalization constant as the probability.

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Electrodisintegration of He^4 Nucleus

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The inelastic continuum of scattered electrons by He^4 nuclei, observed experimentally by the Stanford group, is worked theoretically by using the conventional forms of wave functions for He^4, He^3 and triton, respectively. It is pointed out that two kinds of scattering are involved in the cross section; one is connected with the direct process in which a nucleon can be ejected through the direct interaction with the electromagnetic field induced by the electron and the other one, with the indirect process in which a nucleon is capable of being ejected through the intermediary of nuclear interaction with another nucleon which does interact with the electromagnetic field. The comparison of our results with the experiments appears to be rather good, though qualitatively, allowing for the approximate nature of our method of computation, and then the main nuclear reaction connected with the observed inelastic continuum is shown to be the ejection of nucleon from He^4 except for the low-energy tail where the pion production is observed to take place according to the Stanford group.

§ 1. Introduction

Although the theoretical analysis of elastic scattering of high energy electrons by various nuclei has so far been made extensively by many investigators[1] in connection with the shape of nuclear charge distributions, there have been relatively few who dealt theoretically with inelastic scattering associated with the nuclear excitation and disintegration. In fact, elastic and inelastic scatterings of electrons by deuteron have been worked by Jankus[2] and the nuclear excitations in Be- and C-nuclei by the high energy electron have been dealt with theoretically by Morpurgo[3] and quite recently by Pal and Mukherjee,[4] using the nuclear shell model, whereas we do not have at present any theory treating inelastic scattering of high energy electrons by He^4 for the comparison with the inelastic continuum observed experimentally by the Stanford group.

Scattering of electrons from He^4 nucleus has been experimentally carried out by McAllister and Hofstadter[5] at 188 Mev and by Blankenbecler and Hofstadter[6] at 400 Mev, respectively. Particularly, the latter authors have actually observed, in some detail, the inelastic continuum in the alpha-particle at 400 Mev for scattering angles of 45° and 60°. The most probable reaction in He^4 nucleus accompanied by inelastic electron scattering is conjectured to be the ejection of a nucleon, i. e., a proton or a neutron and then the scattered electron energy must be reduced by, at least, the binding energy of that nucleon in the nucleus in addition to the kinetic energy of an ejected nucleon. The study of such inelastic continuum will, therefore, furnish valuable information about the momentum distribution of nucleons within the nucleus in quite a similar way to the case of the pick-up nuclear reaction. Moreover, the electrodisintegration of He^4 nucleus provides attractive features in two respects : first the interaction between electron and nucleus is of electromagnetic nature, which is now well-established in comparison with the more or less ambiguous features of the nuclear interactions responsible for the ordinary nuclear reactions, and secondly, for He^4 nucleus, we have some knowledge of the ground state wave function, including, in some measure, the correlation among nucleons, through the variational calculation of the binding energy, in contradistinction to the case of heavier nuclei based on the nuclear shell model.

The object of the present work is to compute the inelastic continuum of scattered electrons by He^4 nucleus by using the conventional forms of wave functions for He^4, He^3 and triton. The comparison of our results with the mentioned experimental observations appears to be rather good, though qualitatively, allowing for the approximate nature of our calculation, and thus the main nuclear reaction closely connected with the observed inelastic continuum may be considered to be the ejection of a nucleon from He^4, the other possible modes of disintegrations of He^4 being negligibly small in comparison with the former, at least, at 400 Mev electron bombardment.

§ 2. Cross Section of the Inelastic Continuum Accompanied by a Nucleon Ejection from He^4

Since the incident energy of electrons is high (400 Mev) and the target nucleus (He^4) is very light ($Z/137 \ll 1$), the first Born approximation can be applied safely to our problem and then the electromagnetic perturbation of He^4 nucleus responsible for the reaction may be described by the Møller potential.[7)]

$$A_0=-\frac{4\pi\hbar^2 e}{V}\cdot\frac{(u',\ u)}{(\boldsymbol{p}_e-\boldsymbol{p}_e')^2-(E_e-E_e')^2}\cdot\exp[i(\boldsymbol{p}_e-\boldsymbol{p}_e')\boldsymbol{r}/\hbar-i(E_e-E_e')t/\hbar], \quad (1)$$

$$\boldsymbol{A}=-\frac{4\pi\hbar^2 e}{V}\cdot\frac{(u',\ \alpha u)}{(\boldsymbol{p}_e-\boldsymbol{p}_e')^2-(E_e-E_e')^2}\cdot\exp[i(\boldsymbol{p}_e-\boldsymbol{p}_e')\boldsymbol{r}/\hbar-i(E_e-E_e')t/\hbar], \quad (2)$$

in which $\boldsymbol{p}_e$ and E_e represent the momentum and energy of an incident electron, while $\boldsymbol{p}_e'$ and E_e' those of a scattered electron, respectively. u and u' express the spinors before and after the scattering and α the well-known Dirac's vector spin matrix. V denotes the normalization volume and the other quantities have the usual meaning.

Since the nucleonic motion within the nucleus is dealt approximately in non-relativistic way, the interaction energy of the nucleons with the Møller potential is given by

$$\delta H=\sum_{j=1}^{2}\delta H_{Pj}+\sum_{j=1}^{2}\delta H_{Nj}. \quad (3)$$

δH_{Pj} and δH_{Nj} stand for a proton and a neutron, respectively, which may be written, after a simple calculation, as

$$\delta H_{Pj}=B(\boldsymbol{p}_e,\ \boldsymbol{p}_e')\{(u',\ u)R_j+(u',\ \alpha u)\cdot(\boldsymbol{S}_j+\boldsymbol{T}_j)\}/V, \tag{4}$$

$$\delta H_{Nj}=B(\boldsymbol{p}_e,\ \boldsymbol{p}_e')(u',\ \alpha u)\cdot\boldsymbol{T}_j/V, \tag{5}$$

where

$$R_j=-e^2(1-\Delta E_e/2Mc^2)\cdot\exp[i(\boldsymbol{p}_e-\boldsymbol{p}_e')\boldsymbol{r}_j/\hbar-i\Delta E_e t/\hbar], \tag{6}$$

$$\boldsymbol{S}_j=(e^2/\hbar c)\cdot\exp[i(\boldsymbol{p}_e-\boldsymbol{p}_e')\boldsymbol{r}_j/\hbar-i\Delta E_e t/\hbar](\hbar/i)\boldsymbol{\nabla}_j, \tag{7}$$

$$\boldsymbol{T}_j=(ie^2/2Mc)g_{P\text{ or }N}[\sigma_j\times(\boldsymbol{p}_e-\boldsymbol{p}_e')]\cdot\exp[i(\boldsymbol{p}_e-\boldsymbol{p}_e')\boldsymbol{r}_j/\hbar-i\Delta E_e t/\hbar], \tag{8}$$

$$\Delta E_e=E_e-E_e' \tag{9}$$

and

$$B(\boldsymbol{p}_e,\ \boldsymbol{p}_e')=4\pi\hbar^2/[(\boldsymbol{p}_e-\boldsymbol{p}_e')^2-\Delta E_e^2/c^2]. \tag{10}$$

M denotes the mass of a nucleon, the mass difference of a proton from a neutron being disregarded approximately, σ_{Nj} and σ_{Pj} the Pauli spin matrices of a neutron and a proton, respectively. The g-factors of nucleonic, magnetic moments are given by

$$g_P=+2.7934\pm0.0003 \quad\text{and}\quad g_N=-1.9135\pm0.0003. \tag{10$'$}$$

R_j and $\boldsymbol{S}_j$, which are due to the proton charge, induce the nuclear transition in which the nuclear spin does not change, while $\boldsymbol{T}_j$, due to the magnetic moments of nucleons, gives rise to the nuclear spin-flip during the transition.

The cross section for the nuclear transition is given by

$$d\sigma=V\cdot\frac{2\pi}{\hbar}\cdot\frac{E_e}{c^2p_e}\cdot\frac{1}{2}\sum_{\text{spins}}|(f|\delta H|i)|^2\cdot\rho_F, \tag{11}$$

where the summation extends over the nuclear spins of both ejected nucleon and residual nucleus and, furthermore, over the electronic spins of both incident and scattered electrons, respectively. The factor 1/2 stands for the average with respect to the spin directions of an incident electron. ρ_F denotes the final state density.

As is shown later in the computation of transition matrix components, the following conservations of both momenta and energies of the concerned particles hold.

$$\boldsymbol{k}_e-\boldsymbol{k}_e'=\Delta\boldsymbol{k}_e=\boldsymbol{k}_p+\boldsymbol{k}_r, \tag{12}$$

$$E_e+M_\alpha c^2=E_e'+\frac{\hbar^2k_p^2}{2M}+\frac{\hbar^2k_r^2}{2M_r}+(M+M_r)c^2, \tag{13}$$

where the helium nucleus is assumed to be initially at rest, and $\hbar\boldsymbol{k}_p$ (or $\hbar\boldsymbol{k}_N$) and $\hbar\boldsymbol{k}_r$ represent the momenta of an ejected nucleon and a residual nucleus, respectively, M_r being the mass of a residual nucleus.

The state density of the final state is, as usual, given by

$$\rho_F=\frac{V^2}{(2\pi)^6}\cdot\frac{M_r}{\hbar^2\Delta k_e}k_e'^2dk_e'd\Omega_{k_e'}\cdot k_pdk_pd\varphi_p, \tag{14}$$

by the use of (12) and (13), in which $\boldsymbol{p}_e=\hbar\boldsymbol{k}_e$, $\boldsymbol{p}_e'=\hbar\boldsymbol{k}_e'$, $\Delta k_e=|\boldsymbol{k}_e-\boldsymbol{k}_e'|$ and $d\Omega_{k_e'}$

represents the solid angle into which the electron is scattered, φ_p being one of the polar angles which specify the direction of ejection of a nucleon.

For analytical simplicity, the Gaussian wave functions have been assumed for the nuclei of He^4, He^3 and triton since their use seems to be qualitatively good in the photoelectric disintegrations of the mentioned nuclei, as worked by Gunn and Irving,[8] and Flowers and Mandl,[9] respectively, and, furthermore, through the variational calculations of the corresponding binding energies. Then the total wave functions of both initial and final states of the nucleonic system may be written as follows.

He^4 *at rest (initial state)* :

$$\Psi_i^s(\widetilde{12},\ \widetilde{34}) = \Phi_i(\overline{12},\ \overline{34}) \cdot \frac{1}{2}(\alpha(1)\beta(2)-\alpha(2)\beta(1)) \cdot (\alpha(3)\beta(4)-\alpha(4)\beta(3)), \tag{15}$$

where nucleons 1 and 2 stand for protons, while nucleons 3 and 4 stand for neutrons, respectively. The bar signifies symmetry with respect to the interchange of the pair of particles and the tilde, antisymmetry. α and β express the spin eigenfunctions of the nucleons.

$$\Phi_i(\overline{12},\ \overline{34}) = \frac{1}{\sqrt{V}} N(\mu_\alpha) \exp\Big[-\mu_\alpha \sum_{i>j=1}^{4} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2\Big]. \tag{16}$$

The normalization factor is given by

$$N(\mu_\alpha) = 2^{3/4} 8^{3/2} (\mu_\alpha/\pi)^{9/4}. \tag{16·1}$$

An ejected nucleon+a residual nucleus in a ground state (*final state*) : Since a nucleon's spin is one half and a residual nucleus in a ground state has also one half spin, the final state is in a singlet or triplet state, and the corresponding wave functions are as follows.

(a) Case of proton ejection (proton+triton).

Singlet state ;

$$\Psi_f^s(\widetilde{12},\ \widetilde{34}) = \frac{1}{\sqrt{2}}\{\Phi_f(12,\ \overline{34}) + \Phi_f(21,\ \overline{34})\}$$

$$\times \frac{1}{2}(\alpha(1)\beta(2)-\alpha(2)\beta(1))(\alpha(3)\beta(4)-\alpha(4)\beta(3)). \tag{17}$$

Triplet state ;

$$\Psi_f^t(\widetilde{12},\ \widetilde{34}) = \frac{1}{\sqrt{2}}\{\Phi_f(12,\ \overline{34}) - \Phi_f(21,\ \overline{34})\}$$

$$\times \left\{ \begin{matrix} \alpha(1)\alpha(2) \\ \frac{1}{\sqrt{2}}(\alpha(1)\beta(2)-\alpha(2)\beta(1)) \\ \beta(1)\beta(2) \end{matrix} \right\} \frac{1}{\sqrt{2}}(\alpha(3)\beta(4)-\alpha(4)\beta(3)), \tag{18}$$

$$\Phi_f(12,\ \overline{34})=\frac{1}{V}e^{i\boldsymbol{k}_p\boldsymbol{r}_1}e^{i\boldsymbol{k}_r(\boldsymbol{r}_2+\boldsymbol{r}_3+\boldsymbol{r}_4)/3}N(\mu_t)\exp[-\mu_t\sum_{i>j=2}^{4}(\boldsymbol{r}_i-\boldsymbol{r}_j)^2], \tag{19}$$

$$N(\mu_t)=3^{3/4}8^{1/2}(\mu_t/\pi)^{3/2}. \tag{19·1}$$

(b) Case of neutron ejection (neutron+He³).

For the case of neutron ejection, leaving He³ as a residual nucleus, the corresponding wave functions may be easily obtained through interchanging the role of the proton pair (1, 2) by that of the neutron pair (3, 4) in the equations of (17), (18) and (19).

The initial spin of the four nucleons system is in a singlet state, while its final spin is in a singlet or triplet state. We, therefore, have to deal with a singlet to singlet transition and a singlet to triplet one according to the perturbations (3), (4) and (5).

Thus, it follows, from (11) and (14),

$$d\sigma=V^3\ \frac{\pi}{(2\pi)^6}\ \frac{E_e}{c^2k_e}\ \frac{M_r}{\hbar^4\varDelta k_e}k_e'^2\,dk_e'\,d\Omega_{k_e'}\sum_{\text{spins}}\{|(f^s|\delta H|i^s)|^2+|(f^t|\delta H|i^s)|^2\}k_p\,dk_p\,d\varphi_p, \tag{20}$$

with

$$\left.\begin{aligned}(f^s|\delta H|i^s)&=(\Psi_f^s(\widetilde{12},\ \widetilde{34})|\delta H|\Psi_i^s(\widetilde{12},\ \widetilde{34}))\\(f^t|\delta H|i^s)&=(\Psi_f^t(\widetilde{12},\ \widetilde{34})|\delta H|\Psi_i^s(\widetilde{12},\ \widetilde{34}))\end{aligned}\right\}, \tag{21}$$

and

$$\delta H=B(p_e p_e')\{(u',\ u)\sum_{j=1}^{2}R_j+(u',\ \alpha u)\sum_{j=1}^{2}\boldsymbol{S}_j+(u';\ \alpha u)\sum_{j=1}^{2}\boldsymbol{T}_j\}/V. \tag{22}$$

The first two terms in the curled bracket expression (22) give rise to a singlet to singlet transition, whereas the last term gives rise to a singlet to triplet one, since it involves nucleon's spin operators. The summation over both spin states of the incident and scattered electrons in (20) can be performed in a straightforward way, making use of the projection operator, with the following results.

$$\begin{aligned}\sum_{\text{spins}}|(f^s|\delta H|i^s)|^2=&\left\{\frac{B(p_e,\ p_e')}{V}\right\}^2\frac{1}{E_eE_e'}\\&\times\Big\{(E_eE_e'+c^2(\boldsymbol{p}_e\boldsymbol{p}_e')+\mu^2)|R|^2+(E_eE_e'-c^2(\boldsymbol{p}_e\boldsymbol{p}_e')-\mu^2)|\boldsymbol{U}|^2\\&+cE_e(R(\boldsymbol{U}^*\boldsymbol{p}_e')+R^*(\boldsymbol{U}\boldsymbol{p}_e'))+cE'(R(\boldsymbol{U}^*\boldsymbol{p}_e)+R^*(\boldsymbol{U}\boldsymbol{p}_e))\\&+c^2((\boldsymbol{U}^*\boldsymbol{p}_e')(\boldsymbol{U}\boldsymbol{p}_e)+(\boldsymbol{U}^*\boldsymbol{p}_e)(\boldsymbol{U}\boldsymbol{p}_e'))\Big\},\end{aligned} \tag{23·1}$$

$$\begin{aligned}\sum_{\text{spins}}|(f^t|\delta H|i^s)|^2=&\left\{\frac{B(p_e,\ p_e')}{V}\right\}^2\frac{1}{E_eE_e'}\sum_{m=1}^{3}\Big\{(E_eE_e'-c^2(p_ep_e')-\mu^2)|T_m|^2\\&+c^2((\boldsymbol{T}_m^*\boldsymbol{p}_e)(\boldsymbol{T}_m\boldsymbol{p}_e')+(\boldsymbol{T}_m^*\boldsymbol{p}'_e)(\boldsymbol{T}_m\boldsymbol{p}_e))\Big\},\end{aligned} \tag{23·2}$$

where

$$R=(\Psi_f^s(\widetilde{12},\ \widetilde{34})|\sum_{1}^{2}R_j|\Psi_i^s(\widetilde{12},\ \widetilde{34})), \tag{24·1}$$

$$\boldsymbol{U}=(\Psi_f^s(\widetilde{12},\ \widetilde{34})|\sum_{1}^{2}\boldsymbol{S}_j|\Psi_i^s(\widetilde{12},\ \widetilde{34})), \tag{24·2}$$

$$\boldsymbol{T}_m=(\Psi_f^{t,m}(\widetilde{12},\ \widetilde{34})\,|\sum_1^4\boldsymbol{T}_j|\,\Psi_i^s(\widetilde{12},\ \widetilde{34})), \qquad (24\cdot3)$$

and

$$\mu=mc^2,\ \text{(rest energy of an electron)}.$$

Now we shall treat, at first, the case of proton ejection. Substituting (6), (7), (8), (15), (17) and (18) in (24·1), (24·2) and (24·3) and carrying through the integrations with respect to the nucleon spin coordinates lead to

$$R=-e^2(1-\Delta E_e/2Mc^2)\,(\Phi_f(12,\ \overline{34})+\Phi_f(21,\ \overline{34})\,|\sum_1^2 e^{i\Delta\boldsymbol{k}_e\cdot\boldsymbol{r}_j}|\,\Phi_i(\overline{12},\ \overline{34}))/\sqrt{2}, \qquad (25\cdot1)$$

$$\boldsymbol{U}=(e^2\hbar/iMc)\,(\Phi_f(12,\ \overline{34})+\Phi_f(21,\ \overline{34})\,|\sum_1^2 e^{i\Delta\boldsymbol{k}_e\cdot\boldsymbol{r}_j}\nabla_j|\,\Phi_i(\overline{12},\ \overline{34}))/\sqrt{2}, \qquad (25\cdot2)$$

$$\begin{aligned}\boldsymbol{T}_m=&\,(ie^2\hbar/2Mc)\,g_P\,(\boldsymbol{\eta}_m\times\Delta\boldsymbol{k}_e)\,(\Phi_f(12,\ \overline{34})\\&-\Phi_f(21,\ \overline{34})\,|e^{i\Delta\boldsymbol{k}_e\cdot\boldsymbol{r}_1}-e^{i\Delta\boldsymbol{k}_e\cdot\boldsymbol{r}_2}|\,\Phi_i(\overline{12},\ \overline{34}))/\sqrt{2}\end{aligned} \qquad (25\cdot3)$$

and

$$\eta_1=\left(\frac{-1}{\sqrt{2}},\ \frac{i}{\sqrt{2}},\ 0\right),\quad \eta_2=(0,\ 0,\ 1),\quad \eta_3=\left(\frac{1}{\sqrt{2}},\ \frac{i}{\sqrt{2}},\ 0\right). \qquad (25\cdot4)$$

In the transition matrix components mentioned above, i.e., (25·1) ~ (25·3), two kinds of scattering amplitudes are involved: one is connected with the direct process in which a proton can be ejected through the direct interaction with the electromagnetic field induced by the electron, being represented, for example, by the term of $(\Phi_f(12,\ \overline{34})\,|e^{i\Delta\boldsymbol{k}_e\boldsymbol{r}_1}|\,\Phi_i(\overline{12},\ \overline{34}))$, and the other one is connected with the indirect process in which a proton is capable of being ejected through the nuclear interaction with another nucleon which does interact directly with the electromagnetic field, being given, for example, by the term of $(\Phi_f(21,\ \overline{34})\,|e^{i\Delta\boldsymbol{k}_e\boldsymbol{r}_1}|\,\Phi_i(\overline{12},\ \overline{34}))$. Thus, such direct and indirect processes together with their interference terms can give rise to the total cross section.

By the use of equations (16) and (19), the spatial integrations in (25·1), (25·2), and (25·3) can easily be performed with the following results.

$$R=\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)\,N(\mu_t)}{\mu_\alpha^{3/2}(4\mu_\alpha+3\mu_t)^3}\cdot\frac{1}{\sqrt{V}}\cdot R_0\cdot\delta_{\boldsymbol{k}_p+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}, \qquad (26\cdot1)$$

$$\boldsymbol{U}=\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)\,N(\mu_t)}{\mu_\alpha^{3/2}(4\mu_\alpha+3\mu_t)^3}\cdot\frac{1}{\sqrt{V}}\cdot\boldsymbol{U}_0\cdot\delta_{\boldsymbol{k}_p+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}, \qquad (26\cdot2)$$

$$\boldsymbol{T}_m=\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)\,N(\mu_t)}{\mu_\alpha^{3/2}(4\mu_\alpha+3\mu_t)^3}\cdot\frac{1}{\sqrt{V}}\,\boldsymbol{T}_{m0}\cdot\delta_{\boldsymbol{k}_p+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}, \qquad (26\cdot3)$$

and

$$R_0=-e^2\left(1-\frac{\Delta E_e}{2Mc^2}\right)\cdot\{e^{-(\boldsymbol{k}_p-\Delta\boldsymbol{k}_e)^2/12\mu_\alpha}+e^{-\boldsymbol{k}_p^2/12\mu_\alpha-\Delta\boldsymbol{k}_e^2/6(4\mu_\alpha+3\mu_t)}\}, \qquad (27\cdot1)$$

$$\boldsymbol{U}_0=\frac{e^2\hbar}{Mc}\cdot\Big\{(\boldsymbol{k}_p-\Delta\boldsymbol{k}_e)\,e^{-(\boldsymbol{k}_p-\Delta\boldsymbol{k}_e)^2/12\mu_\alpha}$$

$$-\frac{1}{3}\Big(\boldsymbol{k}_p+\frac{8\mu_\alpha}{4\mu_\alpha+3\mu_t}\Delta\boldsymbol{k}_e\Big)e^{-k_p^2/12\mu_\alpha-\Delta k_e^2/6(4\mu_\alpha+3\mu_t)}\Big\}, \tag{27·2}$$

$$\boldsymbol{T}_{mo}=i\frac{e^2\hbar}{2Mc}g_P(\eta_m\times\Delta\boldsymbol{k}_e)\,\{e^{-(\boldsymbol{k}_p-\Delta\boldsymbol{k}_e)^2/12\mu_\alpha}-e^{-k_p^2/12\mu_\alpha-\Delta k_e^2/6(4\mu_\alpha+3\mu_t)}\}. \tag{27·3}$$

The first term in each expression in the curled bracket of (27·1) to (27·3) represents the contribution from the direct process, while the second term represents that from the indirect process mentioned above, the angular dependence of scattered electron upon those two kinds of terms being different from each other. Substituting (23·1), (23·2), (26·1)∼(26·3) and (27·1)∼(27·3) in (20), we obtain, for the differential cross section of inelastic scattering of electron with the momentum $\hbar k_e'$ into the solid angle $d\Omega_{k_e'}$, accompanied by the proton ejection of momentum $\hbar k_p$ from He^4,

$$d\sigma=\frac{\pi}{(2\pi)^6}\cdot\frac{M_r}{c^2\hbar^4}\cdot\frac{k_e'^2dk_e'd\Omega_{k_e'}}{k_e\Delta k_eE_e'}\cdot\Big\{B(\boldsymbol{p}_e,\ \boldsymbol{p}_e')\cdot\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)N(\mu_t)}{\mu_\alpha^{3/2}(4\mu_\alpha+3\mu_t)^3}\Big\}^2$$

$$\times\Big\{(E_eE_e'+c^2(\boldsymbol{p}_e\cdot\boldsymbol{p}_e')+\mu^2)|R|^2+(E_eE_e'-c^2(\boldsymbol{p}_e\cdot\boldsymbol{p}_e')-\mu^2)|\boldsymbol{U}_0|^2+2cE_eR_0(\boldsymbol{U}_0\cdot\boldsymbol{p}_e')$$

$$+2cE_e'R_0(\boldsymbol{U}_0\cdot\boldsymbol{p}_e)+2c^2(\boldsymbol{U}_0\cdot\boldsymbol{p}_e')(\boldsymbol{U}_0\cdot\boldsymbol{p}_e)$$

$$+\sum_m\big[(E_eE_e'-c^2(\boldsymbol{p}_e\cdot\boldsymbol{p}_e')-\mu^2)|\boldsymbol{T}_{mo}|^2+2c^2(\boldsymbol{T}_{mo}^*\cdot\boldsymbol{p}_e')(\boldsymbol{T}_{mo}\cdot\boldsymbol{p}_e)\big]\Big\}\delta_{\boldsymbol{k}_p+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}k_pdk_pd\varphi_p. \tag{28}$$

For the case of a neutron ejection, the equations of (20), (23·1) and (23·2) are still valid except for the matrix components of R, U and T, which are now replaced by the following quantities.

$$R'=\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)N(\mu_t)}{\mu_\alpha(4\mu_\alpha+3\mu_t)^3}\cdot\frac{1}{\sqrt{V}}\cdot R_0'\cdot\delta_{\boldsymbol{k}_N+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}, \tag{29·1}$$

$$\boldsymbol{U}'=\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)N(\mu_t)}{\mu_\alpha(4\mu_\alpha+3\mu_t)^3}\cdot\frac{1}{\sqrt{V}}\cdot\boldsymbol{U}_0'\cdot\delta_{\boldsymbol{k}_N+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}, \tag{29·2}$$

$$\boldsymbol{T}_m'=\frac{\sqrt{2}\,\pi^{9/2}N(\mu_\alpha)N(\mu_t)}{\mu_\alpha(4\mu_\alpha+3\mu_t)^3}\cdot\frac{1}{\sqrt{V}}\cdot\boldsymbol{T}_{mo}'\cdot\delta_{\boldsymbol{k}_N+\boldsymbol{k}_r,\ \Delta\boldsymbol{k}_e}, \tag{29·3}$$

where

$$R_0'=-2e^2\Big(1-\frac{\Delta E_e}{2Mc}\Big)e^{-k_N^2/12\mu_\alpha-\Delta k_e^2/6(4\mu_\alpha+3\mu_\alpha')}, \tag{30·1}$$

$$\boldsymbol{U}_0'=\frac{e^2\hbar}{Mc}\Big\{-\frac{2}{3}\Big(\boldsymbol{k}_N+\frac{8\mu_\alpha}{4\mu_\alpha+3\mu_t}\cdot\Delta\boldsymbol{k}_e\Big)e^{-k_N^2/12\mu_\alpha-\Delta k_e^2/6(4\mu_\alpha+3\mu_\alpha')}\Big\}, \tag{30·2}$$

$$\boldsymbol{T}'_{mo}=i\frac{e^2\hbar}{2Mc}g_N(\eta_m\times\Delta\boldsymbol{k}_e)\,\{e^{-(\boldsymbol{k}_N-\Delta\boldsymbol{k}_e)^2/12\mu_\alpha}-e^{-k_N{}^2/12\mu_\alpha-\Delta k_e^2/6(4\mu_\alpha+3\mu_\alpha')}\}. \tag{30·3}$$

μ_α' represents the spread-parameter in the He^3 wave function of Gaussian type, corresponding to μ_t for the case of triton, and $\hbar\boldsymbol{k}_r$ in this case denotes the momentum of the center of mass of He^3 nucleus. $\hbar\boldsymbol{k}_N$ expresses the momentum of an ejected neutron. R_0', $\boldsymbol{U}_0'$ and the second term in (30·3) give rise to the indirect process, whereas, the first term in (30·3) alone is responsible for the direct one, which situation is a little different from the case of proton ejection.

In order to compare with the observed energy spectra of inelastically scattered electrons at 45° and 60°, we have to integrate the obtained cross sections for both proton- and neutron-ejections with respect to the momenta of ejected nucleons, i. e., k_p, φ_p or k_N, φ_N, in which case the following relations, derived from the conservation laws (12) and (13), are used.

$$\Delta k_e\cos\theta_p=(M_r+M)\,k_p/2M+[\Delta k_e^2/2-M_r(\Delta E_e+c^2\Delta M)/\hbar^2]/k_p, \tag{31·1}$$

or

$$\Delta k_e\cos\theta_N=(M_r+M)\,k_N/2M+[\Delta k_e^2/2-M_r(\Delta E_e+c^2\Delta M)/\hbar^2]/k_N, \tag{31·2}$$

$$|K-\Delta k_e|\leq\frac{M_r+M}{M}k_p \quad\text{or}\quad \frac{M_r+M}{M}k_N\leq K+\Delta k_e, \tag{31·3}$$

$$K=\left(\frac{M_r}{M}\cdot\frac{2(M_r+M)}{\hbar^2}\right)^{1/2}\left(\Delta E_e+c^2\Delta M-\frac{\hbar^2\Delta k_e{}^2}{2(M+M_r)}\right)^{1/2}, \tag{31·4}$$

$$\Delta M=M_\alpha-(M_r+M). \tag{31·5}$$

The mentioned integrations are performed through a straightforward but lengthy process, and adding the resulting cross sections for both proton and neutron ejection, we get finally, for the cross section of the inelastic continuum irrespective of proton or neutron ejection,

$$\begin{aligned}d\sigma(\boldsymbol{k}_e,\boldsymbol{k}_e')=&\frac{8}{3(6\pi)^{1/2}}\frac{M_r}{M}\frac{1}{\mu_\alpha^{1/2}}\frac{(48\mu_\alpha\mu_t)^2}{(4\mu_\alpha+3\mu_t)^6}\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{E_e}{c\hbar k_e}\right)\left(\frac{M_r}{\hbar\Delta k_e}\right)\frac{k_e'^2\,dk_e'\,d\Omega_{k_e'}}{[\Delta k_e'^2-(\Delta E_e/\hbar c)^2]^2}\\&\times\sum_m\Gamma_m\left\{\left[1+\frac{c^2\hbar^2(\boldsymbol{k}_e\boldsymbol{k}_e')}{E_eE_e'}+\frac{\mu^2}{E_eE_e'}\right]\left[a_m\left\{1-\frac{\Delta E_e}{Mc^2}(1+2D_m)\right.\right.\right.\\&+\left(\frac{\Delta E_e}{Mc^2}\right)^2\left(\frac{1}{4}+2D_m+3C_m+B_m\right)-\frac{1}{2}\left(\frac{\Delta E_e}{Mc^2}\right)^3D_m\right\}-\left(\frac{\Delta E_e}{Mc^2}\right)\frac{g_P{}^2+g_N{}^2}{4}b_m\Bigg]\\&+\left(\frac{\hbar\Delta k_e}{Mc}\right)^2\left[3-\frac{c^2\hbar^2(\boldsymbol{k}_e\boldsymbol{k}_e')}{E_eE_e'}-\frac{3\mu^2}{E_eE_e'}\right]\left[\frac{g_P{}^2+g_N{}^2}{4}b_m-C_ma_m\right]\Bigg\},\end{aligned} \tag{32}$$

$$\left.\begin{aligned}a_1&=1 & b_1&=1\\a_2&=5 & b_2&=1\\a_3&=1 & b_3&=-1\end{aligned}\right\}, \tag{33}$$

$$\left.\begin{aligned}
\Gamma_1&=\frac{3}{t}\exp\left[-\frac{t}{3(1+t)}\frac{1}{\mu_\alpha}\gamma-\frac{t(t-1)}{6(t+1)^2}\frac{\Delta k_e^2}{\mu_\alpha}\right]\sinh\left(\frac{t}{3(1+t)^2}\frac{\delta}{\mu_\alpha}\right)\\
\Gamma_2&=3\exp\left[-\frac{t}{3(t+1)}\frac{1}{\mu_\alpha}\gamma-\frac{t(t-1)}{6(t+1)^2}\frac{\Delta k_e^2}{\mu_\alpha}\right.\\
&\qquad\left.+\frac{(t-3)a/2+(t-1)b}{12(t+1)}\frac{\Delta k_e^2}{\mu_\alpha}\right]\sinh\left(\frac{1}{3(1+t)^2}\frac{\delta}{\mu_\alpha}\right)\\
\Gamma_3&=\frac{12}{t-1}\exp\left[-\frac{t}{3(t+1)}\frac{1}{\mu_\alpha}\gamma-\frac{t(t-1)}{6(t+1)^2}\frac{\Delta k_e^2}{\mu_\alpha}\right.\\
&\qquad\left.+\frac{(t-3)a/2+(t-1)b}{24(t+1)}\frac{\Delta k_e^2}{\mu_\alpha}\right]\sinh\left(\frac{t-1}{6(1+t)^2}\frac{\delta}{\mu_\alpha}\right)
\end{aligned}\right\}\tag{34}$$

$$\left.\begin{aligned}
B_1&=\left[\frac{t(t-1)}{(t+1)^2}+\frac{2t}{t+1}\frac{\gamma}{\Delta k_e^2}+\frac{6\mu_\alpha}{\Delta k_e^2}\right]-\frac{2t}{(t+1)^2}\frac{\delta}{\Delta k_e^2}\coth\left(\frac{t}{3(t+1)^2}\frac{\delta}{\mu_\alpha}\right)\\
B_2&=\frac{1}{9}\left[\frac{a^2(t+1)^2+2a(t+1)+(1-t)}{(t+1)^2}+\frac{2t}{t+1}\frac{\gamma}{\Delta k_e^2}+6[a(t+1)+1]\frac{\mu_\alpha}{\Delta k_e^2}\right.\\
&\qquad\left.-\frac{2a(t+1)+2}{(t+1)^2}\frac{\delta}{\Delta k_e^2}\coth\left(\frac{1}{3(t+1)^2}\frac{\delta}{\mu_\alpha}\right)\right]\\
B_3&=-\frac{1}{3}\left[-\frac{t(a+2+ta)}{(t+1)^2}+\frac{2t}{t+1}\frac{\gamma}{\Delta k_e^2}-\frac{6}{t-1}(a(t+1)-(t-1))\frac{\mu_\alpha}{\Delta k_e^2}\right.\\
&\qquad\left.+\frac{a(t+1)+(1-t)}{(t+1)^2}\frac{\delta}{\Delta k_e^2}\coth\left(\frac{t-1}{6(t+1)^2}\frac{\delta}{\mu_\alpha}\right)\right]
\end{aligned}\right\}\tag{35}$$

$$\left.\begin{aligned}
C_1&=\frac{9(1+t)^2}{t^2}\cdot\frac{\mu_\alpha^2}{\Delta k_e^4}-\frac{3}{t}\cdot\frac{\mu_\alpha\delta}{\Delta k_e^4}\coth\left(\frac{t\delta}{3(1+t)^2\mu_\alpha}\right),\\
C_2&=(1+t)^2\frac{\mu_\alpha^2}{\Delta k_e^4}-\frac{1}{3}\frac{\mu_\alpha\delta}{\Delta k_e^4}\coth\left(\frac{\delta}{3(1+t)^2\mu_\alpha}\right),\\
C_3&=-\frac{12(1+t)^2}{(1-t)^2}\frac{\mu_\alpha^2}{\Delta k_e^4}+\frac{2}{t-1}\frac{\delta\mu_\alpha}{\Delta k_e^4}\coth\left(\frac{(t-1)\delta}{6(1+t)^2\mu_\alpha}\right),
\end{aligned}\right\}\tag{36}$$

$$\left.\begin{aligned}
D_1&=-\frac{3(t+1)}{t}\cdot\frac{\mu_\alpha}{\Delta k_e^2}-\frac{4}{(t+1)^2}-\frac{1}{2}\\
&\qquad+\frac{4}{(1+t)^2}\frac{\delta}{\Delta k_e^2}\coth\left(\frac{t\cdot\delta^2}{3(1+t)^2}\cdot\frac{1}{\mu_\alpha}\right),\\
D_2&=-(t+1)\frac{\mu_\alpha}{\Delta k_e^2}+\frac{4}{3(1+t)^2}-\frac{1}{6}(2a+1)\\
&\qquad+\frac{4}{3(1+t)^2}\cdot\frac{\delta}{\Delta k_e^2}\coth\left(\frac{\delta}{3(1+t)^2\mu_\alpha}\right),
\end{aligned}\right|\tag{37}$$

$$
\left.\begin{aligned}
D_3 &= -\frac{2(1+t)}{t-1} - \frac{4}{3(1+t)^2} - \frac{a}{6} - \frac{1}{3} \\
&\quad + \frac{4}{3(1+t)^2}\,\frac{\delta}{\Delta k_e^2}\cdot\coth\left(\frac{(t-1)\,\delta}{6(1+t)^2\mu_\alpha}\right), \\
t &= \frac{M_r}{M}, \quad a = \frac{8\mu_\alpha}{4\mu_\alpha + 3\mu_t}, \quad b = \frac{6\mu_t}{4\mu_\alpha + 3\mu_t}, \\
\gamma &= \frac{M}{\hbar^2}(\Delta E_e - c^2 \Delta M), \quad \delta = [t\cdot \Delta k_e^2 (2(1+t)\gamma - \Delta k_e^2)]^{1/2}.
\end{aligned}\right\} \tag{38}
$$

For the ultra-relativistic case of electron and, further, for $\Delta E_e/Mc^2 \ll 1$, the above cross section may be approximately reduced into the following form.

$$
\begin{aligned}
d\sigma(\boldsymbol{k}_e,\ \boldsymbol{k}_e') &= \frac{1}{3(6\pi)^{1/2}}\cdot\frac{M_r}{M}\cdot\frac{1}{\mu_\alpha^{1/2}} \\
&\quad\times\frac{(48\mu_\alpha\mu_t)^3}{(4\mu_\alpha+3\mu_t)^6}\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{Mc}{\hbar\Delta k_e}\right)\frac{dk_e' d\Omega_{k_e'}}{k_e^2}\cdot\frac{\cos^2\Theta/2}{\sin^4\Theta/2} \\
&\quad\times\sum_m \Gamma_m\left\{a_m + \left[\frac{g_P^2+g_N^2}{4}b_m - C_m a_m\right]\left(\frac{\hbar\Delta k_e}{Mc}\right)^2\left(\frac{2}{\cos^2\Theta/2}-1\right)\right\},
\end{aligned} \tag{39}
$$

$$
(\boldsymbol{k}_e\cdot\boldsymbol{k}_e') = k_e k_e' \cos\Theta. \tag{39·1}
$$

In the above results, i. e., (32) and (39), we have supposed that $\mu_t = \mu_\alpha'$ and the mass difference of a triton from He^3 nucleus has been approximately neglected for simplicity, since such approximations can hardly be considered to introduce serious error into the qualitative behaviours of our results, in view of the similar binding energies of both nuclei. ($B(He^3) = 7.74$ Mev, $B(H_3) = 8.49$ Mev).

§ 3. Comparisons with the Experiments and Discussions

The inelastic continuum of scattered electrons in helium at 400 Mev for scattering angles of 45° and 60° has been observed, in some detail, by Blankenbecler and Hofstadter.[6] The results are shown in Figs. 1 and 2. As is shown in the qualitative consideration by Hofstadter,[6] the inelastic continuum may be considered to demonstrate, in some measure, the reflection of the momentum distribution of the ejected nucleons within the nucleus in a similar way to the case of deuteron pick-up process. However, the tail part of the high energy side of the continuum is observed to be overlapped with the elastic peak, while in the low energy region the process of pion production is detected to superpose on the mentioned continuum due to the nucleon ejection. Therefore, the inelastic continuum purely due to the nucleon ejection can hardly be determined accurately in its shape and magnitude particularly in both tail parts of the high- and low-energy sides. Furthermore, the observed cross sections are expressed only in arbitrary units in the mentioned figures.

In view of the above situation, we have carried out the numerical calculations of

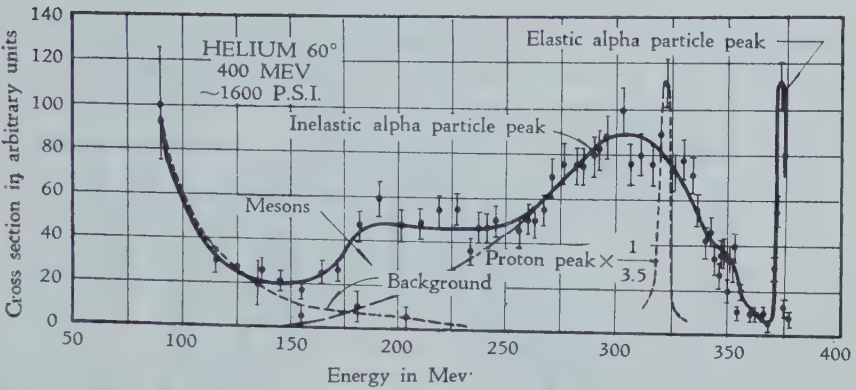


Fig. 1. The elastic and inelastic scatterings of electrons from the alpha particle at 400 Mev and 60°. The area labeled "meson" refers to negative pions produced in the target and emerging with the same momentum as the corresponding scattered electrons. The free proton peak is shown for comparison.

(32) for a number of appropriate values of μ_α and μ_t. The actual calculations, however, reveal that the shape and magnitude of the inelastic continuum become less sensitive to the values of μ_t and, moreover, the variational calculations of the binding energies of alpha-particle and triton, assuming, for example, a Gaussian well with appropriate parameters for the nuclear potential, give approximately $\mu_\alpha \simeq \mu_t$. For the sake of simplicity, therefore, we have assumed that $\mu_\alpha = \mu_t$ in the following numerical calculations, which are shown in Figs. 3 and 4 together with the observed data of the Stanford group. Since the observed cross sections are described in arbitrary units, the scale in the figures of each calculated cross section is estimated through numerically fitting its integrated cross section to the corresponding observed cross section in the energy range from 260 Mev to the theoretical maximum-energy which scales are denoted, in the mentioned figures, for each case. The general trends of the inelastic continuum appear to be well reproduced for the case of $\mu_\alpha^{-1/2} = 4.5$ fermi, allowing for the approximate nature of our calculations. Particularly, the comparison of our results with the observed

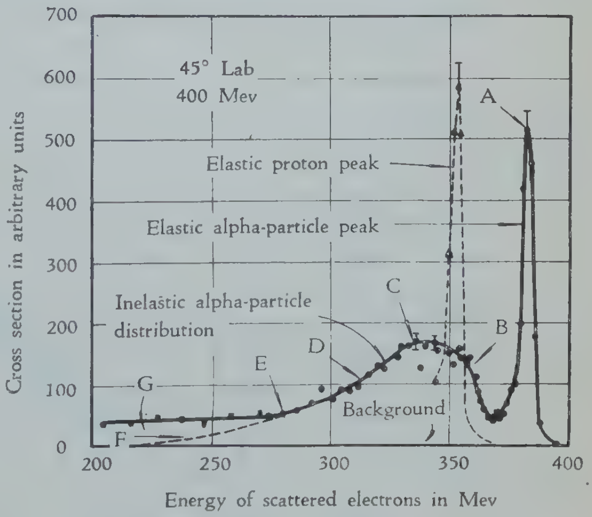


Fig. 2. Electrodisintegration of the alpha particle at 400 Mev and 45°. The elastic peak is shown at A. The inelastic continuum BCDE is related to the momentum distribution of nucleons within the alpha particle. *G* points to negative pion production.

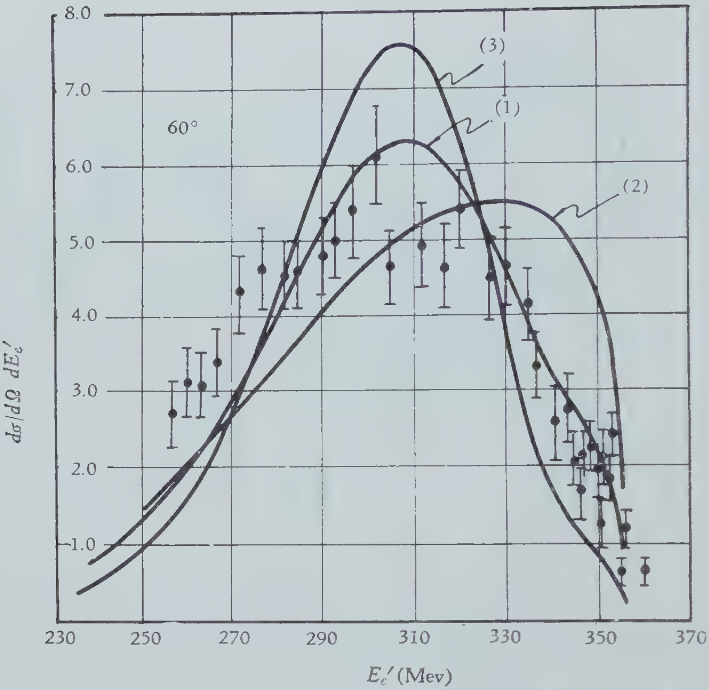


Fig. 3. Comparison of our numerical results with the experiments (60°).
(1) $\mu_\alpha^{-1/2}=4.5$ fermi, scale of the ordinate$=0.313\times10^{-32}$ cm^2/sterad. Mev.
(2) $\mu_\alpha^{-1/2}=3.65$ fermi, scale of the ordinate$=0.347\times10^{-32}$ cm^3/sterad. Mev.
(3) $\mu_\alpha^{-1/2}=5.25$ fermi, scale of the ordinate$=0.321\times10^{-32}$ cm^2/sterad. Mev.

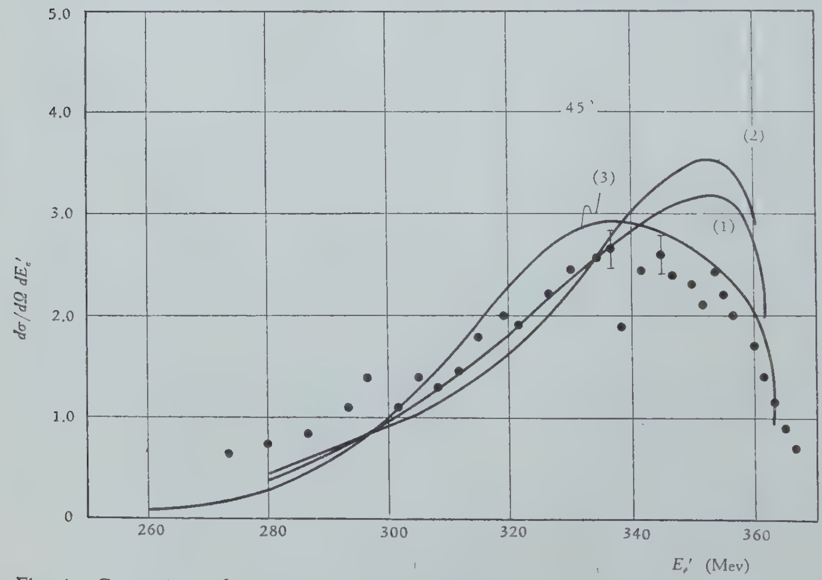
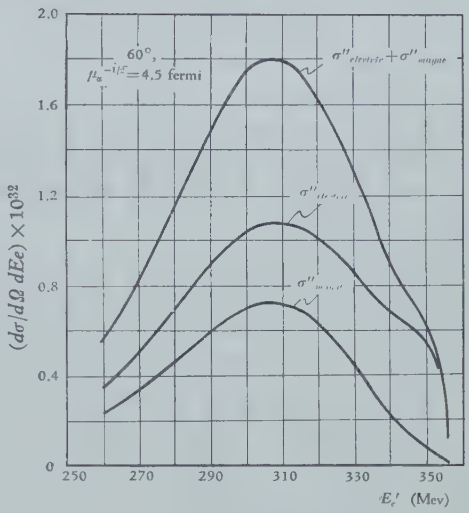


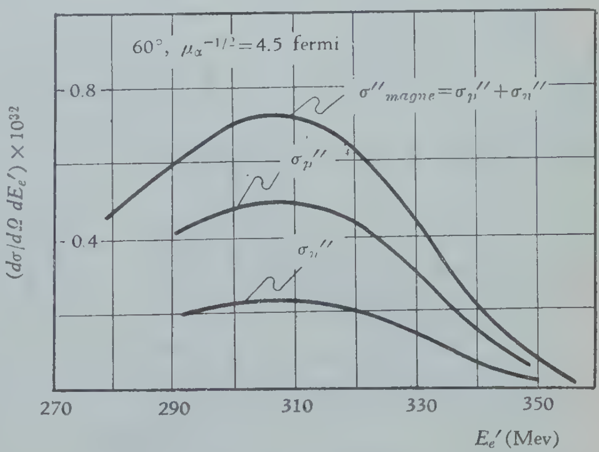
Fig. 4. Comparison of our results with the experiments (45°).
(1) $\mu_\alpha^{-1/2}=4.5$ fermi, scale of the ordinate$=3.41\times10^{-32}$ cm^2/sterad. Mev.
(2) $\mu_\alpha^{-1/2}=3.65$ fermi, scale of the ordinate$=4.75\times10^{-32}$ cm^2/sterad. Mev.
(3) $\mu_\alpha^{-1/2}=5.25$ fermi, scale of the ordinate$=2.88\times10^{-32}$ cm^2/sterad. Mev.

behaviours in the high energy side of the continuum seems to be fairly satisfactory. The appearance of a small hump on the high energy tail of 60° data is well reproduced and may be mainly ascribed to the indirect process mentioned before. In the low energy tails of both 60° and 45° data, however, the appreciable discrepancies compared with the experiments are found to exist. Unfortunately, whether such discrepancies may be attributable to the crudeness of our wave functions or to the mixing-up of the process of meson production can hardly be determined at present. The reliable, experimental discrimination from each other process will be highly desirable for the quantitative comparison in future. In this connection, it should be remarked that we have actually estimated the approximate orders of magnitudes of both contributions from the deuteron emission from He^4 and its break-up process into separate nucleons, which are found to be, at most, ten percent of a nucleon ejection process under consideration and can hardly be responsible for the cause of the above discrepancies.



(a) Cross sections due to the electric and magnetic interactions.

In Figs. 5 and 6 are shown separately the various kinds of contributions to total cross section (32) in order to estimate their relative importances. At 400 Mev and 60°, the contribution from the electric interaction is seen to become a little larger than or approximately comparable with that from the magnetic interaction (a). In the contribution from magnetic interaction (b), the proton ejection is observed to become dominant compared with the neutron, as expected from the orbital current of the former. The direct process in the electric contribution is seen to be predominant, compared with the indirect one,



(b) Proton- and neutron-contributions in the cross section due to the magnetic interaction.

Fig. 5. Relative contributions from the various kinds of interactions (60°). ($\mu_\alpha^{-1/2}=4.5$ fermi)

except for the high-energy tail where the latter process becomes appreciable in comparison with the former (c), giving rise to the small hump. The relative contributions of the electric interaction and the magnetic one as well as those of the direct and indirect ones in the electric interaction are observed to depend sensitively upon the scattering angles, as shown in Fig. 6.

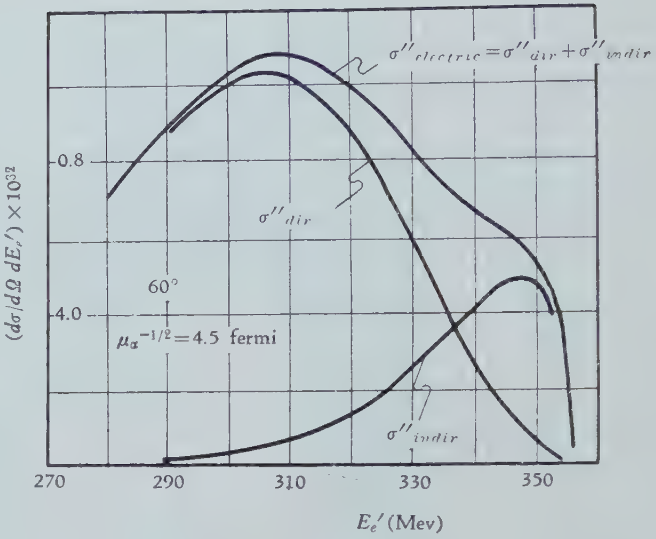


Fig. 5. (c) Contributions from the direct and indirect processes in the cross section due to the electric interaction.

Next, in order to examine the validity of our wave function for He^4, the Gaussian function (16) with a parameter of $\mu_\alpha^{-1/2}=4.5$ fermi has been attempted to calculate the root mean square $\langle r\rangle$, the form factor of elastic scattering and the binding energy of He^4 nucleus, respectively. The calculated value of $\langle r\rangle=1.69$ fermi should be compared with the observed one of 1.61 fermi and the resulting form factor is shown in Fig. 7 in comparison with the experimental results. The comparisons appear to be fairly satisfactory, allowing for our approximate wave functions.

As is well known, the binding energy of the nuclei depends sensitively upon the assumed wave function, and on the shape and parameters of the nuclear potential. For provisional calculations, the two-body nuclear potentials of the Yukawa-, Gaussian- and exponential-types have been assumed and the nuclear parameters involved in each type of central potentials have been determined as usually done by fitting to the observed data of a deuteron binding energy, triplet- and singlet-scattering lengths, respectively. The resulting binding

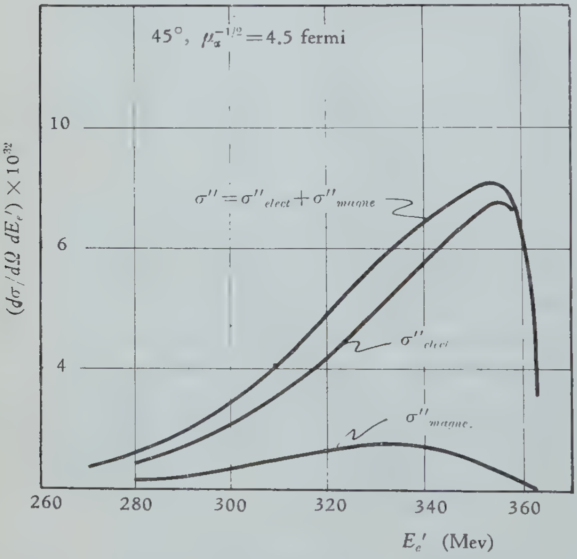


Fig. 6. Relative contributions from the various kinds of interactions (45°). ($\mu_\alpha^{-1/2}=4.5$ fermi)

(a) Cross sections due to the electric and magnetic interactions.

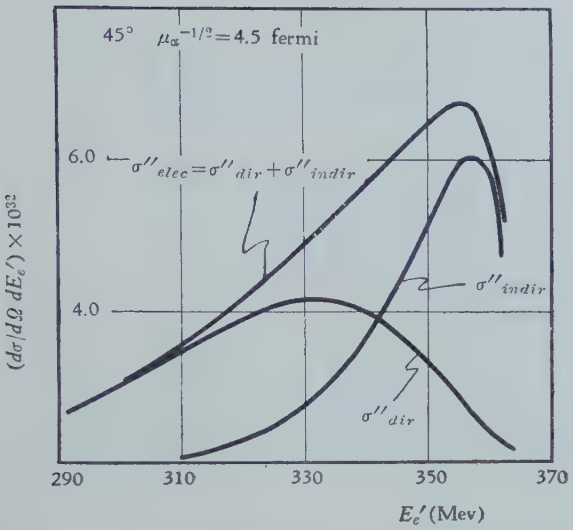


Fig. 6. (b) Contributions from the direct and indirect processes in the cross section due to the electric interaction.

energies of He^4 based on our Gaussian wave function with $\mu_\alpha^{-1/2}=4.5$ fermi have been found to become of 21.9 Mev to 23 Mev according to our assumed shapes of the nuclear potentials, which does not seem to be unreasonable in view of the approximate nature of our wave function, since the recent works by Clark$^{10)}$ and Irving$^{11)}$ show clearly that the mixing of D states wave functions onto an S state one is required essentially for the quantitative calculation of He^4 binding energy. For the similar nuclear potentials, the binding energies of triton, calculated by the similar form of wave function with $\mu_\alpha^{-1/2}=4.5$ fermi, become of 8.7 Mev to 8.5 Mev.

Finally, some remarks should be added of the final state wave function (19), which assumes plane waves for the ejected nucleons. This approximation will be permissible only when the ejected nucleons exist for the greater part of time outside the range of nuclear force of the residual nuclei. The actual situation, however, will be different more or less from such a case and the distortion of the outgoing waves of the ejected nucleon is to be taken into account in the more refined calculation, together with the improvement of He^4 wave function. Furthermore, it should be noted that the D wave function superposed on an S state one is shown to give negligible influence on the inelastic continuum under consideration.

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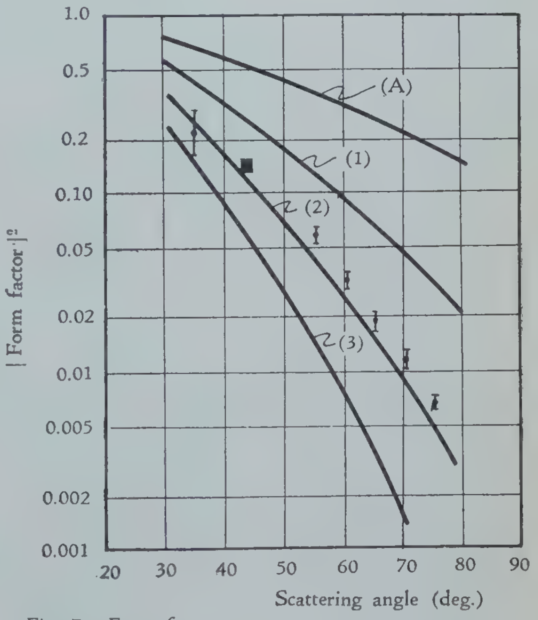

Fig. 7. Form factor.
The curves denoted by (1), (2) and (3) represent the results of our calculations, corresponding to
(1) $\mu_\alpha^{-1/2}=3.65$ fermi, (2) $\mu_\alpha^{-1/2}=4.5$ fermi and (3) $\mu_\alpha^{-1/2}=5.25$ fermi.
The dots express the observed results of the Stanford group. The curve (A) denotes the result of calculation based on a new form of Irving's wave function of a ground state of He^4. (Irving: Proc. Phys. Soc. (London) A **66**, (1953), 17, Rustgi and Levinger: Phys. Rev. **106** (1957), 530).

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Photodisintegration of the Deuteron near 11 Mev

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It is shown that we can account for the photodisintegration of the deuteron near 11 Mev in consistence with nucleon-nucleon scattering at the corresponding energy. Some features of nuclear forces are elucidated in this analysis. It is shown that the phase shift of the singlet even state is somewhat smaller than that previously supposed, and the phase shifts of the triplet odd state are not small at such an energy.

§ 1. Introduction

The investigntion of photodisintegration of the deuteron is very important for understanding of nuclear forces. However, its complete treatment has not yet been performed. The characteristic feature in the photodisintegration of the deuteron is that the isotropic part in the angular distribution is large.[1),2)] Many authors have tried to account for the large isotropic part from different points of view.[3)] However, reasonable explanation has scarcely been given, and fair agreement between theoretical results and the experimental data is obtained only in a few treatments.[4)]

In this paper we shall show that we can account for the photodisintegration of the deuteron near 11 Mev[5)] in consistence with p-p scattering at 18.2 ± 0.2 Mev.[6)] Then some features of nuclear forces can be elucidated in our analysis,[5)] and discussions are made on these properties of nuclear forces.

In the photodisintegration of the deuteron, most of the contribution comes from the electric dipole (e. d.) and the magnetic dipole (m. d.) transitions.[7)] As is well known, the final states of e. d. and m. d. are ^3O (the triplet odd state) and ^1E (the siglet even state),[*][8)] and these states are just the only two states that are concerned with p-p scattering. Now the important point to be paid attention is that there exists large ambiguity in the determination of the phase shifts in p-p scattering,[9)] and on the other hand the phase shifts are very sensistive to the cross sections of photodisintegration of the deuteron. On account of this we make use of the phase shifts, which fit p-p scattering data, for the calculation of the photodisintegration of the deuteron. In this paper the photodisintegration of the deuteron at $E_\gamma=11.3$ Mev corresponding to 18.2 Mev p-p scattering is treated.

In § 2 the formulas for computations are given, and our method is explained. In

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[*] Small contributions from other states are neglected here.

§ 3 we show the calculated results and comparison with experiments. Then the features of ^1E and ^3O brought out by our calculation are illustrated in § 4. Finally the conclusions are summarized in § 5.

§ 2. Preliminary for our work

The symbols used in this paper are as follows.

E_γ : Energy of the incident γ ray in C. M. S.
M : Mean mass of neutron and proton.
ϵ : Binding energy of the deuteron.

$$k^2=M(E_\gamma-\epsilon)/\hbar^2, \quad \alpha^2=M\epsilon/\hbar^2, \quad \mathrm{k}=m_\pi c/\hbar.$$

μ_p : Magnetic moment of proton.
μ_n : Magnetic moment of neutron.
U_d : Radial wave function for $^3S_1$ state of the deuteron.
w_d : Radial wave function for $^3D_1$ state of the deuteron.
f_d : Radial wave function for $^3S_1+{}^3D_1$ state of the deuteron.
U_0 : Radial wave function for $^1S_0$ state.
U_2 : Radial wave function for $^1D_2$ state.
U_{1i} : Radial wave function for ^3P_i state.
$\delta_L{}^J$: Phase shift for state of orbital angular momentum L and total angular momentum J.
σ_{ed} : Electric dipole cross section.
σ_{md} : Magnetic dipole cross section.
σ_T : Total cross section.
σ_{ed}^{ip} : Isotropic cross section arising from e. d.
σ_{md}^{ip} : Isotropic cross section arising from m. d.
σ_{ip} : Total isotropic cross section.
D_s : Hard core radius of ^1E.

At our energy the situation is rather simple (because the energy is low). A number of effects which give merely small contributions can be neglected. The electric quadrupole (e. q.) cross section is about 0.2% of σ_{ed}[7], and the magnetic quadrupole cross section is much smaller.[7] Therefore, these two may be safely neglected.* In σ_{md} the contribution from the exchange current is still small at $E_\gamma \simeq 10$ Mev,[10] and the transition to the continuous $^3S_1+{}^3D_1$ state is only of the order of e. q. near our energy.[8] Therefore the transitions to be considered are only e. d. corresponding to $^3S_1+{}^3D_1 \rightarrow {}^3P_0$, $^3P_1$, $^3P_2$, and m. d. corresponding to $^3S_1 \rightarrow {}^1S_0$, $^3D_1 \rightarrow {}^1D_2$ without including the exchange current.[8]

The formulas for these transitions are as follows.

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* Although the interference term of e. d. with e. q. is not very small, its effect is unimportant as is seen in § 3.

$$\sigma_{ed}=\frac{\pi}{3}\cdot\frac{e^2}{\hbar c}\cdot\frac{ME_\gamma}{\hbar^2}\cdot k\cdot\frac{1}{9}\cdot(I_{10}^2+3I_{11}^2+5I_{12}^2), \tag{1}$$

$$d\sigma_{ed}(\theta)=\frac{\pi}{3}\cdot\frac{e^2}{\hbar c}\cdot\frac{ME_\gamma}{\hbar^2}\cdot k\cdot\frac{d\Omega}{4\pi}\cdot\left[\left\{\frac{1}{9}|e^{i\delta_1^0}I_{10}-e^{i\delta_1^2}I_{12}|^2+\frac{1}{4}|e^{i\delta_1^1}I_{11}-e^{i\delta_1^2}I_{12}|^2\right\}\right.$$
$$\left.+\frac{1}{2}\sin^2\theta\left\{\frac{1}{4}|e^{i\delta_1^1}I_{11}+3e^{i\delta_1^2}I_{12}|^2-\frac{1}{15}|e^{i\delta_1^0}I_{10}-5e^{i\delta_1^2}I_{12}|^2+\frac{1}{15}I_{10}^2\right\}\right] \tag{2}$$

$$I_{10}=\int r\,dr\,U_{10}(U_d-\sqrt{2}\,w_d),\quad I_{11}=\int r\,dr\,U_{11}\Big(U_d+\frac{1}{\sqrt{2}}w_d\Big),$$
$$I_{12}=\int r\,dr\,U_{12}\Big(U_d-\frac{1}{5\sqrt{2}}w_d\Big). \tag{3}$$

$$\sigma_{md}=\frac{\pi}{3}(\mu_n-\mu_p)^2\cdot\frac{e^2}{\hbar c}\cdot\Big(\frac{\hbar}{Mc}\Big)^2\cdot\frac{ME_\gamma}{\hbar^2}\cdot k\cdot(I_{00}^2+I_{02}^2), \tag{4}$$

$$d\sigma_{md}(\theta)=\frac{\pi}{3}(\mu_n-\mu_p)^2\cdot\frac{e^2}{\hbar c}\cdot\Big(\frac{\hbar}{Mc}\Big)^2\frac{ME_\gamma}{\hbar^2}\cdot k\cdot\frac{d\Omega}{4\pi}\cdot$$
$$\left[\left\{I_{00}^2-\sqrt{2}\cos(\delta_0^0-\delta_2^2)\,I_{00}I_{02}+\frac{1}{2}I_{02}^2\right\}+\sin^2\theta\left\{\frac{3}{\sqrt{2}}\cos(\delta_0^0-\delta_2^2)\,I_{00}I_{02}+\frac{3}{4}I_{02}^2\right\}\right], \tag{5}$$

$$I_{00}=\int U_0U_d\,dr,\quad I_{02}=\int U_2w_d\,dr. \tag{6}$$

In what follows we explain our method in detail.

I. Wave function of the deuteron.

In the calculation we used two sorts of wave functions of the deuteron. One is that obtained from phenomenological potential without hard core, which is given by Feshbach and Schwinger.[11] Its D-state probability is 2.5%. The other one is that derived from the pion potential, and it has a small hard core, which is given by $I-O-T-W$.[12] The D-state probability of it is 6.4%.

The former is obtained by adjusting the parameters of the Yukawa type potential without hard core, so that it can account for the low energy and the deuteron data. The latter is derived from the pion potential at $r > 1/k$ (because the inner potential is not well established, and the deuteron data depend on the behavior at $r > 1/k$), and it can reproduce the scattering and the deuteron data. It is interesting to see what sort of differences will appear between these two cases.

The approximate forms, which we used in the calculation, of the wave functions of the deuteron for the above two cases are as follows.

1) f_d of Feshbach and Schwinger.

$$U_d=1.042\sqrt{k}\,(e^{-0.324kr}-e^{-1.870kr}), \tag{7·a}$$

$$w_d=0.2524\sqrt{k}\,(1-0.01939kr)\,e^{-0.428kr}-1.140\sqrt{k}\,e^{-1.546kr}. \tag{7·b}$$

2) f_d of $I-O-T-W$.

$$U_d=1.039\sqrt{\mathrm{k}}\,e^{-0.328kr}-1.392\sqrt{\mathrm{k}}\,e^{-2.360kr}, \tag{8·a}$$

$$w_d=0.02624\sqrt{\mathrm{k}}\,\{1+3/0.328\,\mathrm{k}r+3/(0.328\,\mathrm{k}r)^2\}\,e^{-0.328kr}$$
$$-1.298/r\mathrm{k}\cdot e^{-0.926kr} \quad \text{in} \quad r>1/\mathrm{k}. \tag{8·b}$$

The approximation of w_d of (8·b) is not good at $r<1/\mathrm{k}$, and we used a smooth curve, so that it falls to zero at $r=0$ in stead of (8·b) at $r<1/\mathrm{k}$. (The results are scarcely changed by the form at $r<1/\mathrm{k}$ for low energy reactions.)

II. Calculation of e. d. and m. d.

Now let us elucidate the computation of e. d. and m. d.[13)]

1) E. D. We discuss the e. d. at first. If δ_1^0, δ_1^1, and δ_1^2 are given, σ_{ed} (and σ_{ed}^{ip}) can be computed at once using the formulas (1) ~ (3). Because U_{1i} is small at small distances, and because there is a factor r in I_{1i}, the contribution from small r is unimportant, and we may approximate U_{1i} by

$$kU_{1i}=\frac{\sin(kr+\delta_1^i)}{kr}-\cos(kr+\delta_1^i). \tag{9}$$

There may arise a question that, U_{1i} of (9) tends to ∞ at $r=0$, and therefore the approximation (9) may not be good enough. Nevertheless we can prove that the approximation (9) is not bad. (See Appendix A.)

2) M. D. In the computation of m. d. special attention is needed. As we can see from I_{00} of (6), the contribution from small distances is important for this case. U_0 at small distances is quite unknown, if only δ_0^0 is given. However, we can estimate the lower and the upper limits of I_{00} (and therefore the lower and the upper limits of σ_{md} and σ_{md}^{ip}).

The lower limit of I_{00} can be obtained, if D_s is given (The relation between the phase shifts and the hard core radius will be discussed later), by putting U_0 as

$$kU_0=\sin(kr+\delta_0^0)\,(1-e^{-\zeta(r-D_s)}), \tag{10}$$

and choose ζ so that it satisfies the effective range theory.

The upper limit of I_{00} can be estimated as follows. The simplest way to compute the upper limit is to use

$$kU_0=\sin(kr+\delta_0^0). \tag{10)'}$$

However, to reduce the indeterminancy it is better to use

$$kU_0=\sin(kr+\delta_0^0)\,(1-e^{-\eta(r-D_s)}) \tag{10)''}$$

with suitable values of η. We choosed η so that (10)′ and (10)″ are differed by 1% at $r=1/\mathrm{k}$. Values of ζ and η for various D_s are given in Table 1 (See Appendix B).

Generally speaking, the smaller δ_0^0 is, the larger D_s is. In the calculation we varied D_s for given values of δ_0^0. If σ_{md}^{ip} (and σ_{md}) varies quickly by the variation of D_s (for a given δ_0^0), then our method may become unreliable, because this means that σ_{md}^{ip} (and

Table 1. Values of ζ of (10)′ and η of (10)″ as the function of D_s is shown.

$D_s\,(1/k)$	$\zeta\,(k)$	$\eta\,(k)$
0	1.69	3.6
0.2	2.15	5.8
0.4	2.98	7.7
0.6	4.83	11.5

σ_{md}) cannot be determined only by δ_0^0. (Note that D_s cannot be known by δ_0^0, although smaller δ_0^0 corresponds to larger D_s.) Fortunately the situation is not so, as shown in the calculation in § 4. It should be noted that the magnitude of σ_{md}^{ip} (and σ_{md}) is determined approximately by δ_0^0. Also it is to be remarked that the actual σ_{md}^{ip} (and σ_{md}) is rather close to the lower limit at $E_\gamma=11.3$ Mev, because the energy is not high.

Up to now we have discussed I_{00}. There is no difficulty in the calculation of I_{02}. We can do it at once only if δ_2^2 is known (in our case $\delta_2^2\simeq 0$), because the contribution from small distances is unimportant for such a case.

III. Phase Shifts of p-p Scattering at 18.2 Mev.

We used the phase shifts of p-p scattering at 18.2 Mev given by Clementel et al.[9] and Iwadare et al.[14] to the computation of the photodisintegration of the deuteron at 11.3 Mev. These phase shifts are given in Table 2. The six sets corresponding to $\delta_0^0=43.7°$ and $50.5°$ were given by Clementel et al.[9] by the pure phenomenological analysis of phase shifts. The one corresponding to $\delta_0^0=48.3°$ was given by Iwadare et al.[14] in the investigation of the pion potential. We do not discuss here the methods to obtain these phase shifts (The reader may refer to the original papers). In § 3, we see what results will be got from these sets of phase shifts, and what differences will appear among them.

Table 2. Some sets of phase shifts (in unit of degree) obtained by Clementel et al. and Iwadare et al. to account for p-p scattering at 18.2 Mev.

	δ_0^0	δ_1^0	δ_1^1	δ_1^2
A	43.7	19.4	−7.4	2.8
B	”	−14.6	10.5	−1.5
C	”	24.1	−4.3	−1.1
D	48.3	16.7	−3.3	1.4
A′	50.5	3.3	−4.3	1.5
B′	”	−4.8	3.6	1.9
C′	”	6.3	−3.2	0.3

§ 3. Calculated results

There is yet no good experiment at $E_\gamma\simeq 10$ Mev. The cross section at $E_\gamma=11$ Mev given by Allen is as follows.[15]

$$\sigma_T=1.20\pm0.15 \text{ mb}, \tag{11}$$

and the main part of the angular distribution is

$$a+b\sin^2\theta \tag{12}$$

with

$$a/b=0.05\pm0.02. \tag{13}$$

In what follows we show the calculated result, and compare it with the experiment. For convenience we denote f_d of (7) by f_d(phe), meaning that it is derived from phenomenological potential. Also we denote f_d of (8) by $f_d(\pi)$, by which we mean that it is derived from the pion potential.

The calculated results, when we assume special values of D_s in the computation of σ^{ip}_{md} and σ_{md}, are shown in Table 3 and Table 4. Here we took $D_s=0$ for f_d(phe) and $D_s=0.2/\mathrm{k}$ for $f_d(\pi)$ (Note that, the hard core radius of ^3E is 0 for f_d(phe), and 0.2/k for $f_d(\pi)$.). To show the situation more clearly the values of a/b of Table 3 and 4 are presented in Fig. 1 and Fig. 2.

In these tables and figures the relation between the phase shifts and the hard core radius is not taken into consideration. Although D_s in the case of $\delta_0{}^0=50.5°$ may not be large, D_s in the case of $\delta_0{}^0=43.7°$ might not be small. Therefore we over estimated σ^{ip}_{md} (and σ_{md}) in the latter case. However, this causes only a slight effect on our final results, if we take the effect of hard core into consideration. This point is clarified in § 4. Therefore we may see the aspect by Tables 3, 4 and Figs. 1, 2.

From the calculated results we see, in spite of a large difference between f_d(phe) and $f_d(\pi)$, the results of Table 3 and of Table 4 are quite similar.

From these tables we see at once that 1) although there exist some differences among the calculated values of σ_T, all sets can account for σ_T within the experimental error,* 2) the term $\sin^2\theta\cos\theta$, which is due to the interference of e. d. and e. q., is not interesting** because the calculated results of the seven sets are very similar. Therefore we shall not touch these further in what follows.

The most important one to be paid attention is a/b which is strongly dependent on the features of the phase shifts. We may understand that special conditions are necessary to account for the experimental value of a/b (although the indeterminancy of a/b is still large). Concerning a/b we can comprehend the following properties of nuclear force. 1) a/b's of the first three sets corresponding to $\delta_0{}^0=43.7°$ fall into the required region, 0.05 ± 0.02, while a/b's of the last three sets corresponding to $\delta_0{}^0=50.5°$ do not. Also it seems that the set D corresponding to $\delta_0{}^0=48.3°$ would fail.

It is owing to the largeness of σ^{ip}_{md} and σ^{ip}_{ed} in the first three sets, which are obtained with the smaller $\delta_0{}^0$ and the larger $\delta_1{}^t$, that the first three sets can account for the experimental a/b (within the experimental error). We shall discuss this point again in § 4.

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* There exists important relation between σ_T and the triplet odd potential, about which we shall illustrate in another paper.

** Except a case in which the theoretical value is found to be different from the (future) experimental value.

2) The calculated results are almost the same for both of f_d(phe) and $f_d(\pi)$. Generally speaking, the cross sections for $f_d(\pi)$ are a little larger than those for f_d(phe) at our energy. Because the difference is not large, it shows that the influence of the wave function of the deuteron is small at low energies, as it is expected.

3) From the dependence of σ^{ip}_{md} (and σ_{md}) on $\delta_0^{\,0}$ we see that, the smaller $\delta_0^{\,0}$ is, the larger σ^{ip}_{md} (and σ_{md}) is. It indicates that smaller $\delta_0^{\,0}$ is favourable to the photodisintegration of the deuteron. The physical meaning of this will be illustrated in § 4.

4) From the dependence of σ^{ip}_{ed} on $\delta_1^{\,i}$ we see that as $\delta_1^{\,i}$ are larger (in absolute value) and more separated, σ^{ip}_{ed} is larger. The physical meaning of this will also be discussed § 4.

Table 3. The calculated result for the transitions from f_d(phe) at $E_T=11.3$ Mev using the phase shifts of Table 2 is shown. In the calculation of m. d. we took $D_s=0$. The upper half of the table is for the case when we estimate m. d. by its lower limit, and the lower half is for the case when m. d. is estimated by its upper limit. All cross sections are in unit of 10^{-29} cm^2. Also in the last column the coefficient of $\sin^2\vartheta\cos\vartheta$ (β), which is given by the interference of e. d. and e. q., is given.

	σ_{ed}	σ_{md}	σ_T	σ^{ip}_{ed}	σ^{ip}_{md}	σ_{ip}	a/b	β
A	108.30	2.96	111.26	2.08	2.37	4.45	0.028	0.099
B	114.93	,,	117.89	4.79	,,	7.16	0.043	0.101
C	106.97	,,	109.93	2.48	,,	4.85	0.031	0.100
D	110.72	2.49	113.51	1.44	1.99	3.78	0.023	0.100
A′	108.92	2.27	111.19	1.04	1.81	2.85	0.018	0.100
B′	115.05	,,	117.32	1.76	,,	3.57	0.021	0.100
C′	108.82	,,	111.09	0.93	,,	2.74	0.017	0.101
A	108.30	4.49	112.79	2.08	3.78	5.86	0.037	0.099
B	114.93	,,	119.42	4.79	,,	8.57	0.052	0.101
C	106.97	,,	111.46	2.48	,,	6.26	0.039	0.100
D	110.72	3.93	114.65	1.44	3.32	4.76	0.029	0.100
A′	108.92	3.65	112.57	1.04	3.08	4.12	0.026	0.100
B′	115.05	,,	118.70	1.76	,,	4.84	0.028	0.100
C′	108.82	,,	112.47	0.93	,,	4.01	0.025	0.101

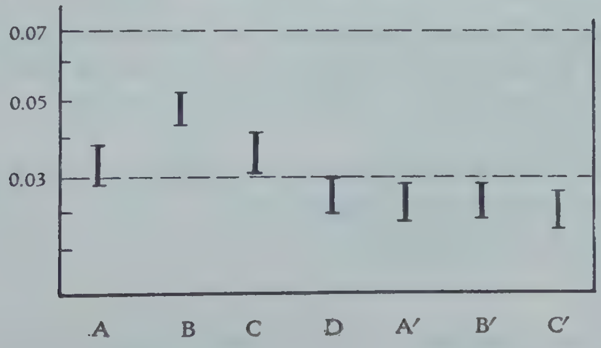

Fig. 1. *a/b* of Table 3 is plotted.

Table 4. The calculated result for the transitions from $f_d(\pi)$ at $E_T=11.3$ Mev is shown. In the calculation of m. d. we took $D_s=0.2/k$. For other contents see the explanation of Table 3.

	σ_{ed}	σ_{md}	σ_T	σ_{ed}^{ip}	σ_{md}^{ip}	σ_{ip}	a/b
A	109.29	3.29	112.58	2.31	2.77	5.08	0.032
B	119.05	,,	122.34	4.82	,,	7.59	0.044
C	108.63	,,	111.92	2.62	,,	5.39	0.034
D	112.20	2.79	114.99	1.66	2.34	4.00	0.024
A′	111.10	2.56	113.66	1.15	2.15	3.30	0.020
B′	118.06	,,	120.62	1.83	,,	3.98	0.023
C′	110.99	,,	113.55	1.07	,,	3.22	0.020
A	109.29	5.04	114.33	2.31	4.39	6.70	0.042
B	119.05	,,	124.09	4.82	,,	9.21	0.053
C	108.63	,,	113.67	2.62	,,	7.01	0.044
D	112.20	4.44	116.64	1.66	3.88	5.54	0.033
A′	111.10	4.14	115.24	1.15	3.62	4.77	0.029
B′	118.06	,,	122.20	1.83	,,	5.45	0.031
C′	110.99	,,	115.13	1.07	,,	4.69	0.028

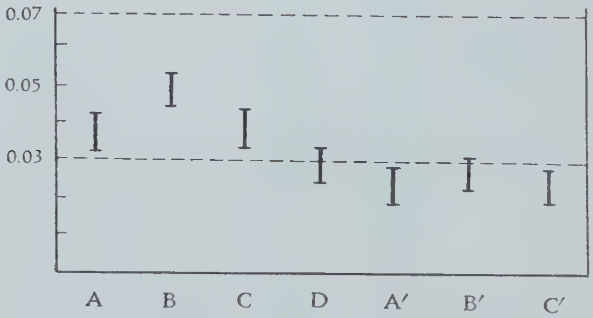

Fig. 2. a/b of Table 4 is plotted.

§ 4. Properties of nuclear forces

In what follows we illustrate the properties of nuclear forces indicated by the calculation of § 3. At first we elucidate the feature of ^1E indicated by σ_{md}^{ip}, secondly of ^3O indicated by σ_{ed}^{ip}, and then about the relation with p-p scattering.

I. Feature of ^1E Indicated by m. d.

In § 3 we saw that the smaller $\delta_0{}^0$ is, the larger σ_{md}^{ip} is. In what follows, we elucidate the reason, and explain the feature of ^1E.

From (5) we see, σ_{md}^{ip} is determined by

$$I_{00}^2-\sqrt{2}\cos(\delta_0{}^0-\delta_2{}^2)\,I_{00}\,I_{02}+\tfrac{1}{2}\,I_{02}^2\,. \qquad (5)'$$

From (6) we can understand that I_{02} is much smaller than I_{00} at low energies, because U_2 is small at $r<1/\alpha$, and w_d is smaller than U_d. Therefore the main part of (5)′

is I_{00}^2. From the first equation of (6) we see that, I_{00} comes from the overlap of U_0 and U_d. From Fig. 3 we can see that I_{00} is determined by the first positive part and the first negative part of U_0 wave. If the overlap of the first positive part of U_0 wave with U_d is much larger than the overlap of the first negative part of U_0 wave with U_d, I_{00} is very large. If it is not the case, I_{00} is small. The former corresponds to smaller $\delta_0{}^0$, and the latter to larger $\delta_0{}^0$.

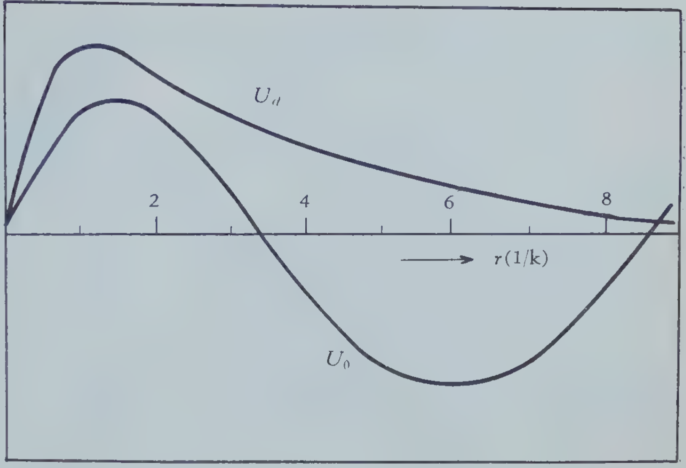

Fig. 3. The overlap of U_d (phe) wave with U_0 wave at 18.2 Mev nucleon-nucleon scattering is plotted. For U_0 we used (10) corresponding to $D_s=0$, i. e. $\sin(kr+\delta_0{}^0)(1-e^{-\zeta r})$ with $\delta_0{}^0=50.5°$.

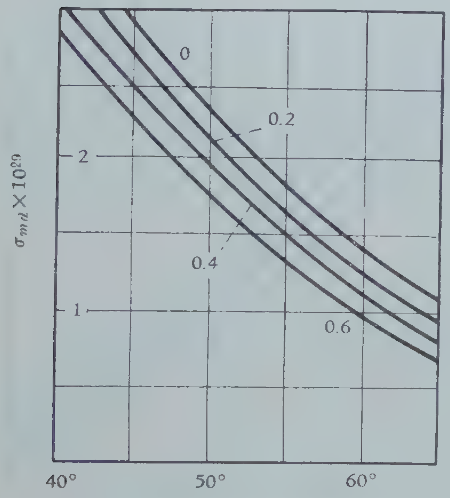

Fig. 4. The lower limit of σ_{md} for the transition, ${}^3S_1\rightarrow{}^1S_0$ (at $E_\gamma=11.3$ Mev) is shown as the function of $\delta_0{}^0$ and D_s. 0, 0.2, 0.4, and 0.6 are D_s in unit of 1/k. They represent the lower limit of σ_{md} corresponding to given D_s.

To make the situation more clear, we show the dependence of m. d. cross sections on $\delta_0{}^0$ in Figs. 4, 5, and 6. In these figures the lower limit of σ^{ip}_{md} and σ_{md} are plotted as the function of $\delta_0{}^0$ and D_s*. Here we used f_d (phe) as the wave function of the deuteron, and varied D_s in $0\sim0.6/k$.

From these figures we can comprehend that, taking the existence of the hard core into consideration,* "If $\delta_0{}^0$ is smaller, the corresponding σ^{ip}_{md} (and σ_{md}) is larger", except a case in which $\delta_0{}^0=f(D_s)$ is changed quite slowly with the variation of D_s at our energy, although it does not seem to be the case (Even if $\delta_0{}^0$ varies so slowly that, when D_s varies from 0 to $0.6/k$, it decreases from 55°# to 45° at our energy,* and σ^{ip}_{md} is still increased by 20% more in this variation). Perhaps the shadowed part of Fig. 5 would represent the relation between the phase shifts and σ^{ip}_{md}.*#

It seems that $\delta_0{}^0$ has been taken somewhat larger at energies higher than 10 Mev.[17)] Speaking in the words of potential, it seems that the singlet even potential would be rather repulsive than it has been supposed thus far (except at low energies).

---

* If $\delta_0{}^0$ is smaller, the corresponding D_s is larger, generally speaking.

# The Yukawa type potential without the hard core gives $\delta_0{}^0\simeq54°$ at 18 Mev.[16]

II. Feature of ^3O indicated by e. d.

Although smaller δ_0^0 is favourable for the photodisintegration of the deuteron, it is still insufficient by itself to account for the large experimental a/b. (See the computation in § 3). Therefore we are forced to expect some part of the isotropic cross section to come from the e. d. transition.

From the formulas (2) and (3) we may see that, generally speaking, if the D-state probability is larger and δ_1^i are larger, σ_{ed}^{ip} is larger. However, there is a limit which we can take about the D-state probability. Furthermore the detailed form of the wave function of the deuteron is unimportant at low energies as we see in § 3. Therefore, to expect large σ_{ed}^{ip}, we infer that for 18 Mev nucleon-nucleon scattering δ_1^i are no longer small and somewhat separated from one another. In the words of potential we may say that the tensor force in ^3O is not small.

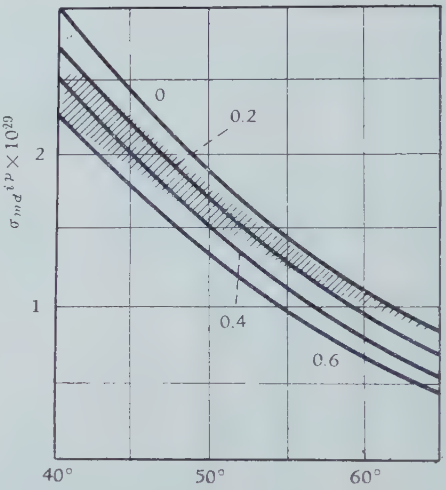

Fig. 5. The lower limit of σ_{md}^{ip} (at $E_\Upsilon=11.3$ Mev) as the function of δ_0^0 and D_s is shown for the case of f_d (phe). The meaning of the figures in the graph is similar to that of Fig. 4. The shadowed part would represent the relation between δ_0^0 and σ_{md}^{ip}.

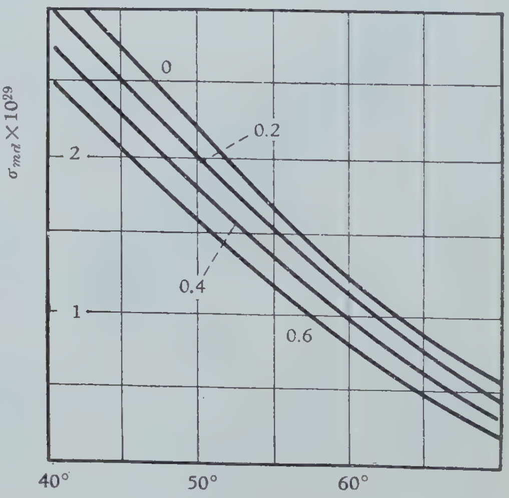

Fig. 6. The lower limit of σ_{md} (at $E_\Upsilon=11.3$ Mev) as the function of δ_0^0 and D_s is shown for the case of f_d (phe). The meaning of the figures is similar to that of Fig. 4.

III. Relation with p-p Scattering

We have seen that if δ_0^0 is smaller, σ_{md}^{ip} is larger, and that if δ_1^i are larger and more separated, σ_{ed}^{ip} is larger. Summing up these two effects, we can account for the experimental value of a/b. Now we discuss the relation with p-p scattering. In p-p scattering $d\sigma(90°)/d\Omega$ must be accounted for by δ_0^0 and the difference among δ_1^0, δ_1^1 and δ_1^2. Therefore if δ_0^0 is smaller, δ_1^0, δ_1^1, and δ_1^2 must be larger in the absolute value and more separated from one another. Thus we may conclude as follows, "To account for both p-p scattering and the photodisintegration of the deuteron in consistence with one another, δ_0^0 would be smaller, and δ_1^0, δ_1^1, δ_1^2 would be larger and more separated from one another, than they have been supposed up to today".

§ 5. Conclusion

We summarize the results obtained in this paper in the following.

1) We can account for the photodisintegration of the deuteron near $E_\gamma = 11$ Mev in consistence with p-p scattering, if we take δ_0^0 to be somewhat smaller than that previously supposed, and the phase shifts of ${}^3P$ states larger than those previously given.

2) Generally speaking, if δ_0^0 is smaller, σ_{md}^{ip} is increased, and makes the isotropic part larger, which is favourable for the photodisintegration of the deuteron. It seems that ${}^1E$ potential would be somewhat more repulsive than previously estimated at energies above 10 Mev for nucleon-nucleon scattering. From the analysis of § 3 and § 4 it seems to be $\delta_0^0 \lesssim 48°$ for 18 Mev nucleon-nucleon scattering.

3) Generally speaking, if the three phase shifts of ${}^3P$ are larger (in the absolute value) and more separated from one another, σ_{el}^{ip} is larger, and it is favourable for the photodisintegration of the deuteron. These phase shifts are no longer small for 18 Mev nucleon-nucleon scattering.

4) To account for both p-p scattering and photodisintegration of the deuteron consistently, δ_0^0 would be smaller, and δ_1^i would be larger and separated from one another, than those previously given.

5) The influence of the wave function of the deuteron is small at $E_\gamma \simeq 11$ Mev.

Similar analysis at higher energies to show up many properties of nuclear forces and further analysis near our energy to reveal a concrete shape of the nuclear potential are now being carried out and will be published in due time.

The author is much indebted to the financial support by the Yomiuri Yukawa fellowship.

Appendix A

U_{1i} of (9) tends to ∞ at $r=0$, whereas U_{1i} must vanish at $r=0$. Therefore we need to examine the accuracy when we approximate U_{1i} by (9). (9) can be rewritten as

$$U_{1i} = r\left(j_1(kr)\cos\delta_1^i - n_1(kr)\sin\delta_1^i\right). \qquad (9)'$$

To estimate the accuracy of approximation, we calculated the contribution of the last term of (9)′ from $r \leq 1/k$ to I_{1i} of (6) (Note that, $\lim_{r\to 0} j_1(kr) = 0$, and $\lim_{r\to 0} n_1(kr) = -1/k^2r^2$).

The integrated form of I_{1i} can be written as

$$I_{1i} = G_i \cos\delta_1^i + H_i \cos\delta_1^i. \qquad (6)'$$

Here G_i is due to $j_1(kr)$, and H_i is due to $n_1(kr)$ of (9)′. At our energy

$$G_i \simeq 2H_i. \qquad (A)$$

In H_i the contribution from $r \leq 1/k$ (we denote it by H_i') can be expressed by

$$2/9 \cdot H_i \geq H_i'. \tag{B}$$

From (6)′, (A) and (B) we get

$$H_i' \sin\delta_1^i / I_{1i} \lesssim 0.04, \quad \text{when} \quad \delta_1^i = 20°,$$

and

$$H_i' \sin\delta_i^1 / I_{1i} \lesssim 0.02, \quad \text{when} \quad \delta_1^i = \pm 5°.$$

Therefore the approximation of (9) is not bad.

Appendix B

Applying the effective range theory,[18] the singlet effective range r_s can be expressed as the function of the singlet scattering length a_s, the hard core radius D_s, and ζ of (10) by

$$r_s/2 = P + Q/\zeta + R/\zeta^2 + S/\zeta^3 \tag{C}$$

Here

$$P = D_s + aD_s^2 + a^2 D_s^3/3, \quad Q = 3/2\,(1 + 2\,aD_s + 2\,a^2 D_s^2),$$

$$R = a/2\,(7 + 13\,aD_s), \quad S = 15\,a^2/2, \quad \text{and} \quad a = -1/a_s \tag{D}$$

If r_s, a_s, and D_s are given, ζ can be computed from (C) and (D). (See Table 1) For r_s and a_s we used $r_s = 2.7 \times 10^{-13}$ cm, and $a_s = -23.68 \times 10^{-13}$ cm.[19]

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Electronic Energy Levels of Polyene Chains

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An attempt is made to treat the problem of absorption wavelength of polyene chains in the case of the alternate single and double bonds. The method of LCUAO MO of Brown and Matsen is found to be useful for the problem. Simple calculations yield the wavelength formula,

$$\lambda=\frac{12398}{A+\sqrt{A^2+B^2\sin^2(\pi/N)}}\times10^{-8}\ \mathrm{cm}$$

where A and B are the parameters of the theory and N indicates the number of π-electrons in the polyene chain. It is clear that this formula has the desired property of the so-called " convergence ". The fit to the experimental data is good. Possible physical implications of the present calculation are discussed in some detail.

§ 1. Introduction

In the one-dimensional free π-electron model of linear long chain molecules the σ-bond skeletons are replaced by suitable deep square-well potentials for the π-electrons. If we identify the lowest excitation energy of the model with the energy of the first absorption of the chain molecule, then it follows that the absorption wavelength is proportional to the chain length, that is to say, the length of the square-well potential L. This free-electron molecular orbital method is applied successfully to long symmetrical odd-atom dyes in which the first absorption wavelength increases by 1000 ± 50 Å for every additional vinyl group in the chain, successive increases being ' noncovergent '.[1] However, this method encounters a serious difficulty in the treatments of unsymmetrical cyanine dyes or polyenes where the absorption wavelength does not increase linearly but ' converge ' with increasing chain length, probably to a finite limiting wavelength of absorption. Several authors attempted to explain this conspicuous property. Araki and Murai[2] explained the convergence of the absorption wavelength of carotenoids by introducing the effect of the interaction between π-electrons into the one-dimensional free-electron model. The method of calculation in their paper is very interesting and the formula for the wavelength of the absorption maxima obtained by them reproduces the convergence property in good agreement with the experimental data of carotenoids. In addition, Araki[3] published a discussion on the physical implication of the calculation and stated that ' the light absorption of long carotenoids is due mainly to the plasma oscillation of π-electrons contained in their conjugated chains '. However, it is not necessarily clear why we need not or should not include the interaction between π-electrons in the case of symmetrical cyanines, while the inclusion is quite essential for carotenoids and it seems rather problematic to explain the limiting energy gap as the excitation of the plasma oscillation in a microscopic electron gas tube, because the frequency seems too small as a so-called plasma oscillation and the mechanism of the light absorption of the plasma oscillation is not clear. Kuhn[4] proposed a modification of the simple square-well potential by introducing the alternation of effective nuclear potential corresponding to the alternate single and double bonds of the chain. This periodicity introduces a Brillouin gap into the electronic energy spectrum, which brings about the convergence of the absorption wavelength in long chains. By varying a parameter V_0, a measure of the degree of the single-double alternation, Kuhn succeeded to treat both non-convergent case of the symmetrical cyanine and convergent cases of the unsymmetrical cyanine and the polyene. Dewar[5] put such alternating bond strengths into an LCAO-type treatment and got a convergent two-parameter formula fitting the polyene spectra. But Bayliss and Rivière[6] objected that such alternations are expected to get smaller with increasing chain length. This point of view seems to appeal to our naive intuitive imagination. Platt[1] supported, however, again the existence of the single-double alternation in his detailed semi-empirical investigation of the present subject. Then, it would be a challenging theoretical problem to confirm that the alternation of the single-double bond character and the bond length does not decrease with increasing chain length. To present a complete answer to this problem is undoubtedly a very difficult task because we must consider the role of σ-electrons together with π-electrons in a sufficiently long chain and see what kind of the bond alternation will be realized. This is a formidable and almost impossible task for the present scheme of the theoretical calculation.

Nevertheless, in our opinion, there are indications that we might be able to say something about the present problem within the simple π-electron approximations. It is known that the simple LCAO method with all bonds equal makes the absorption wavelength proportional to the chain length L for all very long chains, polyenes included, and the order-length relation for carbon-carbon bonds indicates that the inequalities in bond lengths diminish rather rapidly as we go on from the end to the middle part of long chains. However, this situation does not necessarily imply that the simple LCAO MO type treatment is entirely useless for the present problem. We should recognize the positive side of the meaning of the successful order-length relation for shorter chains, where comparisons between the predictions and the experimental data are possible, and generally good agreements are found. It is important to recognize that the use of the molecular orbitals in the form of linear combination of the atomic $2p\pi$ orbital on each carbon atom is useful but not necessarily essential to obtain the appropriate order-length relation. The point is that it is a meaningful approximation to suppose that the one-electron waves are delocalized and quantized in the whole dimension of the molecular system. The close parallelism between LCAO MO and FE (free-electron) MO is the well-known fact. In this regard Platt[7] displayed a very instructive example where σ-bond skeletons of conjugated molecules are replaced by two-dimensional boxes of suitable shapes. For instance polyenes and polyacenes are represented by simple rectangular boxes of suitable sizes in which the π-electrons are moving freely. Platt could reveal an astonishing fact that the total density distribution of π-electrons reproduces rather closely the geometry of the σ-

bond skeleton and further the amount of electron concentration on each bond position is properly in accord with the bond order prediction by the LCAO MO treatment based on the actual bond skeleton. This must be more than accidental and it is natural to expect that this density variation of π-electron cloud should counteract the positions of carbon atoms in the bond skeleton and the counteraction might be estimated to a certain degree by the simple molecular orbital type treatment within the π-electron approximation. This is what we attempt in the present paper. We treat here the polyene chains because we think that this is the most suitable example to reveal the whereabout of the problem.

§ 2. Wavelength formula of polyenes with alternate bond length

Let us first state our problem in terms of the usual LCAO MO method. The system consists of N π-electrons and the σ-bond skeleton containing N carbon nuclei, N being an even number. The secular determinant of N degree to be solved is

$$\begin{vmatrix} \alpha-E & \beta & & & & & & \\ \beta & \alpha-E & \beta' & & & & & \\ & \beta' & \alpha-E & \beta & & & & \\ & & \beta & \alpha-E & & & & \\ & & & & \ddots & & & \\ & & & & & \alpha-E & \beta' & \\ & & & & & \beta' & \alpha-E & \beta \\ & & & & & & \beta & \alpha-E \end{vmatrix}=0 \qquad (1)$$

where α and β, β' are usual Coulomb integral parameter and resonance integral parameters respectively.* It is customarily supposed that $\beta=\beta'$ means the equal bond length and if we want to discuss the case of alternate bond length we have to treat the general case of $\beta \fallingdotseq \beta'$. If we solve the above secular equation, we obtain the energy levels $\{E_j\}$ ($j=1, \cdots, N$) and the lower $N/2$ levels are occupied by the N π-electrons. Then the total π-electron energy of the system E^{π}_{total} is

$$E^{\pi}_{\text{total}}=2\sum_{j=1}^{N/2} E_j, \qquad (2)$$

and the energy gap corresponding to the first absorption maximum is

$$\Delta E=|E_{N/2+1}-E_{N/2}|. \qquad (3)$$

The solution of the above equation (1) was discussed by Lennard-Jones8) and Coulson.9) In the case of $\beta=\beta'$ the solution can be expressed in a compact analytical form :

---

* Here and in the following we neglect the overlap integral between two neighboring orbitals, as usually done in the discussions similar to the present one. Of course, we can include it by writing $\beta-SE$ in place of β. However, it seems to bring about unnecessary complications for the present discussion.

$$E_j = \alpha + 2\beta \cos \frac{j\pi}{N+1}. \tag{4}$$

Then
$$\Delta E = \left| \left(\alpha + 2\beta \cos \frac{N/2+1}{N+1}\pi \right) - \left(\alpha + 2\beta \cos \frac{N/2}{N+1}\pi \right) \right|$$
$$= \left| 4\beta \sin \frac{\pi}{2(N+1)} \right| \tag{5}$$

and evidently

$$\Delta E \simeq \left| \frac{2\pi\beta}{N+1} \right| \to 0 \quad (N:\ \text{Large}). \tag{6}$$

This simply means that the absorption wavelength grows larger and larger and shows no convergence property as the chain becomes longer. It is expected that this unfavorable situation might be remedied in the case of $\beta \neq \beta'$ but unfortunately the solution have not been obtained in a form as compact as in the case of $\beta = \beta'$. Dewar[5] attempted to treat the case of $\beta \neq \beta'$ by a sort of perturbation calculation and obtained a convergent two-parameter formula for the energy gap ΔE fitting the polyene spectra. However, the validity of the perturbational treatment is not necessarily obvious and the information about the other energy levels cannot be obtained in his treatment.

Recently, Brown and Matsen[10] attempted to extend the united-atom concept to conjugated systems by taking the molecular orbitals to be linear combinations of united-atom orbitals located in each bond of the system. It may be called LCUAO MO method after them. Fig. 1 may be helpful to understand the method. A precise form of the π-electron UAO's will be specified in the next section. In the present section they occur only in the Coulomb and resonance integrals which are regarded as semi-empirical parameters. As shown in (1) and Fig. 1, if we want to treat the case of the alternate bond length by the usual LCAO MO method, the constancy of the distances between the adjacent centers of AO's is lost and two values of resonance integrals, β and β', become necessary. The situation is different in the LCUAO MO method, where the distance between the adjacent UAO's can be assumed to be constant approximately and consequently we need two values of the Coulomb integrals, α and α'.

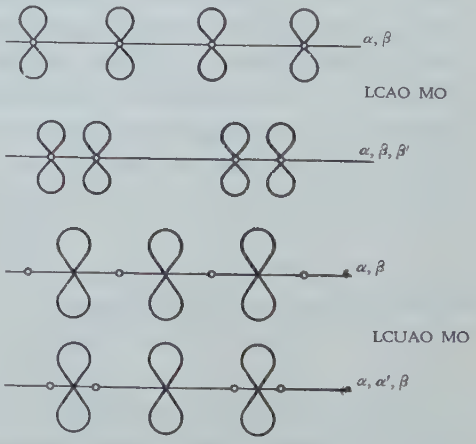

Fig. 1

Thus, the secular determinant to be solved in the LCUAO MO method is, corresponding to (1),

$$D_{N-1}=\begin{vmatrix} \omega & 1 & & & & & \\ 1 & \omega' & 1 & & & & \\ & 1 & \omega & 1 & & & \\ & & 1 & \omega' & & & \\ & & & & \ddots & & \\ & & & & & \omega' & 1 \\ & & & & & 1 & \omega \end{vmatrix}=0. \tag{7}$$

Here
$$\omega=(\alpha-E)/\beta,\quad \omega'=(\alpha'-E)/\beta,$$
$$\alpha=\alpha_0-\epsilon,\quad \alpha'=\alpha_0+\epsilon',\quad \epsilon,\ \epsilon'\geq 0,\quad \beta=\beta_0+\delta,$$

where α_0 and β_0 denote the Coulomb and the resonance integral respectively in the case of equal bond length. For later convenience we take ϵ and ϵ' to be positive. There is some ambiguity in the sign of δ but we need only a plausible relation of $|\epsilon|$, $|\epsilon'|\gg|\delta|$. It is to be noted that the above determinant is of degree $N-1$, instead of N in (1).

Now let us try to solve the equation (7). If we define the following series of determinants :

$$D_1=\omega,$$

$$D_2=\begin{vmatrix} \omega & 1 \\ 1 & \omega' \end{vmatrix}=(\omega\omega')-1,$$

$$D_3=\begin{vmatrix} \omega & 1 & \\ 1 & \omega' & 1 \\ & 1 & \omega \end{vmatrix}=\omega[(\omega\omega')-2],$$

$$D_4=\begin{vmatrix} \omega & 1 & & \\ 1 & \omega' & 1 & \\ & 1 & \omega & 1 \\ & & 1 & \omega' \end{vmatrix}=(\omega\omega')^2-3(\omega\omega')+1,\ \ldots,$$

we obtain the recurrence relations,

$$D_{N-2}=\omega' D_{N-3}-D_{N-4}, \tag{8}$$
$$D_{N-1}=\omega D_{N-2}-D_{N-3},$$

N being an arbitrary even number larger than 5. Then we can prove easily by mathematical induction that D_{N-1} takes the form :

$$D_{N-1}=\omega f_{N-1}[(\omega\omega')], \tag{9}$$

where $f_{N-1}(x)$ means a polynomial of x. In the special case of $\omega=\omega'$, namely $\alpha=\alpha'$, we obtain the equation,

$$D_{N-1}=\omega f_{N-1}[\omega^2]=0. \tag{10}$$

The solution of this equation is

$$\omega_j=(\alpha-E_j)/\beta=-2\cos(j\pi/N). \quad (j=1,\ 2,\ \cdots,\ N-1). \tag{11}$$

Here $\omega_{j=N/2}$ just corresponds to $\omega=0$ in the above equation (10) and this is the highest occupied level if N is supposed to be the number of π electrons in the polyene chain. Then the roots of $f_{N-1}[\omega^2]=0$ is expressed as

$$\begin{aligned}&\omega_j^2=4\cos^2(j\pi/N),\quad j=1,\ \cdots,\ (N/2)-1,\ (N/2)+1,\ \cdots,\ N-1\\&\omega_j^2=\omega_{N-j}^2.\end{aligned} \tag{12}$$

Comparing (9) with (10), we see that the roots of the equation (7) are given as follows,

$$\begin{aligned}&\omega=0\\&(\omega\omega')_j=4\cos^2(j\pi/N),\quad j=1,\ \cdots,\ (N/2)-1,\ (N/2)+1,\ \cdots,\ N-1\\&(\omega\omega')_j=(\omega\omega')_{N-j}.\end{aligned} \tag{13}$$

A more elegant treatment of the secular equation (7) will be found in Appendix. Substituting the relations $\omega=(\alpha-E)/\beta$ and $\omega'=(\alpha'-E)/\beta$ in the above relations, we obtain the energy levels relevant to the present problem :

$$\begin{aligned}&E_j=\alpha_0-\tfrac{1}{2}(\epsilon-\epsilon')-\sqrt{[(\epsilon+\epsilon')/2]^2+(2\beta)^2\cos^2(j\pi/N)},\quad j=1,\ \cdots,\ (N/2)-1\\&E_{N/2}=\alpha_0-\epsilon,\\&E_{N/2+1}=\alpha_0-\tfrac{1}{2}(\epsilon-\epsilon')+\sqrt{[(\epsilon+\epsilon')/2]^2+(2\beta)^2\sin^2(\pi/N)}.\end{aligned} \tag{14}$$

$E_{N/2}$ is the energy of the highest occupied level and $E_{N/2+1}$ is the energy of the lowest vacant level. Then

$$\begin{aligned}\Delta E&=\tfrac{1}{2}(\epsilon+\epsilon')+\sqrt{[(\epsilon+\epsilon')/2]^2+(2\beta)^2\sin^2(\pi/N)}\\&=A+\sqrt{A^2+B^2\sin^2(\pi/N)},\quad A=\tfrac{1}{2}(\epsilon+\epsilon'),\quad B=(2\beta).\end{aligned} \tag{15}$$

According to the relation $\lambda=hc/\Delta E$ the absorption wavelength formula is written as

$$\lambda=\frac{12398}{A+\sqrt{A^2+B^2\sin^2(\pi/N)}}\times10^{-8}\,\mathrm{cm}\quad (A,\ B\ \text{in eV}). \tag{16}$$

A similar formula was presented by W. Kuhn. It is evident that this formula has the desired property of convergence. The limiting wavelength is given to be $\lambda_{N\to\infty}=12398/2A$ and this relation restricts rather narrowly the allowed range of values of the parameter A. With parameter values of $A=1.13$ and $B^2=48$, this formula can fit the experimental data to the same degree as H. Kuhn's formula,[4)] as is shown in Table 1. A more careful choice of the parameter values will bring about a better agreement with the experimental data but too much elaboration on this point does not seem to be worthy.

Table 1. (λ in mμ)

Polyene	N	λ_{exp}	λ_{Kuhn}	$\lambda_{\text{H-H}}$
Butadiene	4	217	233	201
Hexatriene	6	260	282	260
Octatetraene	8	302	320	309
Axerophtene	10	346	350	349
Anhydro Vitamin A	12	369	375	382
β-Carotene	22	451	451	471
Dehydro-β-Carotene	24	475	461	481
Dehydrocopene	30	504	486	502

§ 3. Discussion on alternation of bond length

In this section we shall first try to make a rough estimation of the magnitude of ϵ and ϵ' in (7) by specifying our united-atom orbital for π-electrons. Suppose a system which consists of two carbon cores of effective charge Z and one $2p\pi$-electron. (See Fig. 2)

The two carbon cores, a and b, are separated by a distance of $2R$ and the electron is described by a $2p\pi$-orbital with the center at the middle point between the two carbon nuclei. The energy operator is (in atomic units),

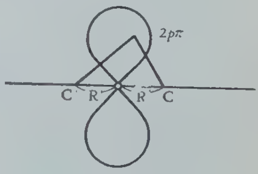

Fig. 2

$$H=-\frac{1}{2}\Delta-\frac{Z}{r_a}-\frac{Z}{r_b} \tag{17}$$

and the wave function is

$$\psi=(2p\pi\,;\,\zeta)=N(2,\,\zeta)re^{-\zeta r}(1/\sqrt{2})[Y_{1,-1}(\theta,\,\varphi)+(-1)Y_{1,+1}(\theta,\,\varphi)], \tag{18}$$

$$Y_{1,\pm1}(\theta,\,\varphi)=\mp\sqrt{\frac{3}{4}}\sin\theta\, e^{\pm i\varphi}\frac{1}{\sqrt{\pi}},$$

where $N(2,\,\zeta)$ is the normalization constant of the radial part. The expectation value of the energy is obtained by a simple calculation to be

$$E(2R)=\int\psi^*H\psi d\tau=\frac{1}{R^2}\frac{\sigma^2}{8}-\frac{Z}{2R}\frac{1}{\sigma^2}\left\{-24+4\sigma^2+e^{-\sigma}(24+24\sigma+8\sigma^2+\sigma^3)\right\},$$

$$\sigma=2\zeta R, \tag{19}$$

and $\partial E/\partial\sigma=0$ yields

$$\sigma\Big/2RZ=(1/\sigma^2)\left\{48-e^{-\sigma}(48+48\sigma+24\sigma^2+7\sigma^3+\sigma^4)\right\}. \tag{20}$$

For the present purpose we assume $Z=3$ and at $2R=1.4\,\text{Å}$ we determine ζ to be 1.3229 by using (20). Then with the fixed values of $Z=3$ and $\zeta=1.3229$ we calculate the energy $E(2R)$ at three values of the carbon-carbon distance, $2R=1.35\,\text{Å}$, $1.4\,\text{Å}$ and $1.45\,\text{Å}$. The result is shown in Table 2.

Table 2.

$2R$ (Å)	E $(2R)$ (a.u.)	ϵ, ϵ'	$\epsilon+\epsilon'$
1.35	−2.179614	$\epsilon=0.044853$	$\epsilon+\epsilon'=0.089311$ a.u.
1.4	−2.134761	$\epsilon'=0.044458$	$=2.43015$ eV
1.45	−2.090303		

Here ϵ and ϵ' are assumed to be given by the differences of adjacent E values as indicated in the third column of Table 2. It is interesting to observe that the value of $\epsilon+\epsilon'=2\varDelta=2.43$ eV estimated above is close to the value $2\varDelta=2.26$ eV used in our wavelength formula to fit the experimental data in the preceding section. By a simple interpolation procedure we find that the variation of bond lengths $1.353\,\text{Å}\leftarrow(1.4\,\text{Å})\rightarrow 1.4465\,\text{Å}$ just corresponds to the value $2\varDelta=\epsilon+\epsilon'=2.26$ eV. In other words, this result may be stated as follows: if the polyene chains with equal bond length of 1.4 Å become chains of alternative bond length in which the shorter bond length is 1.3535 Å and the longer 1.4465 Å, this change brings about the convergence of absorption wavelength and we can obtain the agreement with theory and experiment in polyene series. The bond lengths 1.35 Å and 1.45 Å seem to be reasonable in view of the experimental information on the bond alternation in butadiene.

So far, the preservation of the alternation of bond length in the long polyenes is nothing more than an assumption in the present treatment just as in the treatments of H. Kuhn,[4] Dewar[5] and Platt.[1] The only advantage of the present treatment over others is that we present a reasonable estimation of the parameter value which determines the limiting absorption wavelength of the polyene series.

Now we shall try to look for possible causes to preserve the distinct bond alternation in the longer chains beyond the expectation from the usual order-length relation. From (2) and (14) we obtain

$$E^{\pi}_{\text{total}}=2\sum_{j=1}^{N/2-1}\left\{\alpha_0-\tfrac{1}{2}(\epsilon-\epsilon')-\sqrt{[(\epsilon+\epsilon')/2]^2+[2(\beta_0+\delta)]^2\cos^2(j\pi/N)}\right\}+2(\alpha_0-\epsilon). \tag{21}$$

This corresponds to the case where the chain has shorter bonds on both ends. The energy of the reversed case where longer bonds are found on both ends is easily found to be

$$E^{\pi}_{\text{total}}=2\sum_{j=1}^{N/2-1}\left\{\alpha_0-\tfrac{1}{2}(\epsilon-\epsilon')-\sqrt{[(\epsilon+\epsilon')/2]^2+[2(\beta_0+\delta)]^2\cos^2(j\pi/N)}\right\}+2(\alpha_0+\epsilon'). \tag{22}$$

Then

$$\varDelta=E^{\pi}_{\text{total}}(22)-E^{\pi}_{\text{total}}(21)=2(\epsilon+\epsilon').$$

This energy difference is characteristic of the chain with even number carbon atoms. According to our tentative calculation displayed above $\varDelta\simeq 4\sim 5$ eV. This must be said to be rather small, against our first expectation. It would be possible, however, that this energy difference could work well in the chains of, at least, intermediate length. We know that the nature makes the best use of such a small energy difference on the determination of the spatial structure of the matter. But it would be rather hazardous to assert that this energy difference could preserve the distinct bond alternation even in the very long chains. It would be more natural to expect that the bond alternation would diminish sooner or later in the very long chain in spite of the difference Δ. Then how about the convergence property of the absorption wavelength? Our conjecture on this point is that the effect of correlation between electrons through the long range Coulomb force, just as discussed by Araki and Murai,[2] will arise and compensate the eventual decay of the effect of the above mentioned energy gain in the very long chains. This effect of the electron correlation is expected to be found equally both in the polyene chains discussed here and in the symmetrical and unsymmetrical cyanines but, unfortunately, we lack the experimental data on the very long chains, especially in the case of cyanine dyes.

§ 4. Conclusion

By using LCUAO MO method of Brown and Matsen, we have solved analytically the secular equation for the general case of polyene chains with even number carbon atoms and obtained an absorption wavelength formula as useful as the ones due to the other authors. In addition we have attempted some discussions on the real occurrence of the bond alternation in the longer chains. Our discussion could go hardly beyond mere speculation but we believe that the fact that the polyene chains discussed here contain even number of carbon atoms will play an essential role in this problem. Probably we should prepare a separate explanation for the convergence property of absorption wavelength in the unsymmetrical cyanines, where the end nuclei would play a dominant role and the experimental data exist only for rather shorter chains.

Appendix

Let P_N be the determinant

$$\begin{vmatrix} \omega & 1 & & & \\ 1 & \omega & 1 & & \\ & 1 & \omega & & \\ & & & \ddots & \\ & & & \omega & 1 \\ & & & 1 & \omega \end{vmatrix}. \qquad \text{(A1)}$$

If we write $\omega = -2\cos\theta$, then we find

$$P_N = (-1)^N \frac{\sin(N+1)\theta}{\sin\theta}. \qquad \text{(A2)}$$

Then the roots of $P_N=0$ are all real and are

$$\omega_j=-2\cos\frac{j\pi}{N+1},\quad j=1,\ 2,\ \cdots,\ N. \tag{A3}$$

This is the result obtained by Coulson.9)

Expanding P_N in (A1) in terms of the top row, we find the recurrence formula,

$$P_N=\omega P_{N-1}-P_{N-2}. \tag{A4}$$

From this, the recurrence relation among P_{2n}, P_{2n-2} and P_{2n-4} is found to be

$$P_{2n}=(\omega^2-2)P_{2n-2}-P_{2n-4}. \tag{A5}$$

The initial values are

$$P_2=\begin{vmatrix}\omega & 1\\ 1 & \omega\end{vmatrix}=\omega^2-1$$

$$P_4=\begin{vmatrix}\omega & 1 & & \\ 1 & \omega & 1 & \\ & 1 & \omega & 1\\ & & 1 & \omega\end{vmatrix}=\omega^4-3\omega^2+1.$$

Now let us consider the determinant

$$d_{2n}=\begin{vmatrix}\omega & 1 & & & & \\ 1 & \omega' & 1 & & & \\ & 1 & \omega & 1 & & \\ & & 1 & \omega' & & \\ & & & \ddots & & \\ & & & & \omega & 1\\ & & & & 1 & \omega'\end{vmatrix}. \tag{A6}$$

Expanding in terms of the top row, we get the recurrence relation

$$d_{2n}=\omega D_{2n-1}-d_{2n-2}. \tag{A7}$$

Here

$$D_{2n-1}=\begin{vmatrix}\omega' & 1 & & & \\ 1 & \omega & 1 & & \\ & 1 & \omega' & & \\ & & \ddots & & \\ & & & \omega & 1\\ & & & 1 & \omega'\end{vmatrix}. \tag{A8}$$

Then, again expanding in terms of the top row, we find

$$D_{2n-1}=\omega' d_{2n-2}-D_{2n-3}. \tag{A9}$$

From (A7) and (A9) we obtain the recurrence formula

$$d_{2n}=(\omega\omega'-2)\,d_{2n-2}-d_{2n-4}. \tag{A10}$$

The initial values are

$$d_2=\begin{vmatrix} \omega & 1 \\ 1 & \omega' \end{vmatrix}=(\omega\omega')-1,$$

$$d_4=\begin{vmatrix} \omega & 1 & & \\ 1 & \omega' & 1 & \\ & 1 & \omega & 1 \\ & & 1 & \omega' \end{vmatrix}=(\omega\omega')^2-3\,(\omega\omega')+1.$$

Comparing (A10) with (A5), we observe that, if we write $(\omega\omega')^{1/2}=-2\cos\theta$, d_{2n} can be expressed as follows,

$$d_{2n}=\frac{\sin(2n+1)\,\theta}{\sin\theta}. \tag{A11}$$

Then the equation $d_{2n}=0$ yields the formula

$$(\omega\omega')_j=4\cos^2\frac{j\pi}{2n+1}, \quad j=1,\ 2,\ \cdots,\ 2n. \tag{A12}$$

Next, let us return to the recurrence relation (A7). By using (A11), we get

$$\omega D_{2n-1}=d_{2n}+d_{2n-2}=[\sin(2n+1)\,\theta+\sin(2n-1)\,\theta]/\sin\theta=\omega\omega'\sin 2n\theta/\sin 2\theta,$$

namely

$$D_{2n-1}=\omega'\frac{\sin 2n\theta}{\sin 2\theta}. \tag{A13}$$

Then the equation $D_{2n-1}=0$ gives the formulas,

$$\omega'=0, \quad \sin 2n\theta/\sin 2\theta=0. \tag{A14}$$

If we rewrite $\omega'\to\omega$ and $\omega\to\omega'$, then D_{2n-1} here is essentially the same determinant as D_{2n-1} in § 2 and from (A13) we readily obtain the results given in (13) of § 2.

Finally we consider the following determinant

$$\Delta_{2n}(\delta,\ \delta')=\begin{vmatrix} \omega+\delta & 1 & & & & \\ 1 & \omega' & 1 & & & \\ & 1 & \omega & 1 & & \\ & & 1 & \omega' & & \\ & & & & \ddots & \\ & & & & \omega & 1 \\ & & & & 1 & \omega'+\delta' \end{vmatrix}. \tag{A15}$$

By using similar techniques as described above, we reach the following expression.

$$\Delta_{2n}(\delta,\ \delta')=(1/\sin 2\theta)\Big\{2\sin(2n+1)\theta\cdot\cos\theta+2\,\delta\delta'\sin(2n-1)\theta\cdot\cos\theta$$
$$+(\omega\delta'+\omega'\delta)\sin 2n\theta\Big\}. \qquad \text{(A16)}$$

If we could solve the equation $\Delta_{2n}(\delta,\ \delta')=0$ analytically in the general case of arbitrary values of δ and δ', we could attempt some interesting treatments of the symmetrical and unsymmetrical cyanine dyes, because, in the method of LCUAO MO, δ and δ' might be able to describe the effects of various end nuclei of the dye chains and ω and ω' the alternation of bond character in the unsymmetrical cyanine chain. Unfortunately, it seems that the formula (A16) can be factorized only at the restricted values of δ and δ', namely $\delta=\delta'=-1,0,1$:

(I) $\delta=\delta'=-1$; $\Delta_{2n}=(\omega\omega'-\omega-\omega')\sin 2n\theta/\sin 2\theta$,

(II) $\delta=\delta'=0$; $\Delta_{2n}=d_{2n}=\sin 2n\theta/\sin\theta$,

(III)* $\delta=\delta'=+1$; $\Delta_{2n}=(\omega\omega'+\omega+\omega')\sin 2n\theta/\sin 2\theta$.

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* This case is discussed by Wallis in his recent publication entitled "Effect of Free Ends on the Vibration Frequencies of One-Dimensional Lattices" (R. F. Wallis, Phys. Rev. **105** (1957), 540). This paper came to our notice after the completion of our present calculation.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On the Ground State Spin of Ti^{47} and Mn^{55}

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October 7, 1957

Lawson and Uretsky[1] have shown that it is successful to compute the energy splittings for the configuration $(f_{7/2})^n$ by making use of the excitation energies for the $(f_{7/2})^{\pm 2}$ or $(f_{7/2})^4$ given by experimental measurements, in nuclei with $(f_{7/2})^n$ protons and 28 (closed shell) neutrons. Talmi[2] has also taken the same procedure in nuclei with $(f_{2/7})^n$ neutrons and 20 (closed shell) protons. In these nuclei it is worthwhile to note that the energy splitting $E(2)$ between the first excited state and the ground state for $(f_{7/2})^{\pm 2}$ shows a nearly constant value; 1.4∼1.6Mev. This regularity would give a sensitive effect to lowering the state $(7/2)$ than the state $(5/2)$ of nuclei for $(f_{7/2})^{\pm 3}$ configuration; (Ca^{43}, Ca^{45}, V^{51} and Mn^{53}). On the contrary, in nuclei: Fe^{56}, Cr^{54}, Ti^{46} and Ti^{48} the experimental values of $E(2)$ fall in the range from 0.85 to 1.0. In the first approximation, one neglects the effects of configurational mixing, and assumes that the first four excited states assigned ($J=0$, 2, 4, 6) of these nuclei would be pure and represented by the wave function $\psi(f_{7/2}^{\pm 2}(J), j^2(0):J)$, $J=0$, 2, 4, and 6 respectively. With the concept of the fractional parentage coefficients one computes the energy splittings of the eigenvalues for the pure states $\psi(f_{7/2}^{\pm 3}(J), j^2(0):J)$, $J=7/2$, $5/2$, $3/2, \cdots$ which are assumed to correspond to the low lying levels for the nuclei Mn^{55} and Ti^{47}, by adopting the known experinental data for the above $(f_{7/2})^{\pm 2}$ configuration. One obtains the following equations for the energy splittings,

$$E(7/2)-E(5/2)=-1.416\,E(2)+0.568\,E(4)+0.1E(6), \qquad (1)$$

$$E(3/2)-E(7/2)=0.225E(2)+1.608\,E(4)-1.083E(6). \qquad (2)$$

Here the absolute values of $E(J)$, $J=2$, 4, and 6 are about 0.85, 2.00, and 3.00 respectively. For these values the left-hand sides of the equations (1) and (2) are all positive. From the above treatment one finds that the crossing over of the $(7/2)$ and the $(5/2)$ levels depends sensitively on the energy splitting $E(2)$.

A summary of our calculations together with the known experimental information is contained in the table. For the experimentally unknown value $E(6)$ in Fe^{56} or Ti^{46}, a reasonably assumed one, 3.00, is replaced, which is assigned by an asterisk. It will now be predicted that the ground state spin of V^{49} would also be $5/2$ from the arguments for Ti^{48}. And for Cr^{49}, the measurement of $E(2)$ for Cr^{50} is desirable. Flowers[3] investigated to explain a crossing over of the $(7/2)$ and the $(5/2)$ levels by increasing the range of nuclear forces. Comparing with our treatment, an unusually long range force $(\sqrt{\alpha}=1.8\sim1.9)$ would be correct to assume for $(f_{7/2})^n$ configurations. But unfortunately, it is impossible to explain the theoretical level $(3/2)$; $l=3$ lying just above the $(7/2)$ level in both treatments from our present experimental data.

known levels (Mev)		
Fe^{56}	Ti^{46}	Ti^{48}
0.0 (0+)	0.0 (0+)	0.0 (0+)
0.845 (2+)	0.89 (2+)	0.99 (2+)
2.08 (4+)	2.01 (4+)	2.32 (4+)
3.00* (6+)	3.00* (6+)	3.35 (6+)

$E(7/2)-E(5/2)$ (Mev.)

	Calculated	Experimental
Ti^{47} (Ti^{46})	0.18	0.16
Ti^{47} (Ti^{48})	0.25	0.16
Mn^{55}	0.28	0.13

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A Comment on Bardeen's Theory of Superconductivity

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October 12, 1957

In recent papers on superconductivity, Bardeen, Cooper, and Schrieffer[1] have claimed that their theory based on the electron-phonon interaction results in an energy gap for electronic excitation. This seems to us erroneous, because by taking linear combinations of their excited states we can easily set up new states whose energies are continuously distributed far below the ground state Ψ_0 proposed by them.

Following their notation, they have started from the interaction Hamiltonian

$$H_2=-\sum_{k,k',q}V(k,k',q)\ c^*_{k+q,\sigma}\ c^*_{k-q,\sigma'}\ c_{k',\sigma'}c_{k,\sigma} \tag{1}$$

which is the second order term in the perturbation expansion of the electron-phonon interaction. As they have done, we shall only retain those scatterings in (1) which take place within a shell $\varDelta$ of width $\hbar\omega(\omega\quad\omega_0$: the Debye cut-off) at the Fermi surface and there replace $V(k,k',q)$ by its average $V>0$. Then terms with $\sigma=\sigma'$ in (1) cancel each other and *all* the remaining terms with $\sigma\neq\sigma'$ give rise to matrix elements of the same sign, contrary to the assertion of Bardeen et al. Hence, by the same argument made by them in assuming virtual pairs, we can take the state $\Psi'_0=\sum\Psi_\alpha'/\sqrt{A}$, where α runs over *all* those configurations in which electrons and holes are excited within the shell $\varDelta$, A being the total number of such configurations. Obviously we have

$$\langle\Psi_0'H_2\Psi_0'\rangle\sim-V(\varDelta N)^3 \tag{2}$$

where $\varDelta N\sim N\hbar/\zeta$ is the number of one-electron states within $\varDelta$, N being the total number of electrons and ζ being the Fermi energy. Since V is proportional to N^{-1}, the energy (2) is proportional to N^2. This physically means that the electron system will collapse if N is increased indefinitely.

If we add the fourth order term to (1), the divergence will be even more drastic. Similar situations can occur whenever we deal with nondiagonal terms of a perturbation exspansion of a Hamiltanian $H=\sum g^n H^{(n)}$, g being the coupling constant. As the total number of particles $N\to\infty$, the n-th order term $N^{-1} \langle \Psi H^{(n)} \Psi \rangle$ will converge only for a particular set of wave functions, so that the variation method cannot be applied to the expanded Hamiltonian as has been done by Bardeen et al. The reason why they have obtained their ground state together with a finite energy gap is simply that they have just omitted the terms with $k+k'\neq 0$ in (1) and retained their their reduced Hamiltonian H_{red} only. Though they have shown that the second order perturbation of $H_2 - H_{\text{red}}$ on their ground state is not appreciable, we can take linear combinations of their excited states whose energies are continuously distributed from $-V(\Delta N)^3$ as mentioned above.

Thus we have to conclude that the existence of an energy gap, which is essential to obtaining all Bardeen's results on superconducting properties, has not yet been proved from the first principle. We should not start from the Hamiltonian (1) unless we apply perturbation theory in the systematic way, but go back to the original Hamiltonian which explicitly contains the electron-phonon interaction. Recently one of he present authors (T. K.) has reformulated the theory of Bardeen and Pines[2)] without making use of the perturbational canonical transformation which is bound to lead to the Hamiltonian (1). The new theory of Bardeen et al. should rather be regarded as a phenomenological model with an energy gap. As such, it is no doubt interesting and suggestive.

We are much indebted to Professor Bardeen for sending their preprints as well as for his kind remarks.

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Angular correlation of Internal Bremsstrahlung and Successive Nuclear Gamma-Ray in the Radiative K-Capture

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The internal bremsstrahlung* in the radiative capture has analogous features to the usual β-ray in many respects. It is a process given by the second order terms with respect to e, in contrast to the non-radiative electron-capture.[1)] By analogies between I.B. and β-ray, a new method will be given for the study of the capture process, a field of research which yet remains little explored. The spectra of I.B. were already observed for various elements[2)]. Most of those were not followed by successive nuclear γ-rays. These spectra were compared with the results theoretically derived by Glauber and Martin[3] who treated I.B. from K- as well as L-electron capture taking into account the effect of nuclear coulomb potential on the propagation of the captured electron.

Experiments concerning the angular distribution of I.B., e.g., the directional correlation with successive γ-rays, or an expeperiment corresponding to Wu's[4] on the violation of the symmetry for the reflection of space, would sometimes give better information than β-ray, because the scattering of photon is smaller, and its intrinsic spin is larger than those of electrons.

The directional correlation function of I.B. and a successive nuclear γ-ray is expressed by

$$W=\sum_{M_a} P(M_i, M_a), \quad M_i=\pm 1, \quad (1)$$

when the K-capture is succeeded by an $E2$ γ-transition to the final state of spin zero**. $P(M_i, M_a)$ is the square of matrix element for the transition from the initial state with the magnetic quantum number M_a to the intermediate state $M_i=\pm 1$. By virtue of vanishing final spin, these two signs correspond respectively to the senses of the circular polarizations of $E2$ γ-ray. The case of $J_a=4$, $J_i=2$, and $M_i=1$ is considered in the following calculations. Eq. (1) is then expressed as

$$\begin{aligned} W(\theta) &= (A_0+(1/3)A_2)P_0(\cos\theta) \\ &\quad +(1/21)A_2P_2(\cos\theta)+W_a, \\ W_a(\theta) &= (2/21)^{1/2}(B_1-(1/5)B_3)P_1(\cos\theta) \\ &\quad +(2/105)B_3P_3(\cos\theta). \qquad (2) \end{aligned}$$

The coefficients A's and B's are derived by means of perturbation, and they include the energy of I.B., k and also the coupling constants C_T, C_T', C_A, and C_A', but none of the angular variables[5]. The term $W_a(\theta)$ is to be added if the successive γ-ray is circularly polarized completely in the left-handed sense and its sign is reversed for the right-handed one. The coulombic effect on the propagation function may be put aside for the considerably high energy of I.B.

In the present paper, these are calculated for a unique first forbidden radiative K-capture only, as a simple example. The coefficient of the isotropic term is normalized to unity. Then $W(\theta)$ is reduced to

$$\begin{aligned} W(\theta) &= 1+a_2P_2(\cos\theta)+W_a(\theta), \\ W_a(\theta) &= b_1P_1(\cos\theta)+b_3P_3(\cos\theta), \end{aligned} \qquad (3)$$

where a's and b's are given in Tables I and II for the case of Y^{88} (available energy $E_d=1.85$Mev)***. In addition, the asymmetric part $W_a(\theta)$ vanishes when $C_T=-C_T'=\pm C_A=\mp C_A'$, and is effective when $C_A=C_A'=0$, and $C_T=-C_T'$.

The intensity ratio of radiative to non-radiative capture is of the order of 10^{-4} $(E_d/mc^2)^2$ for the first forbidden transition This means the large N/S ratio of the order of $10^3\sim 10^4$ in the coincidence observation even for the simplest decay scheme such as $La^{138}(E_d=2.1$ Mev$)$[6]. A very large resolving power under the discrimination of the energy of I.B. is, therefore, necessary for the coincidence experiment. This difficulty would rather be easily overcome than the trouble due to the unavoidable large scattering of β-electrons, provided that a few technical improvements are achieved.

Similarly it will be possible to derive the formulae corresponding to $W_a(\theta)$ for the case of aligned initial nuclear spin. Experimentally, the alignment of nuclear spin is nearer to our hands than the polarization of nuclear gammas. The asymmetry would, however, be reduced on account of the mixing both of initial and intermediate states.

Table I. Anisotropy of the angular correlation of I.B. and nuclear gamma for the case of Y^{88}. $C_T=-C_T'$, $C_A=C_A'=0$, $\Delta M_i=-1$. a's and b's are defined by Eq. (3).

k (Mev)	a_2	b_1	b_3
1.22	0.018	−0.26	0.11
1.42	0.098	−0.75	0.10
1.62	0.13	−0.80	0.084
1.82	0.14	−0.95	0.093

Table II. Anisotropy of the angular correlation of I.B. and nuclear gamma for the case of Y^{88}. $C_T=-C_T'=-C_A=C_A'$. a's and b's are defined by Eq. (3). $b_1=b_3=0$.

k (Mev)	a_2
1.22	0.097
1.42	0.12
1.62	0.14
1.82	0.13

---

* Abbreviated as I.B. hereafter.

** The quantizing axis is chosen parallel to the direction of the successive γ-ray.

*** Y^{88} is adopted here as the only known case of unique first forbidden case with a successive E2 γ-ray, but its decay scheme is too complicated. If the unique first forbidden capture is not persisted, there are several samples of simple decay suitable to be observed.

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Note added in proof We have just learned that Cutkosky (Phys. Rev. **107** (1957) 330; **95** (1954), 1222) suggested the same problem.

Spin and Size of Λ^0 Particle

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October 19, 1957

The ratio of nonmesonic to mesonic decay of He^4 hyperfragments as derived by Ruderman and Karplus[1] is

$$Q^{(-)}=0.4(17)^l(E_b)^{1/2} \qquad (1)$$

where l is the pion angular momentum and E_b is the binding energy of the Λ^0 particle in Mev. The observed ratio is 1.25 and the binding energy is approximately 2 Mev[2]. Thus $l=0$ and 1 are strongly favored and from this the spin of the Λ^0 particle was estimated to be 1/2 or 3/2. In deriving Eq. (1) the above authors assumed the Λ^0 particle to be a point but when one assumes a finite size for the emitter the internal conversion coefficient is no longer so sensitively dependent on the orbital momentum of the emitted particle and it is desired to examine to what extent the size of the Λ^0 particle affects the conclusions on the spin of the Λ^0 particle.

Let us rewrite Eq. (1) as

$$Q^{(-)}=0.4\,(17)^l\,C_l(R)\,(E_b)^{1/2} \qquad (2)$$

and $C_l(R)$ shall be the measure of the effect of the size of the Λ° particle. The internal conversion is calculated by taking the pion-nucleon coupling

$$g\int d^3r\psi^*(\boldsymbol{r})\,\tau_i\sigma\cdot\nabla\varphi_i(\boldsymbol{r})\,\psi(\boldsymbol{r}) \qquad (3)$$

where the standard notations are used. In performing this integral, the motion of the proton in the hyperfragment will be neglected and the ejected proton will be assumed to travel as a free wave. The pion wave function will be decomposed into partial waves and we shall assume no distortion of the outgoing waves due to the size of the emitter. Then the effect of the finite size of the Λ^0 particle will be reflected only in the domain of the integral, Eq. (3). Since only the radial part of Eq. (3) is affected,

$$C_l(R)=\frac{|\int_R^\infty r^2 j_l(kr)\,(d/dr)h_l^{(1)}(qr)\,dr|^2}{|\int_0^\infty r^2 j_l(kr)\,(d/dr)h_l^{(1)}(qr)\,dr|^2} \qquad (4)$$

where k and q are as given in reference 1 and j and $h^{(1)}$ are spherical Bessel function and spherical Hankel function of the first kind, respectively. By numerical integration the following values were obtained. $C_1(0.4)=0.27$, $C_1(0.6)=0.11$, $C_1(0.8)=0.044$; $C_2(1.0)=0.037$, $C_2(1.2)=0.0060$; $C_3(1.0)=0.072$, $C_3(1.5)=0.0065$, $C_3(2.0)=0.00012$. R is measured in the unit of $\hbar/\mu c=1.4\times10^{-13}$cm. On the other hand, from the experimentally obtained values of $Q^{(-)}$ and E_b and from Eq. (2), the expected values of $C_l(R)$ are 0.13, 0.0078 and 0.00046 for $l=1$, 2 and 3, respectively. These situations will correspond to $R=0.6$, 1.2 and 1.8.

Thus the conclusion of Ruderman and Karplus that l must be one is safe so long as the radius of the Λ^0 particle is no larger than 0.8×10^{-13}cm. In order for a higher l value to be possible, a radius of 1.7×10^{-13}cm or more is required for the Λ^0 particle.

The author is thankful to Professors M. Karplus for helpful discussions.

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The Utility of the "Individual Γ_5 Transformation"

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October 25. 1957

Recently, Lee and Yang[1] have proposed a speculation that the nonconservation of parity is due to the peculiarity of neutrino. This two component neutrino theory is compatible with the available asymmetry data[2][3]. However, it seems to us that the asymmetry is common in all weak interactions participating in decays. This viewpoint is supported by θ-τ dilemma and the universality of coupling constant in weak interactions[4]. We now discuss a model according to which parity conservation is violated necessarily in all weak interactions.

We first assume that *all interactions are invariant under the individual Γ_5 transformation*[5]:

$$\psi_a \to \eta_a^a \gamma_5 \psi_a, \quad \psi_b \to \eta_b^a \psi_b, \quad \psi_c \to \eta_c^\alpha \psi_c, \cdots$$

$$m_a \to -m_a, \quad m_b \to m_b, \quad m_c \to m_c, \cdots.$$

(η's are phase factors : $|\eta|^2=1$.
where we regard the six species ν, e, μ, p, n and Λ as elementary or different particles. Other stronglies are considered to be composite of p, n, Λ and their antiparticles[6]. All interactions are, therefore, of Fermi type. If a and a' are different, the spinor pair $(\bar{\psi}_a\ O\ \psi_{a'})$ should contain a factor $(1 \pm \gamma_5)$ because of the invariance assumed, while if a and a' are the same, $(\bar{\psi}_a\, O\, \psi_{a'})$ need not contain $(1 \pm \gamma_5)$.

We further assume that *the interaction of the type*

$$(\textit{coupling constant})\quad (\bar{\psi}_a O_i \psi_{a'})\,(\bar{\psi}_{a''} O_i' \psi_{a'''})$$

is strong, if $a=a'$ and $a''=a'''$, and weak if otherwise. As an exception, however, we should admit that such combination as $a=a'$ and $a''=a'''$, if they contain leptons, are missing or much weaker than the weak interactions for some unknown reasons.

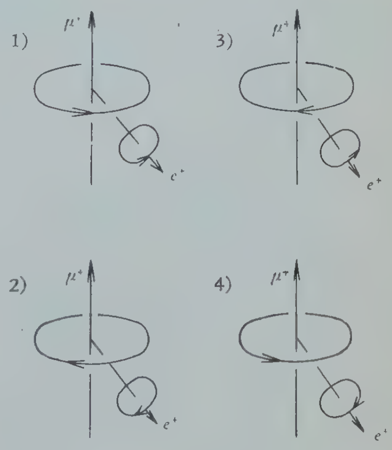
These two assumptions lead immediately that the parity and the strangeness cannot be conserved and the charge conjugation invariance is violated in all weak interactions, while in strong interactions, where the strangeness is conserved, the parity can be conserved. Thus a correlation seems to be established among the parity nonconservation, the strangeness selection rule and the strength of the interactions.

Although there is a substantial difference between the predictions of these assumptions and the two component neutrino theory, both fit equally well to experimental data available up to date. We know the experimental fact that e^+ in the μ-e decay predominates backwards relative to the direction of motion of μ^+. This fact and our assumptions lead to the four possible relations as shown in Fig. 1, between the directions of motion of μ^+ and e^+, and their spins. If the spin of μ^+ is antiparallel to its motion, then either the two component neutrino theory or the lepton number conservation law must be abandoned, while in our case we may choose 2) or 3) in Fig. 1, admitting the lepton number conservation law. For (STP) coupling, we get the relation

$$\lambda = 1 - (16/9)\rho,$$

where λ is the asymmetry coefficient and ρ is the Michel parameter; (*angular distribution of e^+ relative to the direction of motion of μ^+*) $=$ *const.* $(1+\lambda \cdot r \cdot \cos\theta)$, r being the degree of polarization of μ^+ at the instant of its decay, $\theta = \angle(\boldsymbol{p}_e, \boldsymbol{p}_\mu)$.

Our theory can also explain the β-decay data just as two component neutrino theory does. The details of this letter will be published in a later issue of this journal.



for (VA) coupling   for (STP) coupling

Fig. 1. In our case both (VA) and (STP) couplings, in the charge retention order, are permitted, but cannot be mixed.

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Low Lying Levels of P^{32} and *jj* Coupling

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October 28, 1957

Dalton, Hinds and Parry[1] have recently studied the energy levels of P^{32}, using the reaction the reaction $P^{31}(dp)P^{32}$. The angular distributions of the proton groups leading to several levels in P^{32} have been measured and analysed in terms of the deuteron stripping theory of Butler[2]. The results of this experiment, as well as those of the previous experiments[3], for the ground state doublet of P^{32} are in very good agreement with the predictions of the *jj* coupling shell model[4]. However, the presence of as many as four $l=0$ transitions below 4 Mev and a large number of $l=1$ transitions leading to excited states of P^{32} lead Dalton, Hinds and Parry to suggest that the ground state of P^{31} and even the low excited states of P^{32} do not have simple *jj* coupling shell model configurations.

In this note we attempt to understand the results of the $P^{31}(dp)$ experiment in terms of only small deviations from *jj* coupling. Regarding the $l=1$ transitions, it should be pointed out that such a situation occures in several nuclei in this region and does not necessarily imply a large impurity in the wave function of the ground state of the target nucleus. For example, the $K^{39}(dp)K^{40}$ experiment[5] also shows a large number of $l=1$ transitions, even though the ground state of K^{39} should have a fairly pure *jj* coupling configuration, $(d_{3/2})^{-1}$. Hence, although this phenomena is as yet difficult to understand, it does not call for assuming large impurities in P^{31} ground state wave function.

For the ground state of P^{31} we assume the wave function to consist of a dominant component (a) $(s_{1/2})^{3}_{1/2}$ and a small admixture of the states (b) $(s_{1/2})(d_{3/2})_0^{2}$ and (c) $[(s_{1/2})(d_{3/2})^{2}_{1}]_{1/2}$,* as required by a coupling scheme close to the *jj* coupling. Admixture of a component such as $[(s_{1/2})^{2}_{1}(d_{3/2})]_{1/2}$ is ruled out, since it would lead to the ground state doublet of P^{32} by capture of an $s_{1/2}$ neutron, and the experiments[1,3] indicate less than 5% of $l=0$ mixture in this transition.**

A qualitative scheme for the energy levels of P^{32} below ~4 Mev may be easily constructed on the basis of simple *jj* coupling configurations; a rough estimate of the excitation energies can be made on the basis of the observed single particle states[3] in S^{33}. We now make some brief comments on the results of Dalton, Hinds and Parry.

---

*) M. Umezawa (Prog. Theor. Phys. 8(1952), 509) has introduced the component (b) to explain the magnetic moment of P^{31}, whereas Huby and Newns (Proc. Phys. Soc. A68 (1955), 758) introduce the state (c) to explain the $P^{31}(dn)$ results.

**) We also consider the $d_{5/2}$ shell as filled and undisturbed. Inclusion of the simplest states involving break up of this core of 28 nucleons would require consideration of components like $[(d_{5/2})^{-1}(d_{3/2})_2^{2}]_{1/2}$ which would imply a complete breakdown of the *jj* coupling scheme.

1. Component (a) leads to the ground state doublet of P^{32} by capture of a $d_{3/2}$ neutron.
2. Doublet states in P^{32} belonging to the configuration $(s_{1/2})(f_{7/2})$ at an excitation energy of ∼3 Mev may be reached by capture of a $f_{7/2}$ neutron in component (a). Cross-section for an $l=3$ transition should be rather small. It is not surprising that this doublet is not well-identified. 3.45Mev level may be one component of this doublet.
3. A number of $l=1$ transitions may be observed due to capture of a $p_{3/2}$ or a $p_{1/2}$ neutron in components (a), (b) or (c). For excitation energies above 4 Mev. the levels of P^{32} are quite close, and hence a considerable amount of configuration mixing may be expected. This perhaps explains comparable magnitudes for the reduced widths of most of these transitions. The doublets of the configurations $(s_{1/2})(p_{3/2})$ and $(s_{1/2})(p_{1/2})$ may be expected at ∼3.5 Mev and ∼4.5 Mev energies; the former may be the levels observed at 3.259 and 3.318 Mev.
4. The (b) component may capture an $s_{1/2}$ meutron and lead to the states $(s_{1/2})^2{}_{0,1}$ $(d_{3/2})^2{}_0$ by an $l=0$ transition. These states may be identified with the observed 0.515 and 1.154 Mev levels. The (c) component may also lead to the state $(s_{1/2})_0^2$ $(d_{3/2})_1^2$ by capture of an $s_{1/2}$ neutron ($l=0$). This state may be one of the 2.177 —2.227 Mev doublet. Since these levels are not resolved, it may be that only one of them shows $l=0$ transition; the other one may be quite weak. Small reduced widths observed for these $l=0$ transitions indicate the components (b) and (c) are indeed small as required if the *jj* coupling is a good first approximation.
5. A number of weak states observed below 4 Mev could be reached by capture of a $d_{3/2}$ neutron in components (b) and (c). Cross-section for such transitions would be quite small.

We thus find that it is possible to understand the results of the $P^{31}(dp)P^{32}$ experiment for the low levels of P^{32}, at least qualitatively, in terms of a coupling scheme reasonably close to *jj* coupling.

Detailed quantitative calculations for the level scheme of P^{32} are now being carried out with a view to check the validity of the scheme presented above.

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